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TOPOLOGICAL STRUCTURES OF NON-NEWTONIAN METRIC SPACES

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ABSTRACT. The non-Newtonian calculi which are provide a wide variety of mathematical tools for use in science, engineering, and mathematics, appear to have considerable potential for use as alternatives to the classical calculus of Newton and Leibnitz. Every property in classical calculus has an analogue in non-Newtonian calculus. Recently, metric spaces are defined depending on the non-Newtonian calculus. We introduce, in this work, some properties the notion of the non-Newtonian metric spaces and investigate topological structures of non-Newtonian metric spaces.

1. INTRODUCTION

The non-Newtonian calculi are useful mathematical tools in science, engineering and mathematics and provide a wide variety of possibilities, as a different perspective. Specific fields of application include: fractal theory, image analysis (e.g., in bio-medicine), growth/decay processes (e. g., in economic growth, bacterial growth, and radioactive decay), finance (e.g., rates of return), the theory of elasticity in economics, marketing, the economics of climate change, atmospheric temperature, wave theory in physics, quantum physics and gauge theory, signal processing, information technology, pathogen counts in treated water, actuarial science, tumor therapy in medicine, materials science/engineering, demographics, differential equations (including a multiplicative Lorenz system and Runge-Kutta methods), calculus of variations, finite-difference methods, averages of functions, means of two positive numbers, weighted calculus, meta-calculus, approximation theory, least-squares methods, multivariable calculus, complex analysis, functional analysis, probability theory, utility theory, Bayesian analysis, stochastics, decision making, dynamical systems, chaos theory, and dimensional spaces.

Since these calculi has emerged, it has become a seriously alternative to the classical analysis developed by Newton and Leibnitz. Just like the classical analysis,

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Non-Newtonian calculi have many varieties as a derivative, an integral, a natural average, a special class of functions having a constant derivative, and two Fundamental Theorems which reveal that the derivative and integral are 'inversely' related. However, the results of obtained by non-Newtonian calculus has also significantly different from the classical analysis. For example, infinitely many non-Newtonian calculi have a nonlinear derivative or integral.

These calculi, which are mentioned above, are geometric calculus, bigeometric calculus, harmonic calculus, biharmonic calculus, quadratic calculus, and biquadratic calculus. In the geometric calculus and the bigeometric calculus from within of these calculi, the derivative and integral are both multiplicative. The geometric derivative and the bigeometric derivative are closely related to the well-known logarithmic derivative and elasticity, respectively. Also, the linear functions of classical calculus are the functions which having a constant derivative and besides the exponential functions in the geometric calculus are the functions which having a constant derivative, the power functions in the bigeometric calculus are the functions which having a constant derivative. Among the non-Newtonian calculi, geometric and bigeometric calculi have been often used.

The non-Newtonian calculi were developed by Michael Grossman and Robert Katz, and it were written to nine books related to the non-Newtonian calculi. Grossman and Katz published first book concerning with non-Newtonian calculus at 1972.

Recently, studies related with non-Newtonian calculus have increased. Especially, these studies are emerging in the field of applied mathematics.

Nonlinear multiplicative algorithm of Runge-Kutta type for solving multiplicative differential equations is prensented in [1]. The multiplicative Rössler system has been briefly examined to test the method proposed.

In [3], based on the ideas of the differentiation and integration which are two basic operation of Newton-Leibniz calculus obtained several results in non-Newtonian Analysis: Multiplicative mean value theorem, multiplicative tests for monotonicity, multiplicative tests for local extremum, multiplicative Taylor's Theorem for one variable and two variables, multiplicative chain rule, fundamental theorem of multiplicative calculus, multiplicative integration by parts. In addition, some applications on multiplicative calculus are demonstrated, in [3].

Rıza et. al. [12], discuss derivation of multiplicative finite difference methods for the numerical approximation of multiplicative and Volterra-type linear differential equations.

Uzer [16] has extended the multiplicative calculus to the complex valued functions and was interested in the statements of some fundamental theorems and the concepts of multiplicative complex calculus. Uzer, also has demonstrated some analogies between the multiplicative complex calculus and classical calculus by the theoretical and numerical examples.

The multiplicative version of Adams Bashforth-Moulton algorithms for the numerical solution of multiplicative differential equations is proposed by Mısırlı and Gürefe [11].

Bashirov and Rıza [4] have studied on the multiplicative differentiation for complex-valued functions and established the multiplicative Cauchy-Riemann conditions.

Quite recently, some authors have also worked on the classical sequence spaces and related topics by using non-Newtonian calculus [5, 7, 8, 9, 10, 13, 14, 15].

Çakmak and Başar [5] constructed the field $\mathbb{R}(N)$ of non-Newtonian real numbers and the concept of non-Newtonian metric. Also, in [5], triangle and Minkowski's inequalities of non-Newtonian calculus are given and the spaces of bounded, convergent, null convergent and $p-$ absolutely summable sequences in the sense of non-Newtonian calculus are defined.

