ON A THREE-DIMENSIONAL SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

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ABSTRACT. We investigate the solutions to the following system of nonlinear difference equations,
\[
\begin{align*}
    x_{n+1} &= f\left(\frac{z_n}{x_{n-1}}\right), \\
    y_{n+1} &= f\left(\frac{y_n}{x_n}\right), \\
    z_{n+1} &= f\left(\frac{y_n}{z_{n-1}}\right)
\end{align*}
\]
for \( n \in \mathbb{N}_0 \),
where \( x_{-1} = \alpha, y_{-1} = \beta, z_{-1} = \gamma, x_0 = \lambda, y_0 = \mu, \) and \( z_0 = \omega \) are positive real numbers.

1. INTRODUCTION

There are various results on systems of difference equations, see [1, 4, 10, 11, 2]. Understanding the theory and dynamics of such systems play a crucial role in mathematics, physics, and biology, see [8, 7, 3].

Consider the following system of difference equations,
\[
\begin{align*}
    x_{n+1} &= f\left(\frac{z_n}{x_{n-1}}\right), \\
    y_{n+1} &= f\left(\frac{y_n}{x_n}\right), \\
    z_{n+1} &= f\left(\frac{y_n}{z_{n-1}}\right)
\end{align*}
\]
for \( n \in \mathbb{N}_0 \),
where \( x_{-1} = \alpha, y_{-1} = \beta, z_{-1} = \gamma, x_0 = \lambda, y_0 = \mu, \) and \( z_0 = \omega \) are positive numbers.

Next are some papers on periodic and positive solutions to three-dimensional systems of nonlinear difference equations:

Tarek F. Ibrahim studied in [5] the periodic solutions of the following three-dimensional max-type cyclic system of difference equations
\[
\begin{align*}
    x_{n+1} &= \max\left\{ \frac{\alpha}{x_n}, y_n \right\}, \\
    y_{n+1} &= \max\left\{ \frac{\alpha}{y_n}, z_n \right\}, \\
    z_{n+1} &= \max\left\{ \frac{\alpha}{z_n}, x_n \right\}
\end{align*}
\]
M. R. S. Kulenović and Z. Nurkanović in [6] studied the global behavior of the following rational system of nonlinear difference equations

\[ x_{n+1} = \frac{a + x_n}{b + y_n}, \quad y_{n+1} = \frac{c + y_n}{d + z_n}, \quad z_{n+1} = \frac{e + z_n}{f + x_n}. \]

Stevo Stević in [9] studied the stability of the following rational system of nonlinear difference equations

\[ x_{n+1} = \frac{a_1x_{n-2}}{b_1y_nz_{n-1}x_{n-2} + c_1}, \quad y_{n+1} = \frac{a_2y_{n-2}}{b_2z_nx_{n-2} + c_2}, \quad z_{n+1} = \frac{a_3z_{n-2}}{b_3x_ny_{n-2} + c_3}. \]

2. Assumptions

The function \( f \) will have one of the following forms:

\[ f(t) = 1 \quad (2) \]
\[ f(t) = t \quad (3) \]
\[ f(t) = \begin{cases} A, & \text{if } t > 0 \\ B, & \text{if } t < 0 \end{cases} \quad (4) \]
\[ f(t) = \begin{cases} At, & \text{if } t < 0 \\ Bt, & \text{if } t > 0 \end{cases} \quad (5) \]

where \( A, B \in \mathbb{R} \) such that \( A^2 + B^2 \neq 0 \).

3. Main Results

**Theorem 3.1.** Let (2) hold and suppose that \( x_{-1}, y_{-1}, z_{-1}, x_0, y_0, \) and \( z_0 \) are positive real numbers. Also, let \( \{x_n, y_n, z_n\} \) be a solution of the system of equations (1) with \( x_{-1} = \alpha, y_{-1} = \beta, z_{-1} = \gamma, x_0 = \lambda, y_0 = \mu, \) and \( z_0 = \omega \). Then all solutions of (1) are of the following:

