EXISTENCE OF POSITIVE SOLUTIONS FOR ITERATIVE SYSTEMS OF NONLINEAR m-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract. In this paper, we establish the existence of positive solutions for the iterative system of nonlinear dynamic equations on time scales
\[ y_i^{\Delta\Delta}(t) + p_i(t)f_i(y_{i+1}(t)) = 0, \quad 1 \leq i \leq n, \quad t \in [t_1, \sigma(t_m)]T, \]
\[ y_{i+1}(t) = y_1(t), \quad t \in [t_1, \sigma(t_m)]T, \]
satisfying the m-point boundary conditions
\[ y_i(t_1) = 0, \]
\[ \alpha y_i(\sigma(t_m)) + \beta y_i(\sigma(t_m)) = \sum_{k=2}^{m-1} y_i^{\Delta}(t_k), \quad 1 \leq i \leq n, \]
by applying Guo–Krasnosel'skiĭ fixed point theorem.

1. Introduction

The theory of time scales was introduced by Hilger [18] not only to unify continuous and discrete theory, but also to provide an accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. This theory [1, 5, 6] can be applied to various real life situations like epidemic models, stock markets and mathematical modeling of physical and biological systems.

The existence of positive solutions of boundary value problems have created a great deal of interest due to wide applicability in both theory and applications. By using fixed point theorems in cones, Fink and Gatica [7], Wang [27], Zhou and Xu [28], Henderson et al. [9, 10, 11, 12] have studied existence of positive solutions for system of nonlinear boundary value problems associated with ordinary differential equations. Agarwal and O’Regan [2], Sun et al. [25, 26], Henderson et al. [13, 14, 15, 16, 17] considered the system of nonlinear boundary value problems associated with difference equations and established the existence of positive solutions to boundary value problems by using various techniques. In recent years, much attention is paid in establishing the existence of positive solutions for iterative
systems of nonlinear boundary value problems on time scales by the researchers. For some recent contributions, we refer to \[3, 4, 19, 21, 22, 23, 24\].

Motivated by the papers mentioned above, in this paper, we establish the existence of positive solutions for the iterative systems of second order nonlinear dynamic equations on time scales

\[
\begin{align*}
  y_i^\Delta(t) + p_i(t)f_i(y_{i+1}(t)) &= 0, \quad 1 \leq i \leq n, \quad t \in [t_1, \sigma(t_m)]_T, \\
  y_{n+1}(t) &= y_1(t), \quad t \in [t_1, \sigma(t_m)]_T,
\end{align*}
\]

satisfying the \(m\)-point boundary conditions

\[
\begin{align*}
  y_i(t_1) &= 0, \\
  \alpha y_i(\sigma(t_m)) + \beta y_i^\Delta(\sigma(t_m)) &= \sum_{k=2}^{m-1} y_k^\Delta(t_k), \quad 1 \leq i \leq n,
\end{align*}
\]

where \(T\) is the time scale with \(t_1, t_2, \cdots, t_{m-1}, \sigma(t_m), \sigma^2(t_m) \in T\), \(0 \leq t_1 < t_2 < \cdots < t_{m-1} < \sigma(t_m)\), \(\alpha > 0\), \(\beta > m - 2\) are real numbers and \(m \geq 3\). We assume the following conditions hold throughout the paper:

(A1) \(f_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is continuous for \(1 \leq i \leq n\),

(A2) \(p_i : [t_1, \sigma(t_m)]_T \rightarrow \mathbb{R}^+\) is continuous and \(p_i\) does not vanish identically on any closed subinterval of \([t_1, \sigma(t_m)]_T\) for \(1 \leq i \leq n\),

(A3) \(\alpha\) and \(\beta\) are positive constants such that \(\alpha > \frac{\beta}{t_2 - t_1}\) and \(\beta > m - 2\).

We define the nonnegative extended real numbers \(f_{i0}\) and \(f_{i\infty}\) by

\[
\begin{align*}
  f_{i0} &= \lim_{x \to 0^+} \frac{f_i(x)}{x} \quad \text{and} \quad f_{i\infty} = \lim_{x \to \infty} \frac{f_i(x)}{x}, \quad \text{for } 1 \leq i \leq n,
\end{align*}
\]

and assume that they will exist. When \(f_{i0} = 0\) and \(f_{i\infty} = \infty\) for \(1 \leq i \leq n\) is the called super linear case and \(f_{i0} = \infty\) and \(f_{i\infty} = 0\) for \(1 \leq i \leq n\) is called the sub linear case.