Türkmen and Başar [15] have studied the classical sequence spaces and related topics in the sense of geometric calculus. Tekin and Başar [14] used the non-Newtonian complex calculus instead of non-Newtonian real calculus and geometric calculus and presented some important inequalities such triangle, Minkowski, and some other inequalities in the sense of non-Newtonian complex calculus which are frequently used. In [13, 14], the spaces of bounded, convergent, null convergent and $p-$ absolutely summable sequences given in the sense of non-Newtonian calculus.

Kadak [7] and Kadak et. al. [8, 9] have determined Köthe-Toeplitz duals and matrix transformations between certain sequence spaces over the non-Newtonian complex field.

In [10], classical paranormed sequence spaces have been introduced and proved that the spaces are $*-$complete. By using the notion of multiplier sequence, the $\alpha-$, $\beta-$ and $\gamma-$ duals of certain paranormed spaces have been computed and their basis have been constructed in [10].

In this paper, we give some new topological definitions with respect to non-Newtonian calculus and study some topological properties of non-Newtonian metric spaces.

Throughout the paper, we will use the abbreviation NN for the expression "non-Newtonian".

2. non-Newtonian Metric Spaces

We know that the ordinary metric spaces are more important and fundamental in functional analysis and topology. For many years, mathematicians interested with those spaces. Recently, Çakmak and Başar [5] defined and study some properties of NN-metric spaces. In this section, we study some properties of NN-metric spaces.

The function $\alpha : \mathbb{R} \to \mathbb{R}^+$ is called a *generator*, if this function is one-to-one.

Each generator generates exactly one arithmetic and, conversely, each arithmetic is generated by exactly one generator. As a generator, we choose the function exp from $\mathbb{R}$ to the set $\mathbb{R}^+$ of positive reals, that is to say,

$$\alpha : \mathbb{R} \to \mathbb{R}^+ \qquad \text{and} \qquad \alpha^{-1} : \mathbb{R}^+ \to \mathbb{R} x \mapsto \alpha(x) = e^x = y$$
$$y \mapsto \alpha^{-1}(y) = \ln y = x$$

If we choose $I(x) = x$ for all $x \in \mathbb{R}$, then we called that the function I is identity function. We know that inverse of the identity function is itself. Now, if we take $\alpha = I$ and $\alpha = \exp$, then the generator α generates the classical and geometric arithmetics, respectively.

The $\alpha-$positive numbers are the numbers $x \in A$ such that $\dot{0}\dot{<}x$ and the $\alpha-$negative numbers are those for which $x\dot{<}\dot{0}$. The $\alpha-$zero, $\dot{0}$, and the $\alpha-$one, $\dot{1}$, turn out to be $\alpha(0)$ and $\alpha(1)$. The $\alpha-$integers consist of $\dot{0}$ and all the numbers that result by successive $\alpha-$addition of $\dot{1}$ and $\dot{0}$ and by successive $\alpha-$subtraction of $\dot{1}$ and $\dot{0}$. Thus, the $\alpha-$integers turn out to be the following:

$$\cdots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \cdots$$

Therefore, we have $\dot{x} = \alpha(x)$ for each integer. If we choose $\dot{x}$ is an $\alpha-$integer, then $\dot{n} = \underbrace{\dot{1}\dot{+}\dot{1}\dot{+}...\dot{+}\dot{1}}$.

The set $\mathbb{R}^+(N)$ of NN-real numbers is defined as $\mathbb{R}^+(N) = \{\alpha(x) : x \in \mathbb{R}^+\}$.

Now, we define the $\alpha-$artihmetic operations and ordering relation as follows:

$$\begin{array}{ll}
\alpha - \textit{addition} & x\dot{+}y = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\}, \\
\alpha - \textit{subtraction} & x\dot{-}y = \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\}, \\
\alpha - \textit{multiplication} & x\dot{\times}y = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\}, \\
\alpha - \textit{division} & x\dot{/}y = \alpha\{\alpha^{-1}(x)/\alpha^{-1}(y)\}, \\
\alpha - \textit{order} & x\dot{<}y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y).
\end{array}$$

The binary operations $(\dot{+})$ addition and $(\dot{\times})$ multiplication for the set $\mathbb{R}(N)$ of NN-real numbers are defined by

$$\dot{+} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$(x, y) \mapsto x\dot{+}y = \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\}$$

$$\dot{\times} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$(x, y) \mapsto x\dot{\times}y = \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\}.$$

Lemma 1[5] $(\mathbb{R}(N), \dot{+}, \dot{\times})$ is a complete field.

The $\alpha-$square of a number $x \in A \subset \mathbb{R}(N)$ is denoted by $x\dot{\times}x = x^{2_N}$. For each $\alpha-$nonnegative number u, the symbol $\sqrt{x}^N$ will be used to denote $u = \alpha\{\sqrt{\alpha^{-1}(x)}\}$

which is the unique $\alpha-$square is equal to x, which means that $u^{2_N} = x$.