\[
egin{align*}
    x_{12n+1} &= \frac{1}{\beta}, & y_{12n+1} &= \frac{1}{\gamma}, & z_{12n+1} &= \frac{1}{\alpha} \\
    x_{12n+2} &= \frac{1}{\mu}, & y_{12n+2} &= \frac{1}{\omega}, & z_{12n+2} &= \frac{1}{\lambda} \\
    x_{12n+3} &= \gamma, & y_{12n+3} &= \alpha, & z_{12n+3} &= \beta \\
    x_{12n+4} &= \omega, & y_{12n+4} &= \lambda, & z_{12n+4} &= \mu \\
    x_{12n+5} &= \frac{1}{\alpha}, & y_{12n+5} &= \frac{1}{\beta}, & z_{12n+5} &= \frac{1}{\gamma} \\
    x_{12n+6} &= \frac{1}{\lambda}, & y_{12n+6} &= \frac{1}{\mu}, & z_{12n+6} &= \frac{1}{\omega} \\
    x_{12n+7} &= \beta, & y_{12n+7} &= \gamma, & z_{12n+7} &= \alpha \\
    x_{12n+8} &= \mu, & y_{12n+8} &= \omega, & z_{12n+8} &= \lambda \\
    x_{12n+9} &= \frac{1}{\gamma}, & y_{12n+9} &= \frac{1}{\alpha}, & z_{12n+9} &= \frac{1}{\beta} \\
    x_{12n+10} &= \frac{1}{\omega}, & y_{12n+10} &= \frac{1}{\lambda}, & z_{12n+10} &= \frac{1}{\mu} \\
    x_{12n+11} &= \alpha, & y_{12n+11} &= \beta, & z_{12n+11} &= \gamma \\
    x_{12n+12} &= \lambda, & y_{12n+12} &= \mu, & z_{12n+12} &= \omega.
\end{align*}
\]
Proof. The result holds for \( n = 0 \). Now suppose the result is true for some \( k > 0 \), we have the following:

\[
x_{12k+1} = \frac{1}{\beta}, \quad y_{12k+1} = \frac{1}{\gamma}, \quad z_{12k+1} = \frac{1}{\alpha}
\]

\[
x_{12k+2} = \frac{1}{\mu}, \quad y_{12k+2} = \frac{1}{\omega}, \quad z_{12k+2} = \frac{1}{\lambda}
\]

\[
x_{12k+3} = \gamma, \quad y_{12k+3} = \alpha, \quad z_{12k+3} = \beta
\]

\[
x_{12k+4} = \omega, \quad y_{12k+4} = \lambda, \quad z_{12k+4} = \mu
\]

\[
x_{12k+5} = \frac{1}{\alpha}, \quad y_{12k+5} = \frac{1}{\beta}, \quad z_{12k+5} = \frac{1}{\gamma}
\]

\[
x_{12k+6} = \frac{1}{\lambda}, \quad y_{12k+6} = \frac{1}{\mu}, \quad z_{12k+6} = \frac{1}{\omega}
\]

\[
x_{12k+7} = \beta, \quad y_{12k+7} = \gamma, \quad z_{12k+7} = \alpha
\]

\[
x_{12k+8} = \mu, \quad y_{12k+8} = \omega, \quad z_{12k+8} = \lambda
\]

\[
x_{12k+9} = \frac{1}{\gamma}, \quad y_{12k+9} = \frac{1}{\alpha}, \quad z_{12k+9} = \frac{1}{\beta}
\]

\[
x_{12k+10} = \frac{1}{\omega}, \quad y_{12k+10} = \frac{1}{\lambda}, \quad z_{12k+10} = \frac{1}{\mu}
\]

\[
x_{12k+11} = \alpha, \quad y_{12k+11} = \beta, \quad z_{12k+11} = \gamma
\]

\[
x_{12k+12} = \lambda, \quad y_{12k+12} = \mu, \quad z_{12k+12} = \omega.
\]

Also, for \( k + 1 \) we have the following:

\[
x_{12k+13} = \frac{1}{y_{12k+11}} = \frac{1}{\beta}, \quad y_{12k+13} = \frac{1}{y_{12k+12}} = \frac{1}{\gamma}, \quad z_{12k+13} = \frac{1}{y_{12k+13}} = \frac{1}{\alpha}
\]

\[
x_{12k+14} = \frac{1}{y_{12k+12}} = \frac{1}{\mu}, \quad y_{12k+14} = \frac{1}{y_{12k+13}} = \frac{1}{\omega}, \quad z_{12k+14} = \frac{1}{y_{12k+14}} = \frac{1}{\lambda}
\]

\[
x_{12k+15} = \gamma, \quad y_{12k+15} = \frac{1}{y_{12k+14}} = \frac{1}{\omega}, \quad z_{12k+15} = \gamma
\]

\[
x_{12k+16} = \omega, \quad y_{12k+16} = \frac{1}{y_{12k+15}} = \frac{1}{\alpha}, \quad z_{12k+16} = \alpha
\]