The rest of the paper is organized as follows. In Section 2, we construct the Green’s function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green’s function. In Section 3, we establish the existence of at least one positive solution of the boundary value problem (1)-(2) by using the Guo–Krasnosel’skii fixed point theorem for operators on a cone in a Banach space. Finally as an application, we demonstrate our results with an example.

2. Green’s Function and Bounds

In this section, we construct the Green’s function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green’s function.

Let \(G(t, s)\) be the Green’s function for the homogeneous boundary value problem

\[
\begin{align*}
  -y_1^\Delta(t) &= 0, \quad t \in [t_1, \sigma(t_m)]_T, \\
  y_1(t_1) &= 0, \\
  \alpha y_1(\sigma(t_m)) + \beta y_1^\Delta(\sigma(t_m)) &= \sum_{k=2}^{m-1} y_k^\Delta(t_k), \quad m \geq 3
\end{align*}
\]
Lemma 1. [22, 24] Let \( d = \alpha(\sigma(t_m) - t_1) + \beta - m + 2 \neq 0 \). Then the Green’s function \( G(t, s) \) for the homogeneous boundary value problem (3)-(4) is given by

\[
G(t, s) = \begin{cases} 
G_1(t, s), & t_1 \leq s \leq \sigma(s) \leq t_2, \\
G_2(t, s), & t_2 \leq s \leq \sigma(s) \leq t_3, \\
\vdots & \\
G_{m-2}(t, s), & t_{m-2} \leq s \leq \sigma(s) \leq t_{m-1}, \\
G_{m-1}(t, s), & t_{m-1} \leq s \leq \sigma(s) \leq \sigma(t_m),
\end{cases}
\]

where

\[
G_j(t, s) = \begin{cases} 
\frac{1}{d}[(\alpha_1(\sigma(t_m) - t) + \beta_1 - m + j + 1)(\sigma(s) - t_1) + (j - 1)(t - \sigma(s))], & \sigma(s) \leq t, \\
\frac{1}{d}(t - t_1)(\alpha_1(\sigma(t_m) - \sigma(s)) + \beta_1 - m + j + 1), & t \leq s,
\end{cases}
\]

for \( j = 1, 2, \cdots, m - 1 \).

Lemma 2. [22, 24] Assume that the condition (A3) is satisfied. Then the Green’s function \( G(t, s) \) of (3)-(4) is positive, for all \((t, s) \in (t_1, \sigma(t_m))_T \times (t_1, t_m)_T \).

Theorem 3. [22, 24] Assume that the condition (A3) is satisfied. Then the Green’s function \( G(t, s) \) in (5) satisfies the following inequality,

\[
g(t)G(\sigma(s), s) \leq G(t, s) \leq G(\sigma(s), s), \text{ for all } (t, s) \in [t_1, \sigma(t_m)]_T \times [t_1, t_m]_T,
\]

where

\[
g(t) = \min \left\{ \frac{\sigma(t_m) - t}{\sigma(t_m) - t_1}, \frac{t - t_1}{\sigma(t_m) - t_1} \right\}.
\]

Lemma 4. [22, 24] Assume that the condition (A3) is satisfied and \( s \in [t_1, t_m]_T \). Then the Green’s function \( G(t, s) \) in (5) satisfies

\[
\min_{t \in [t_{m-1}, \sigma(t_m)]_T} G(t, s) \geq kG(\sigma(s), s),
\]

where

\[
k = \frac{\beta - m + 2}{\alpha(\sigma(t_m) - t_1) + \beta - m + 2} < 1.
\]