The $\alpha-$absolute value of a number $x \in A \subset \mathbb{R}(N)$ is defined as $\alpha\{|\alpha^{-1}(x)|\}$ and is denoted by $|x|_N$. For each number $x \in A \subset \mathbb{R}(N)$, $\sqrt{x^{2_N}}^N = |x|_N = \alpha\{|\alpha^{-1}(x)|\}$. In this case,

$$|x|_N = \begin{cases} x & , \quad x\dot{>}\alpha(0) \\ \alpha(0) & , \quad x = \alpha(0) \\ \alpha(0)\dot{-}x & , \quad x\dot{<}\alpha(0). \end{cases}$$

Let x and y be any two numbers. The NN-distance between these numbers is defined by

$$|x\dot{-}y|_N = \alpha\{|\alpha^{-1}(x) - \alpha^{-1}(y)|\} = \alpha\{|\alpha^{-1}(y) - \alpha^{-1}(x)|\} = |y\dot{-}x|_N.$$

Lemma 2[5] $|x\dot{\times}y|_N = |x|_N\dot{\times}|y|_N$ for $x, y \in \mathbb{R}(N)$.

Lemma 3(NN-triangle inequality)[5] Let $x, y \in \mathbb{R}(N)$. Then, $|x\dot{+}y|_N\dot{\leq}|x|_N\dot{+}|y|_N$.

Definition 4[5] Let X be a non-empty set and $d_N : X \times X \to \mathbb{R}^+(N)$ be a function such that for all $x, y, z \in X$;
(i) $d_N(x, y) = 1$ if and only if $x = y$
(ii) $d_N(x, y) = d_N(y, x)$
(iii) $d_N(x, y)\dot{\leq}d_N(x, z)\dot{+}d_N(z, y)$.

Then, the map d_N is called *non-Newtonian metric* (NNM) and the pair (X, d_N) is called *non-Newtonian metric space* (NNMS).

Definition 5[5] Let $X = (X, d_N)$ be a NNMS and (x_n) be any sequence in X.
(i) A sequence (x_n) is said to be *NN-convergent* if for every given $\varepsilon\dot{>}\alpha(0)$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $x \in X$ such that $d_N(x_n, x)\dot{<}\varepsilon$ for all $n > n_0$ and is denoted by ${}^N\lim_{n\to\infty} x_n = x$ or $x_n \to^N x$ as $n \to \infty$.
(ii) A sequence (x_n) is said to be *NN-Cauchy* if for every given $\varepsilon\dot{>}\alpha(0)$, there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d_N(x_n, x_m)\dot{<}\varepsilon$ for all $m, n > n_0$.
(iii) The space X is said to be *NN-complete* if every NN-Cauchy sequence in X converges.

Lemma 6[5] Let $X = (X, d_N)$ be a NNMS. Then,
(i) A NN-convergent sequence in X is bounded and its limit is unique.
(ii) A NN-convergent sequence in X is a Cauchy sequence in X.

From the definition of NN-Cauchy sequence and Lemma 2, we can give the following corollary:

Corollary 7 A NN-Cauchy sequence is bounded.

Lemma 8(Rearrangement of the NN-triangle inequality) Suppose $X = (X, d_N)$ is a NNMS and $x, y, z \in X$. Then

$$\left|d_N(x, y)\dot{-}d_N(y, z)\right|_N \dot{\leq}d_N(x, z)$$

Proof. The triangle inequality with the NNM axioms yields first $d_N(x,y)\dot{\leq}d_N(x,z)\dot{+}d_N(z,y)$ and second $d_N(z,y)\dot{\leq}d_N(z,x)\dot{+}d_N(x,y)$. Using the symmetry axiom, rearrangement of the first of these two inequalities gives

$$d_N(x,y)\dot{-}d_N(y,z)\dot{\leq}d_N(x,z)$$

and rearrangement of the second gives $d_N(y,z)\dot{-}d_N(x,y)\dot{\leq}d_N(x,z)$. These last two inequalities together prove the lemma.

Definition 9 Let $X=(X,d_N)$ be a NNMS. The space X is said to be *bounded* if there is a constant $M\dot{>}\dot{0}$ such that $d_N(x,y)\dot{\leq}M$ for all $x,y\in X$. The space X is said to be unbounded if it is not bounded.

Theorem 10 Let $X_1=(X_1,d_{N1})$ and $X_2=(X_2,d_{N2})$ be two NNMS. Then, $X=(X,d_N)$ is also a NNMS, where $X=X_1\times X_2$ and

$$d_N(x,y)=\max\{d_{N1}(x_1,y_1),d_{N2}(x_2,y_2)\} \tag{1}$$

for $x,y\in X$, where $x=(x_1,x_2)$, $y=(y_1,y_2)$. Additionally, X is bounded independent of boundedness of X_1 and X_2.