\[
x_{12k+17} = \frac{1}{y_{12k+15}} = \frac{1}{\alpha}, \quad y_{12k+17} = \frac{1}{y_{12k+16}} = \frac{1}{\mu}, \quad z_{12k+17} = \frac{1}{y_{12k+17}} = \frac{1}{\beta}
\]

\[
x_{12k+18} = \frac{1}{y_{12k+16}} = \frac{1}{\mu}, \quad y_{12k+18} = \frac{1}{y_{12k+17}} = \frac{1}{\gamma}, \quad z_{12k+18} = \frac{1}{y_{12k+18}} = \frac{1}{\omega}
\]

\[
x_{12k+19} = \frac{1}{y_{12k+17}} = \frac{1}{\beta}, \quad y_{12k+19} = \frac{1}{y_{12k+18}} = \frac{1}{\gamma}, \quad z_{12k+19} = \frac{1}{y_{12k+19}} = \frac{1}{\beta}
\]

\[
x_{12k+20} = \frac{1}{y_{12k+18}} = \frac{1}{\mu}, \quad y_{12k+20} = \frac{1}{y_{12k+19}} = \frac{1}{\gamma}, \quad z_{12k+20} = \frac{1}{y_{12k+20}} = \frac{1}{\alpha}
\]

\[
x_{12k+21} = \frac{1}{y_{12k+19}} = \frac{1}{\gamma}, \quad y_{12k+21} = \frac{1}{y_{12k+20}} = \frac{1}{\omega}, \quad z_{12k+21} = \frac{1}{y_{12k+21}} = \frac{1}{\lambda}
\]

\[
x_{12k+22} = \frac{1}{y_{12k+20}} = \frac{1}{\omega}, \quad y_{12k+22} = \frac{1}{y_{12k+21}} = \frac{1}{\alpha}, \quad z_{12k+22} = \frac{1}{y_{12k+22}} = \frac{1}{\mu}
\]

\[
x_{12k+23} = \frac{1}{y_{12k+21}} = \frac{1}{\gamma}, \quad y_{12k+23} = \frac{1}{y_{12k+22}} = \frac{1}{\beta}, \quad z_{12k+23} = \frac{1}{y_{12k+23}} = \frac{1}{\lambda}
\]

\[
x_{12k+24} = \frac{1}{y_{12k+22}} = \frac{1}{\mu}, \quad y_{12k+24} = \frac{1}{y_{12k+23}} = \frac{1}{\beta}, \quad z_{12k+24} = \frac{1}{y_{12k+24}} = \frac{1}{\omega}.
\]
Therefore the result is true for every \( k \in \mathbb{N}_0 \). This concludes the proof.

**Theorem 3.2.** Suppose that (2) hold and let \( \{x_n, y_n, z_n\} \) be a solution of the system of equations (1). Also, assume that \( x_{-1}, y_{-1}, z_{-1}, x_0, y_0, \) and \( z_0 \) are positive real numbers. Then all solutions of (1) are periodic with period twelve.

**Proof.** By (2), we have the following equal:

\[
x_{n+1} = \frac{f(z_n)}{y_{n-1}}, \quad y_{n+1} = \frac{f(x_n)}{z_{n-1}}, \quad z_{n+1} = \frac{f(y_n)}{x_{n-1}}
\]

\[
x_{n+2} = \frac{f(z_{n+1})}{y_n} = \frac{1}{y_n}, \quad y_{n+2} = \frac{f(x_{n+1})}{z_n} = \frac{1}{z_n}, \quad z_{n+2} = \frac{f(y_{n+1})}{x_n} = \frac{1}{x_n}
\]

\[
x_{n+3} = \frac{f(z_{n+2})}{y_{n+1}} = z_{n-1}, \quad y_{n+3} = \frac{f(x_{n+2})}{z_{n+1}} = x_{n-1}, \quad z_{n+3} = \frac{f(y_{n+2})}{x_{n+1}} = y_{n-1}
\]

\[
x_{n+4} = \frac{f(z_{n+3})}{y_{n+2}} = z_n, \quad y_{n+4} = \frac{f(x_{n+3})}{z_{n+2}} = x_n, \quad z_{n+4} = \frac{f(y_{n+3})}{x_{n+2}} = y_n
\]

\[
x_{n+5} = \frac{f(z_{n+4})}{y_{n+3}} = \frac{1}{x_{n-1}}, \quad y_{n+5} = \frac{f(x_{n+4})}{z_{n+3}} = \frac{1}{y_{n-1}}, \quad z_{n+5} = \frac{f(y_{n+4})}{x_{n+3}} = \frac{1}{z_{n-1}}
\]