Now, we express the solution of the boundary value problem (1)-(2) in to an equivalent integral equation, see [3]. Therefore, an \( n \)-tuple \((y_1(t), y_2(t), \cdots, y_n(t))\) is a solution of the boundary value problem (1)-(2) if and only if

\[
y_i(t) = \int_{t_1}^{\sigma(t_m)} G(t, s)p_i(s)f_i(y_{i+1}(s))\Delta s, \quad 1 \leq i \leq n, \quad t \in [t_1, \sigma(t_m)]_T,
\]

where

\[
y_{n+1}(t) = y_1(t), \quad t \in [t_1, \sigma(t_m)]_T.
\]
In particular

\[ y_1(t) = \int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1 \left( \int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots f_{n-1} \left( \int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \right) \cdots \Delta s_2 \right) \Delta s_1, \]

\[ t \in [t_1, \sigma(t_m)]_\mathbb{T}. \]

To establish the existence of positive solutions for the boundary value problem (1)-(2), we will employ the following Guo–Krasnosel’skii fixed point theorem [8, 20].

**Theorem 5.** [8, 20] Let \( X \) be a Banach Space, \( \kappa \subseteq X \) be a cone and suppose that \( \Omega_1, \Omega_2 \) are open subsets of \( X \) with \( 0 \in \Omega_1 \) and \( \overline{\Omega}_1 \subset \Omega_2 \). Suppose further that \( T : \kappa \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \kappa \) is completely continuous operator such that either

(i) \( \|Tu\| \leq \|u\|, \ u \in \kappa \cap \partial \Omega_1 \) and \( \|Tu\| \geq \|u\|, \ u \in \kappa \cap \partial \Omega_2, \) or

(ii) \( \|Tu\| \geq \|u\|, \ u \in \kappa \cap \partial \Omega_1 \) and \( \|Tu\| \leq \|u\|, \ u \in \kappa \cap \partial \Omega_2 \) holds.

Then \( T \) has a fixed point in \( \kappa \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

### 3. Positive Solutions in a Cone

In this section, we establish criteria for the existence of at least one positive solution of the boundary value problem (1)-(2).

For our construction, let \( B = \{x \mid x \in C[t_1, \sigma(t_m)]_\mathbb{T}\} \) with the norm

\[ \|x\| = \sup_{t \in [t_1, \sigma(t_m)]_\mathbb{T}} |x(t)|. \]

Then \((B, \|\cdot\|)\) is a Banach space, we refer [8, 13]. Define a cone \( P \subset B \) by

\[ P = \left\{ x \in B \mid x(t) \geq 0 \text{ on } [t_1, \sigma(t_m)]_\mathbb{T} \text{ and } \min_{t \in [t_{m-1}, \sigma(t_m)]_\mathbb{T}} x(t) \geq k\|x\| \right\}, \]

where \( k \) is given in (7).

Now, we define an integral operator \( T : P \to B \), for \( y_1 \in P \), by

\[ Ty_1(t) = \int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1 \left( \int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots f_{n-1} \left( \int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \right) \cdots \Delta s_2 \right) \Delta s_1. \]  

(8)

Notice from (A1), (A2) and Lemma 2 that, for \( y_1 \in P \), \( Ty_1(t) \geq 0 \) on \([t_1, \sigma(t_m)]_\mathbb{T}\). Also, for \( y_1 \in P \), we have from Theorem 3, that

\[ Ty_1(t) \leq \int_{t_1}^{\sigma(t_m)} G(\sigma(s_1), s_1)p_1(s_1)f_1 \left( \int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots \right) \]

\[ f_{n-1} \left( \int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \right) \cdots \Delta s_2 \right) \Delta s_1. \]
so that

\[
\| Ty_1 \| \leq \int_{t_1}^{\sigma(t_m)} G(\sigma(s_1), s_1)p_1(s_1)f_1 \left( \int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots f_{n-1} \right) \left( \int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \right) \cdots \Delta s_2 \Delta s_1. 
\]

(9)

Next, if \( y_1 \in P \), we have from Lemma 4 and (9) that

\[
\min_{t \in [t_{m-1}, \sigma(t_n)]} Ty_1(t) = \\
\min_{t \in [t_{m-1}, \sigma(t_n)]} \left\{ \int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1 \left( \int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots f_{n-1} \right) \left( \int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \right) \cdots \Delta s_2 \Delta s_1 \right\} \\
\geq k \int_{t_1}^{\sigma(t_m)} G(\sigma(s_1), s_1)p_1(s_1)f_1 \left( \int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots f_{n-1} \right) \left( \int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \right) \cdots \Delta s_2 \Delta s_1 \\
\geq k\| Ty_1 \|.
\]

Hence, \( Ty_1 \in P \) and so \( T : P \to P \). Further, the operator \( T \) is completely continuous operator by an application of the Ascoli-Arzela Theorem.