Proof. It is easily checked that the conditions nonnegativity and symmetry are satisfied. Now, we will prove that the condition triangle inequality for $x,y,z\in X=X_1\times X_2$. Then, we have

$$\begin{aligned}
d_N(x,y) &= \max\{d_{N1}(x_1,y_1),d_{N2}(x_2,y_2)\}\\
&\dot{\leq}\max\{d_{N1}(x_1,z_1)\dot{+}d_{N1}(z_1,y_1),d_{N2}(x_2,z_2)\dot{+}d_{N2}(z_2,y_2)\}\\
&\dot{\leq}\max\{d_{N1}(x_1,z_1),d_{N2}(x_2,z_2)\}\dot{+}\max\{d_{N1}(z_1,y_1),d_{N2}(z_2,y_2)\}\\
&= d_N(x,y)+d_N(z,y).
\end{aligned}$$

for all $x=(x_1,x_2),y=(y_1,y_2)\in X$. Hence, (X,d_N) is a NNMS.

It is trivial that X is bounded, since $d_N(x,y)=\max\{d_{N1}(x_1,y_1),d_{N2}(x_2,y_2)\}$ is finite.

The completeness property is inherited by products of ordinary metric spaces. That is, the product of countably many complete metric spaces is complete. We can carry this idea to NNMS, as below:

Theorem 11 Let $X_1=(X_1,d_{N1})$ and $X_2=(X_2,d_{N2})$ be two NNMS and $X=X_1\times X_2$. Define the NNM d_N as in (1) for $x,y\in X$, where $x=(x_1,x_2)$, $y=(y_1,y_2)$. Then, (X,d_N) is complete if and only if X_1 and X_2 are complete.

Definition 12 Let $X\subset\mathbb{R}^+(N)$. The set X is NN-bounded above if there is a number $M\in\mathbb{R}^+(N)$, say, called an NN-upper bound of X, such that $x\dot{\leq}M$ for all $x\in X$. The set X is NN-bounded below if there is a number $m\in\mathbb{R}^+(N)$, say, called an NN-lower bound of X, such that $m\dot{\leq}x$ for all $x\in X$.

Theorem 13 Let $X\subset\mathbb{R}^+(N)$.
(i) The NN-*supremum* of X is M, which is denoted by $\sup_N X$, if and only if $x\dot{\leq}M$ for all $x\in X$ and for $\varepsilon\dot{>}\alpha(0)$, there exists at least point $x\in X$ such that $|M\dot{-}x|_N\dot{<}\varepsilon$.

(ii) The NN-*infimum* of X is m, which is denoted by $\inf_N X$, if and only if $m\dot{\leq}x$ for all $x \in X$ and for $\varepsilon\dot{>}\alpha(0)$, there exists at least point $x \in X$ such that $|x\dot{-}m|_N\dot{<}\varepsilon$.

Proof. We only prove (i). Let NN-*supremum* of X is M. It is easily checked that the expression "$x\dot{\leq}M$ for all $x \in X$" is hold because of the definition of supremum. Now, we will prove that the expression "for $\varepsilon\dot{>}\alpha(0)$, there exists at least point $x \in X$ such that $|M\dot{-}x|_N\dot{<}\varepsilon$" is hold. Assume that there is no element $x \in X$ such that the condition (i) is holds. Thus, $|M\dot{-}\varepsilon|_N$ is also an NN-upper bound for the set X. This is a contradiction and M is NN-supremum of X.

Conversely, suppose that the condition is holds. Then, we prove that the statement "The NN-*supremum* of X is M" holds. Due to the expression $x\dot{\leq}M$ for all $x \in X$, we obtain that M is an NN-upper bound of X. Now, we suppose that NN-*supremum* of X is not M. Then, it is easily seen that $M\dot{>}\sup_N X$. If we choose $\varepsilon = (M\dot{-}\sup_N X)\dot{>}\alpha(0)$, then there exists at least a number $x\dot{>}\sup_N X$. This is a contradiction and we say that the NN-*supremum* of X is M.

Theorem 14 Let $X \neq \emptyset$. The NN-supremum and NN-infimum of X are both unique, if there exist. Nevertheless, if there exist NN-supremum and NN-infimum of X, then $\inf_N X\dot{\leq}\sup_N X$.

Proof. We suppose that M and $M^{'}$ are suprema of X. If $M\dot{\leq}M^{'}$, then $M^{'}$ is an upper bound of X and M is aleast upper bound. Similarly, $M^{'}\dot{\leq}M$ and so $M = M^{'}$. If m and $m^{'}$ are infima of X, then $m\dot{\geq}m^{'}$, since $m^{'}$ is a lower bound of X and m is a greatest lower bound, similarly $m^{'}\dot{\geq}m$ and so $m = m^{'}$.