\[
x_{n+6} = \frac{f(z_{n+5})}{y_{n+4}} = \frac{1}{x_n}, \quad y_{n+6} = \frac{f(x_{n+5})}{z_{n+4}} = \frac{1}{y_n}, \quad z_{n+6} = \frac{f(y_{n+5})}{x_{n+4}} = \frac{1}{z_n}
\]

\[
x_{n+7} = \frac{f(z_{n+6})}{y_{n+5}} = y_{n-1}, \quad y_{n+7} = \frac{f(x_{n+6})}{z_{n+5}} = z_{n-1}, \quad z_{n+7} = \frac{f(y_{n+6})}{x_{n+5}} = x_{n-1}
\]

\[
x_{n+8} = \frac{f(z_{n+7})}{y_{n+6}} = y_n, \quad y_{n+8} = \frac{f(x_{n+7})}{z_{n+6}} = z_n, \quad z_{n+8} = \frac{f(y_{n+7})}{x_{n+6}} = x_n
\]

\[
x_{n+9} = \frac{f(z_{n+8})}{y_{n+7}} = \frac{1}{z_{n-1}}, \quad y_{n+9} = \frac{f(x_{n+8})}{z_{n+7}} = \frac{1}{x_{n-1}}, \quad z_{n+9} = \frac{f(y_{n+8})}{x_{n+7}} = \frac{1}{y_{n-1}}
\]

\[
x_{n+10} = \frac{f(z_{n+9})}{y_{n+8}} = \frac{1}{z_n}, \quad y_{n+10} = \frac{f(x_{n+9})}{z_{n+8}} = \frac{1}{x_n}, \quad z_{n+10} = \frac{f(y_{n+9})}{x_{n+8}} = \frac{1}{y_n}
\]

\[
x_{n+11} = \frac{f(z_{n+10})}{y_{n+9}} = x_{n-1}, \quad y_{n+11} = \frac{f(x_{n+10})}{z_{n+9}} = y_{n-1}, \quad z_{n+11} = \frac{f(y_{n+10})}{x_{n+9}} = z_{n-1}
\]

\[
x_{n+12} = \frac{f(z_{n+11})}{y_{n+10}} = x_n, \quad y_{n+12} = \frac{f(x_{n+11})}{z_{n+10}} = y_n, \quad z_{n+12} = \frac{f(y_{n+11})}{x_{n+10}} = z_n.
\]

This concludes the proof.

To see the periodic behavior of \( \{x_n, y_n, z_n\} \), observe the following three diagrams with \( x_1 = 1, x_2 = 2, y_1 = 3, y_2 = 4, z_1 = 5, \) and \( z_2 = 6 \):
Theorem 3.3. Let \[ x_{-1}, y_{-1}, z_{-1}, x_0, y_0, z_0 \] be positive real numbers. Also, let \( \{x_n, y_n, z_n\} \) be a solution of the system of equations with \( x_{-1} = \alpha, y_{-1} = \beta, z_{-1} = \gamma, x_0 = \lambda, y_0 = \mu, \) and \( z_0 = \omega. \) Then all solutions of \( \{1\} \) are of the following:

\[
\begin{align*}
x_{6n-5} &= \frac{\omega}{\beta}, \quad y_{6n-5} = \frac{\lambda}{\gamma}, \quad z_{6n-5} = \frac{\mu}{\alpha} \\
x_{6n-4} &= \frac{1}{\alpha}, \quad y_{6n-4} = \frac{1}{\beta}, \quad z_{6n-4} = \frac{1}{\gamma} \\
x_{6n-3} &= \frac{1}{\lambda}, \quad y_{6n-3} = \frac{1}{\mu}, \quad z_{6n-3} = \frac{1}{\omega} \\
x_{6n-2} &= \frac{\beta}{\omega}, \quad y_{6n-2} = \frac{\gamma}{\lambda}, \quad z_{6n-2} = \frac{\alpha}{\mu} \\
x_{6n-1} &= \alpha, \quad y_{6n-1} = \beta, \quad z_{6n-1} = \gamma \\
x_{6n} &= \lambda, \quad y_{6n} = \mu, \quad z_{6n} = \omega.
\end{align*}
\]