**Theorem 6.** Assume that the conditions (A1)-(A3) are satisfied. If \( f_{i0} = 0 \) and \( f_{i\infty} = \infty \), for \( 1 \leq i \leq n \) hold, then the boundary value problem (1)-(2) has at least one positive solution.

**Proof.** Let \( T \) be the cone preserving, completely continuous operator that was defined in (8). From the definitions of \( f_{i0} = 0 \), \( 1 \leq i \leq n \), there exist \( \eta_1 > 0 \) and \( H_1 > 0 \) such that, for each \( 1 \leq i \leq n \),

\[ f_i(x) \leq \eta_1 x, \quad 0 < x \leq H_1, \]

where \( \eta_1 \) satisfies

\[
\eta_1 \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s)p_i(s)\Delta s \leq 1. 
\]

(10)
Let \( y_1 \in P \) with \( \|y_1\| = H_1 \). Then from Theorem 3, for \( t_1 \leq s_{n-1} \leq \sigma(t_m) \), we have
\[
\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \\
\leq \int_{t_1}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \\
\leq \int_{t_1}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)\eta_1y_1(s_n)\Delta s_n \\
\leq \eta_1 \int_{t_1}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)\|y_1\|\Delta s_n \\
\leq \|y_1\| = H_1.
\]

It follows in a similar manner from Theorem 3, for \( t_1 \leq s_{n-2} \leq \sigma(t_m) \),
\[
\int_{t_1}^{\sigma(t_m)} G(s_{n-2}, s_{n-1})p_{n-1}(s_{n-1}) \\
f_{n-1}\left( \int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \right)\Delta s_{n-1} \\
\leq \int_{t_1}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_{n-1})p_{n-1}(s_{n-1})\eta_1H_1\Delta s_{n-1} \leq H_1.
\]
Continuing with this bootstrapping argument, we have, for \( t_1 \leq t \leq \sigma(t_m) \),
\[
\int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1\left( \int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots f_{n}(y_1(s_n))\Delta s_1 \cdots \Delta s_2 \right)\Delta s_1 \leq H_1,
\]
so that, for \( t_1 \leq t \leq \sigma(t_m) \),
\[
Ty_1(t) \leq H_1.
\]
Hence, \( \|Ty_1\| \leq H_1 = \|y_1\| \). If we set
\[
\Omega_1 = \{ x \in B \mid \|x\| < H_1 \},
\]
then
\[
\|Ty_1\| \leq \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_1. \quad (11)
\]
Further, since \( f_{i,\infty} = \infty, 1 \leq i \leq n \), there exist \( \eta_2 > 0 \) and \( \overline{\mu}_2 > 0 \) such that, for each \( 1 \leq i \leq n \),
\[
f_i(x) \geq \eta_2 x, \ x \geq \overline{\mu}_2,
\]
where \( \eta_2 \) satisfies
\[
k^2\eta_2 \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s)p_i(s)\Delta s \geq 1. \quad (12)
\]
Let
\[
H_2 = \max \left\{ 2H_1, \frac{\overline{\mu}_2}{k} \right\}.
\]
Choose \( y_1 \in P \) and \( \| y_1 \| = H_2 \). Then,

\[
\min_{t \in [t_{m-1}, \sigma(t_m)]} y_1(t) \geq k \| y_1 \| \geq H_2.
\]

From Lemma 4, for \( t_1 \leq s_{n-1} \leq \sigma(t_m) \), we have

\[
\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n
\]

\[
\geq k \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_n)p_n(s_n)\eta y_1(s)\Delta s_n
\]

\[
\geq k^2 \eta_2 \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_n)p_n(s_n)\|y_1\|\Delta s_n
\]

\[
\geq \| y_1 \| = H_2.
\]