If $\inf_N X$ and $\sup_N X$ exist, then X is nonempty. We choose $x \in X$. Hence $\inf_N X\dot{\leq}x\dot{\leq}\sup_N X$, since $\inf_N X$ is a lower bound and $\sup_N X$ is an upper bound. It follows that $\inf_N X\dot{\leq}\sup_N X$.

Theorem 15 Let $X \neq \emptyset$, $Y \neq \emptyset$ and $X \subset Y$. If $\sup_N X$ and $\sup_N Y$ exist, then $\sup_N X\dot{\leq}\sup_N Y$ and if $\inf_N X$ and $\inf_N Y$ exist, then $\inf_N X\dot{\geq}\inf_N Y$.

Proof. Since $\sup_N Y$ is an upper bound of Y and $X \subset Y$, it follows that $\sup_N Y$ is an upper bound of X, so $\sup_N X\dot{\leq}\sup_N Y$. The proof for the infimum is similar.

Definition 16 We suppose that $X = (X, d_N)$ be a NNMS and $A \subset X$. We define the diameter of A to be $\sup_N\{d_N(x,y) : x, y \in A\}$ and we denote $diam_N(A)$.

As in the ordinary metric space, the diameter is dependent on the metric.

Theorem 17 Let $X = (X, d_N)$ be a NNMS and $A, B \subset X$ for which $A \subseteq B$. Then, $diam_N(A)\dot{\leq}diam_N(B)$.

Proof. Since $A \subseteq B$, we have $\{d_N(a,b) : a, b \in A\} \subseteq \{d_N(c,d) : c, d \in B\}$ and so, by the Theorem 2, $\sup_N\{d_N(a,b) : a, b \in A\}\dot{\leq}\sup_N\{d_N(c,d) : c, d \in B\}$, which is precisely the inequality that is required.

Theorem 18 Let $\omega(N) = \{x = (x_k) : x_k \in \mathbb{R}(N) \text{ for all } k \in \mathbb{N}\}$ denote the space of all sequences over the NN-field $\mathbb{R}(N)$ ([5]) and d_N^ω be a function on $\omega(N)$ defined by

$$d_N^\omega(x,y) = ^N \sum_{k=1}^{\infty} A_k \dot{\times} \left(\frac{|x_k \dot{-} y_k|_N}{\alpha(1) \dot{+} |x_k \dot{-} y_k|_N} N \right)$$

for all $x, y \in \omega(N)$, where ${}^N \sum_{k=1}^{\infty} A_k$ is NN-convergent and $A_k \dot{>} \alpha(0)$ for all $k \in \mathbb{N}$. Then, $(\omega(N), d_N^\omega)$ is a bounded NNMS.

Proof. We can prove the similar way of Theorem 5.1 in [5] that the d_N^ω is a NNM and $(\omega(N), d_N^\omega)$ is a NNMS. Let $x, y \in \omega(N)$, then we have

$$A_k \frac{|x_k \dot{-} y_k|_N}{\alpha(1) \dot{+} |x_k \dot{-} y_k|_N} N \dot{<}^N A_k \dot{\leq}^N \max_{k \in \mathbb{N}} A_k \ for\ all\ k\ \in \mathbb{N}.$$

Therefore, the NNMS $(\omega(N), d_N^\omega)$ is bounded.

3. Open and Closed Balls-Sets in NNMS

Definition 19 Assume that $X = (X, d_N)$ be a NNMS, A be a subset of X and $x \in X$. The *distance from x to A* defined by $D_N(x, A) = \inf_N \{d_N(x, a) : a \in A\}$. Let $a \in X$. Then, a is called a NN-*boundary point* of A in X if and only if $D_N(a, A) = \alpha(0) = D_N(a, A^c)$, where A^c is a complement of A. The collection of NN-boundary points of A in X is called the NN-boundary of A in X and denoted by $\partial_N A$.

Theorem 20 Suppose that X is a NNMS and A is a subset of X. Then $\partial_N A = \partial_N (A^c)$.

Proof. Since $(A^c)^c = A$, we have, for each $a \in X$, $dist_N(a, (A^c)^c) = dist_N(a, A)$. Thus, $dist_N(a, A) = \alpha(0) = dist_N(a, A^c)$ if and only if $dist_N(a, A^c) = \alpha(0) = dist_N(a, (A^c)^c)$. In other words, $a \in \partial_N A$ if and only if $a \in \partial_N (A^c)$.

Definition 21 Let $X = (X, d_N)$ be a NNMS, $x \in X$. For each $r \dot{>} \dot{0}$, we define the NN-*open ball* in X centered at the point x and with radius r to be the set $B_N(x; r) = \{y \in X : d_N(x, y) \dot{<} r\}$; and the NN-*closed ball* in X centered at the point x and with radius r to be the set $\overline{B}_N(x; r) = \{y \in X : d_N(x, y) \dot{\leq} r\}$.