Proof. The result holds for \( n = 0. \) Now suppose the result is true for some \( k > 0, \) we have the following:

\[
\begin{align*}
x_{6k-5} &= \frac{\omega}{\beta}, \quad y_{6k-5} = \frac{\lambda}{\gamma}, \quad z_{6k-5} = \frac{\mu}{\alpha} \\
x_{6k-4} &= \frac{1}{\alpha}, \quad y_{6k-4} = \frac{1}{\beta}, \quad z_{6k-4} = \frac{1}{\gamma} \\
x_{6k-3} &= \frac{1}{\lambda}, \quad y_{6k-3} = \frac{1}{\mu}, \quad z_{6k-3} = \frac{1}{\omega} \\
x_{6k-2} &= \frac{\beta}{\omega}, \quad y_{6k-2} = \frac{\gamma}{\lambda}, \quad z_{6k-2} = \frac{\alpha}{\mu} \\
x_{6k-1} &= \alpha, \quad y_{6k-1} = \beta, \quad z_{6k-1} = \gamma \\
x_{6k} &= \lambda, \quad y_{6k} = \mu, \quad z_{6k} = \omega.
\end{align*}
\]

Also, for \( k + 1 \) we have the following:

\[
\begin{align*}
x_{6k+1} &= \frac{f(x_{6k})}{y_{6k-1}} = \frac{z_{6k}}{y_{6k-1}} = \frac{\omega}{\beta} \\
y_{6k+1} &= \frac{f(x_{6k})}{z_{6k-1}} = \frac{x_{6k}}{z_{6k-1}} = \frac{\lambda}{\gamma} \\
z_{6k+1} &= \frac{f(y_{6k})}{x_{6k-1}} = \frac{y_{6k}}{x_{6k-1}} = \frac{\mu}{\alpha} \\
x_{6k+2} &= \frac{f(z_{6k+1})}{y_{6k}} = \frac{z_{6k+1}}{y_{6k}} = \frac{\mu/\alpha}{\mu} = \frac{1}{\alpha} \\
y_{6k+2} &= \frac{f(z_{6k+1})}{z_{6k}} = \frac{x_{6k+1}}{z_{6k}} = \frac{\omega/\beta}{\omega} = \frac{1}{\beta} \\
z_{6k+2} &= \frac{f(y_{6k+1})}{x_{6k}} = \frac{y_{6k+1}}{x_{6k}} = \frac{\lambda/\gamma}{\lambda} = \frac{1}{\gamma} \\
x_{6k+3} &= \frac{f(z_{6k+2})}{y_{6k+1}} = \frac{z_{6k+2}}{y_{6k+1}} = \frac{1/\gamma}{1/\lambda} = \frac{1}{\lambda} \\
y_{6k+3} &= \frac{f(z_{6k+2})}{z_{6k+1}} = \frac{x_{6k+2}}{z_{6k+1}} = \frac{1/\alpha}{1/\mu} = \frac{1}{\mu}
\end{align*}
\]
Therefore the result is true for every $k \in \mathbb{N}_0$. This concludes the proof. □

To see the periodic behavior of $\{x_n, y_n, z_n\}$, observe the following three diagrams with $x_1 = 1$, $x_2 = 2$, $y_1 = 3$, $y_2 = 4$, $z_1 = 5$, and $z_2 = 6$: 

![Plot of x(n)](image1)

![Plot of y(n)](image2)

![Plot of z(n)](image3)
Theorem 3.4. Let \( [1] \) hold with \( A, B < 0 \) and suppose that \( x_1, y_1, z_1, x_0, y_0, \) and \( z_0 \) are positive real numbers. Also, let \( \{x_n, y_n, z_n\} \) be a solution of the system of equations \( [1] \) with \( x_1 = \alpha, y_1 = \beta, z_1 = \gamma, x_0 = \lambda, y_0 = \mu, \) and \( z_0 = \omega \). Then all solutions of \( [1] \) are the following:

\[
x_{12n+1} = \frac{A}{\beta} \left( \frac{A}{B} \right)^{3n}, \quad y_{12n+1} = \frac{A}{\gamma} \left( \frac{A}{B} \right)^{3n}, \quad z_{12n+1} = \frac{A}{\alpha} \left( \frac{A}{B} \right)^{3n}
\]

\[
x_{12n+2} = \frac{B}{\mu} \left( \frac{B}{A} \right)^{3n}, \quad y_{12n+2} = \frac{B}{\omega} \left( \frac{B}{A} \right)^{3n}, \quad z_{12n+2} = \frac{B}{\lambda} \left( \frac{B}{A} \right)^{3n}
\]