Remark 22 It is clear that if we choose $\alpha(0) \dot{<} r_1 \dot{<} r_2$, then $B_N(x; r_1) \subset B_N(x; r_2)$.

Example 23 Let $a, b \in \mathbb{R}^+(N)$, $a \dot{<} b$ and $\alpha = \exp$. The NN-open ball of $\mathbb{R}^+(N)$ with the NN-usual metric in Corollary 3.5 in [5] is the NN-open interval defined by

$$(a, b)_N = B_N \left(e^{\frac{a+b}{2}}; e^{\frac{b-a}{2}} \right).$$

Proof. Let $(a+b)/2 = u$ and $(b-a)/2 = v$. We can write

$$(a, b)_N = B\left(e^u; e^v\right) = \left\{ e^y \in \mathbb{R}^+(N) : d_N(e^u, e^y) \dot{<} e^v \right\}.$$

It is easy to see that

$$|e^u \dot{-} e^y|_N = \alpha \left\{ |\alpha^{-1}(e^u) - \alpha^{-1}(e^y)| \right\} = \alpha \left\{ |u - y| \right\} = e^{|u-y|}$$

and we can obtain

$$e^{|u-y|}\dot{<}e^{v} \Rightarrow |u-y|<v \Rightarrow u-v<y<u+v \Rightarrow a<y<b.$$

Example 24 Let $a, b \in \mathbb{R}^+(N)$, $a\dot{<}b$ and $\alpha = \exp$. The NN-closed ball of $\mathbb{R}^+(N)$ with the NN-usual metric in Corollary 3.5 in [5] is the NN-closed interval defined by

$$[a,b]_N = \overline{B}_N\left(e^{\frac{a+b}{2}}; e^{\frac{b-a}{2}}\right).$$

Theorem 25 Let $X = (X, d_N)$ be a NNMS, $a \in X$ and $r \in \mathbb{R}^+(N)$. Then,
(i) $\partial_N(B_N(a;r)) \subseteq \{x \in X : d_N(x,a) = r\}$,
(ii) $\partial_N(\overline{B}_N(a;r)) \subseteq \{x \in X : d_N(x,a) = r\}$,
(iii) $B_N(a;r)$ is NN-open in X,
(iv) $\overline{B}_N(a;r)$ is NN-closed in X.

Proof. Let B be a ball and $u \in \partial_N B$. Then, from Definition 3, $D_N(u,B) = \alpha(0)$ and $D_N(u,B^c) = \alpha(0)$. Let $s = d_N(u,a)$. For each $w \in B$, we have $d_N(a,w)\dot{\leq}r$. From Lemma 2, we have $d_N(u,w)\dot{\geq}d_N(u,a)\dot{-}d_N(a,w)\dot{\geq}s\dot{-}r$. Therefore

$$\alpha(0) = D_N(u,B) = \inf\{d_N(u,w) : w \in B\}\dot{\geq}s\dot{-}r$$

and we obtain $s\dot{\leq}r$. Similarly, for each $v \in B^c$, we have $d_N(v,a)\dot{\geq}r$, which, again from Lemma 2, implies that $d_N(u,v)\dot{\geq}d_N(v,a)\dot{-}d_N(u,a)\dot{\geq}r\dot{-}s$. Therefore

$$\alpha(0) = D_N(u,B^c) = \inf\{d_N(u,v) : v \in B^c\}\dot{\geq}r\dot{-}s.$$

We also obtain $r\dot{\leq}s$. The two inequalities give us $r = s$ and, since u is arbitrary in $\partial_N B$, we have proved that (i) and (ii).

The conditions (iii) and (iv) follow by definition, because $B_N(a;r)$ contains none of these boundary points and $\overline{B}_N(a;r)$ contains all of them.

Following theorem can be proved similar to ordinary metric space and hence we omit the details.

Theorem 26 Let $X = (X, d_N)$ be a NNMS and $x \in X$, $r\dot{>}\alpha(0)$. Then,
(i) X and $\emptyset$ are NN-open sets,
(ii) Finite intersection and arbitrary union of NN-open balls $B_N(x;r)$ is NN-open,
(iii) Arbitrary intersection and finite union of NN-closed balls in a NNMS is NN-closed.

Corollary 27 Let $X = (X, d_N)$ be a NNMS. The NNM X is a topological space with respect to the set of all NN-open sets.

4. NNM Topology

Theorem 28 Let $X = (X, d_N)$ be a NNMS, τ be a given topology on X and $B_N(x;\varepsilon)$ be a NN $\varepsilon-$ball. The collection $\mathcal{C}_N = \{B_N(x;\varepsilon) : x \in X, \varepsilon\dot{>}\alpha(0)\}$ of all NN-balls is a basis for a topology τ on X.

Proof. It is clear that $X \subset \bigcup_{x\in X} B_N(x;\varepsilon)$. Now, we choose the NN-open balls as $B_N(x;\varepsilon)$ and $B_N(y;\varepsilon)$ for $x, y \in X$. Let $a \in B_N(x;\varepsilon) \cap B_N(y;\varepsilon)$. Then there is a NN-ball $B_N(x;\varepsilon_n)$ for some $\varepsilon_n \dot{>} \alpha(0)$ such that $B_N(a;\varepsilon_1) \subset B_N(x;\varepsilon)$ and $B_N(a;\varepsilon_2) \subset B_N(x;\varepsilon)$. Take $\varepsilon_n = \min\{\varepsilon_1, \varepsilon_2\}$. Then, from Remark 3, $B_N(a;\varepsilon_n) \subset B_N(x;\varepsilon) \cap B_N(y;\varepsilon)$, as we desired.

Now, we construct new topological definitions related to NNM.

Definition 29 Let $X = (X, d_N)$ be a NNMS.
(i) The NNMS X together with a topology τ generated by NNM d_N is called a *NNM topological space* and τ is called a *NNM topology* on X.
(ii) A NNM topological space is said to be *non-Newtonian metrizable*(NN-metrizable), if there exists a NNM d_N on X that induces the topology of X. A NNMS X is *NN-metrizable space* together with the NNM d_N that induces the topology of X.
(iii) A set F is $\tau-$open in X in the NNM topology τ induced by the NNM d_N if and only if for each $x \in F$, there is a $\delta \dot{>} \alpha(0)$ such that $B_N(x;\delta) \subset F$. In similar, a set G in X is called $\tau-$closed if its complement X/G is $\tau-$open.

Çakmak and Başar [5] defined the convergence of a sequence in a NNMS. Now, we investigate the relation between the NNM topology τ and the topology of NNM convergence in X.

Theorem 30 The topology of NNM convergence and the NNM topology on a NNMS are equivalent.

Proof. We must show that a sequence in X converges with respect to the topology of NNM convergence if and only if it converges with respect to the NNM topology on X.

Let $\varepsilon \dot{>} \alpha(0)$ and consider an NN $\varepsilon-$ball $B_N(x;\varepsilon)$ in X. Suppose that a sequence (x_n) in X converging to a point $x \in X$ with respect to the topology of NNM convergence. We show that for sufficiently large value of n, x_n is in $B_N(x;\varepsilon)$. From the definition of the NNM convergence, we know, for $\varepsilon \dot{>} \alpha(0)$, there exist an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $d_N(x,y) \dot{<} \varepsilon$ for all $n > n_0$. By the definition of an NN-open ball $B_N(x;\varepsilon)$, this implies that for all $n \dot{\geq} n_0$, $x_n \in B_N(x;\varepsilon)$.

Conversely, we assume that the sequence (x_n) in X converges to a point $a \in X$ with respect to the NNM topology τ on X. Then there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $x_n \in B_N(a;\varepsilon)$ for all $n > n_0$. From the definition of the NN-ball $B_N(a;\varepsilon)$ implies that $d_N(x_n, a) \dot{<} \varepsilon$ for all $n > n_0$. Thus, $x_n \to a$ with respect to the topology of NNM convergence if and only if $x_n \to a$ in the NNM topology τ on X. This completes the proof.

Lemma 31 Let d is an ordinary metric on X and d_N is a function which defined by

$$d_N(x,z) = d(x,y) \dot{+} d(y,z) \tag{2}$$

for all $x, y, z \in X$. The function d_N satisfies NNM conditions. So d_N is a NNM on X.

Proof. Let $x \in X$ and $r \dot{>} 1$. The NN-ball $B_N(x; r)$ coincides with the ball with respect to ordinary metric space (X, d). In this case, the topology generated by the NN-metric d_N is equivalent to the topology generated by the ordinary metric d on X.

Theorem 32 If the topological space X is metrizable, then it is NN-metrizable.

Proof. Let X be a metrizable space. Then, there exists an ordinary metric d on X that induces the topology of X. Define the NNM d_N on X by 2. Therefore, the metric d_N generate the same topology onthat of X. Hence X is NN-metrizable. This is what we wished to show.

5. Topological Properties

In this section, we investigate the topological properties of a NNMS X equipped with the NNM topology τ.

A topological space X satisfies the T_0-*separation axiom*, or is a T_0-*space*, if for each pair x_0 and y_0 of distinct points of X there exists a neighborhood of at least one point which does not contain the other.

A topological space X satisfies the T_1-*separation axiom*, or is a T_1-*space*, if for each pair x_0 and y_0 of distinct points of X there exists a neighborhoods Ux_0 and Uy_0 of x_0 and y_0 such that y_0 is not in Ux_0 and x_0 is not in Uy_0.

A topological space X satisfies the *Hausdorff space*, or is a T_2-*space*, if whenever x_0 and y_0 are distinct points of X, there exist disjoint neighborhoods of x_0 and y_0 (the topology on X is then called a Hausdorff topology).