\[
x_{12n+3} = \gamma \left( \frac{B}{A} \right)^{3n+1}, \quad y_{12n+3} = \alpha \left( \frac{B}{A} \right)^{3n+1}, \quad z_{12n+3} = \beta \left( \frac{B}{A} \right)^{3n+1}
\]

\[
x_{12n+4} = \omega \left( \frac{A}{B} \right)^{3n+1}, \quad y_{12n+4} = \lambda \left( \frac{A}{B} \right)^{3n+1}, \quad z_{12n+4} = \mu \left( \frac{A}{B} \right)^{3n+1}
\]

\[
x_{12n+5} = \frac{A}{\alpha} \left( \frac{A}{B} \right)^{3n+1}, \quad y_{12n+5} = \frac{A}{\beta} \left( \frac{A}{B} \right)^{3n+1}, \quad z_{12n+5} = \frac{A}{\gamma} \left( \frac{A}{B} \right)^{3n+1}
\]

\[
x_{12n+6} = \frac{B}{\lambda} \left( \frac{B}{A} \right)^{3n+1}, \quad y_{12n+6} = \frac{B}{\mu} \left( \frac{B}{A} \right)^{3n+1}, \quad z_{12n+6} = \frac{B}{\omega} \left( \frac{B}{A} \right)^{3n+1}
\]

\[
x_{12n+7} = \beta \left( \frac{B}{A} \right)^{3n+2}, \quad y_{12n+7} = \gamma \left( \frac{B}{A} \right)^{3n+2}, \quad z_{12n+7} = \alpha \left( \frac{B}{A} \right)^{3n+2}
\]

\[
x_{12n+8} = \mu \left( \frac{A}{B} \right)^{3n+2}, \quad y_{12n+8} = \omega \left( \frac{A}{B} \right)^{3n+2}, \quad z_{12n+8} = \lambda \left( \frac{A}{B} \right)^{3n+2}
\]

\[
x_{12n+9} = \frac{A}{\gamma} \left( \frac{A}{B} \right)^{3n+2}, \quad y_{12n+9} = \frac{A}{\alpha} \left( \frac{A}{B} \right)^{3n+2}, \quad z_{12n+9} = \frac{A}{\beta} \left( \frac{A}{B} \right)^{3n+2}
\]

\[
x_{12n+10} = \frac{B}{\omega} \left( \frac{B}{A} \right)^{3n+2}, \quad y_{12n+10} = \frac{B}{\lambda} \left( \frac{B}{A} \right)^{3n+2}, \quad z_{12n+10} = \frac{B}{\mu} \left( \frac{B}{A} \right)^{3n+2}
\]

\[
x_{12n+11} = \alpha \left( \frac{B}{A} \right)^{3n+3}, \quad y_{12n+11} = \beta \left( \frac{B}{A} \right)^{3n+3}, \quad z_{12n+11} = \gamma \left( \frac{B}{A} \right)^{3n+3}
\]

\[
x_{12n+12} = \lambda \left( \frac{A}{B} \right)^{3n+3}, \quad y_{12n+12} = \mu \left( \frac{A}{B} \right)^{3n+3}, \quad z_{12n+12} = \omega \left( \frac{A}{B} \right)^{3n+3}
\]

Proof. The result follows by the principle of mathematical induction. \( \square \)

Corollary 3.1. If \( A \neq B \) and \( A, B < 0 \), then the solutions of \( [1] \) are oscillatory and nonperiodic.

Theorem 3.5. Suppose \( [5] \) hold and let \( \{x_n, y_n, z_n\} \) be a solution of the system of equations \( [1] \). Also, assume that \( x_1, y_1, z_1, x_0, y_0, \) and \( z_0 \) are positive real numbers with \( A > 0 \) and \( B < 0 \). Then all solutions of \( [1] \) are periodic with period twelve.
Proof. Let \((\ldots, \cdot, \cdot)\) be the pair of solutions of (1), then the following set
\[
\left\{ \begin{array}{l}
(\alpha, \beta, \gamma), (\lambda, \mu, \omega) \\
(\beta A, \gamma B, \alpha A), (\gamma B, \alpha A, \beta B, A, \mu), \\
(\alpha A, \beta B, \gamma A), (\beta B, \gamma A, \alpha A).
\end{array} \right.
\]
is periodic with period twelve. This concludes the proof. □

To see the periodic and oscillatory behavior of \(\{x_n, y_n\}\), observe the following three diagrams with \(A = 1, B = -1, x_1 = 1, x_2 = 2, y_1 = 3, y_2 = 4, z_1 = 5, z_2 = 6\):

References


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