A topological space X satisfies the *regular space*, if for every point $x \in X$ and every closed set $F \subset X$ and x is not in F, there are neighborhoods Ux of x and UF of F such that $Ux \cap UF = \emptyset$. A topological space X is called T_3-space, if X is a regular T_1-space.

A topological space X satisfies the *normal space*, if every pair of disjoint closed sets in X have disjoint neighborhoods. A topological space X is called T_4-space, if X is a normal T_1-space.

Theorem 33 The NNMS X is a T_0-space.

Proof. We choose x_0 and y_0 in X such that $x_0 \neq y_0$. Then, $d_N(x_0, y_0) = r$ for some $r \dot{>} \alpha(0)$. We take an open ball $B_N(x_0, r)$ in X. Therefore, by the definition y_0 is not in $B_N(x_0, r)$. Thus, we say that the NNMS X is a T_0-space.

Theorem 34 The NNMS X is a T_1-space.

Proof. Let x_0 and y_0 in X such that $x_0 \neq y_0$. Suppose that $d_N(x_0, y_0) = r_1 \dot{>} \alpha(0)$, and consider a ball $B_N(x_0, r_1)$ in X. It is clear that y_0 is not in $B_N(x_0, r_1)$.

Similarly, suppose that $d_N(x_0, y_0) = r_2 \dot{>} \alpha(0)$, and consider a ball $B_N(y_0, r_2)$ in X, then x_0 is not in $B_N(y_0, r_2)$. This completes the proof.

Theorem 35 The NNMS X is a T_2-space.

Proof. Suppose that x_0 and y_0 be distinct points for in X. Hence, from by the definition of NNM, $d_N(x_0, y_0) = \varepsilon \dot{>} \alpha(0)$. Consider the NN-open balls $B_{N1} = (x_0, \varepsilon/3)$ and $B_{N2} = (y_0, \varepsilon/3)$ which centered at x_0 and y_0, respectively. We claim that $B_{N1} \cap B_{N2} = \emptyset$. If $p \in B_{N1} \cap B_{N2}$, then $d_N(x_0, p) \dot{<} \varepsilon/3$ and $d_N(y_0, p) \dot{<} \varepsilon/3$. Hence, by the NN-triangle inequality,

$$d_N(x_0, y_0) \dot{\leq} d_N(x_0, p) \dot{+} d_N(y_0, p) \dot{<} \frac{2}{3}\varepsilon.$$

But this result contradicts the fact that $d_N(x_0, y_0) = \varepsilon$. Thus, B_{N1} and B_{N2} are disjoint, i.e., x_0 and y_0 belong to the disjoint NN-open balls B_{N1} and B_{N2}, respectively. Accordingly, NNMS X is Hausdorff.

Theorem 36 The NNMS X is normal.

Proof. From Theorem 5, the NNMS X is a Hausdorff space. If we choose U and V are two disjoint closed subsets of the NNMS X, then

$$U_X = \{x \in X : d_N(x, U) \dot{<} d_N(x, V)\} \tag{3}$$

and

$$V_X = \{x \in X : d_N(x, V) \dot{<} d_N(x, U)\} \tag{4}$$

are two disjoint neighborhoods of U and V, respectively. This shows that any two closed disjoint subsets of NNMS X can be separated with disjoint neighborhoods. Hence X is normal.

Theorem 37 The NNMS X is regular.

Proof. It is known that a normal space is regular. Since the NNMS X is normal, then the NNMS X is regular.

6. Conclusion

The purpose of this paper is given to topological structure of non-Newtonian metric spaces which was initiated by Cakmak and Basar [5].

Non-Newtonian calculus is an alternative to the classical calculus of Newton and Leibnitz. Every property and concept in classical calculus has an analogue in non-Newtonian calculus.

It is made following studies in this paper: In Section 2, It is studied some properties of the NNMS which is given by Cakmak and Basar [5] and is given some new definition belonging to NNMS. Section 3 devoted to the open balls and sets in NNMS. In this section, it is given NN-boundary points and sets, NN-open balls and sets, NN-closed balls and sets. In Section 4, the definitions of a basis for a topology on X, NN-topological space, NN-metrizability are given and studied some properties. Finally, in Section 5, the Separation Axioms with respect to the NNM are defined and proved the NNMS X is T_0, T_1, Hausdorff, normal and regular space.

We should note that, as a natural continuation of this paper, one can study the other properties of NNMS for example compactness, category of NNMS, continuity etc. As shown in [5] and this paper, the concept of NNMS has brought a different perspective to the metric space theory. Therefore, this concept can also be studied to the fixed point theory, as in metric fixed metric theory and so it can constructed the NNM fixed point theory. As is well known, in recent years, the study of metric fixed point theory has been widely researched because of the this theory has a fundamental role in various areas of mathematics, science and economic studies.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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