It follows in a similar manner from Lemma 4, for \( t_1 \leq s_{n-2} \leq \sigma(t_m) \),

\[
\int_{t_1}^{\sigma(t_m)} G(s_{n-2}, s_{n-1})p_{n-1}(s_{n-1})
\]

\[
\geq k \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_{n-1})p_{n-1}(s_{n-1})\eta_2 H_2 \Delta s_{n-1}
\]

\[
\geq k^2 \eta_2 H_2 \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_{n-1})p_{n-1}(s_{n-1})\Delta s_{n-1} \geq H_2.
\]

Again, using a bootstrapping argument, we have

\[
\int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1\left(\int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2)\ldots\right)
\]

\[
f_n(y_1(s_n))\Delta s_n \cdots \Delta s_2 \Delta s_1 \geq H_2,
\]

so that

\[
T y_1(t) \geq H_2 = \| y_1 \|.
\]

Hence, \( \| Ty_1 \| \geq \| y_1 \| \). So, if we set

\[
\Omega_2 = \{ x \in B \mid \| x \| < H_2 \},
\]

then

\[
\| Ty_1 \| \geq \| y_1 \|, \text{ for } y_1 \in P \cap \partial \Omega_2.
\]

Applying Theorem 5 to (11) and (13), it follows that \( T \) has a fixed point \( y_1 \in P \cap (\Omega_2 \setminus \Omega_1) \). As such, setting \( y_{n+1} = y_1 \), we obtain a positive solution \( (y_1, y_2, \ldots, y_n) \) of (1)-(2). \( \Box \)
Theorem 7. Assume that the conditions (A1)-(A3) are satisfied. If \( f_{i0} = \infty \) and \( f_{i\infty} = 0 \), for \( 1 \leq i \leq n \) hold, then the boundary value problem (1)-(2) has at least one positive solution.

Proof. Let \( T \) be the cone preserving, completely continuous operator that was defined in (8). Since \( f_{i0} = \infty \), 1 \( \leq i \leq n \), there exist \( \eta_3 > 0 \) and \( H_3 > 0 \) such that, for each \( 1 \leq i \leq n \),

\[
f_i(x) \geq \eta_3 x, \quad 0 < x \leq H_3,
\]

where \( \eta_3 \geq \eta_2 \) and \( \eta_2 \) is given in (12).

Let \( y_1 \in P \) with \( \|y_1\| = H_3 \). Then from Lemma 4, for \( t_1 \leq s_{n-1} \leq \sigma(t_m) \), we have

\[
\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) \Delta s_n \\
\geq \int_{t_{m-1}}^{\sigma(t_m)} G(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) \Delta s_n \\
\geq k \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_n), s_n) p_n(s_n) \eta_3 y_1(s) \Delta s_n \\
\geq k^2 \eta_3 \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_n), s_n) p_n(s_n) \|y_1\| \Delta s_n \\
\geq \|y_1\| = H_3.
\]

It follows in a similar manner from Lemma 4, for \( t_1 \leq s_{n-2} \leq \sigma(t_m) \),

\[
\int_{t_1}^{\sigma(t_m)} G(s_{n-2}, s_{n-1}) p_{n-1}(s_{n-1}) \\
f_n-1 \left( \int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) \Delta s_n \right) \Delta s_{n-1} \\
\geq k \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_n), s_n) p_{n-1}(s_{n-1}) \eta_3 H_3 \Delta s_{n-1} \\
\geq k^2 \eta_3 H_3 \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_n), s_n) p_{n-1}(s_{n-1}) \Delta s_n \geq H_3.
\]

Continuing with this bootstrapping argument, it follows that

\[
\int_{t_1}^{\sigma(t_m)} G(t, s_1) p_1(s_1) f_1 \left( \int_{t_1}^{\sigma(t_m)} G(s_1, s_2) p_2(s_2) \cdots f_n(y_1(s_n)) \Delta s_n \cdots \Delta s_2 \right) \Delta s_1 \geq H_3,
\]

so that

\[
Ty_1(t) \geq H_3 = \|y_1\|.
\]

Hence, \( \|Ty_1\| \geq \|y_1\| \). So, if we set

\[
\Omega_3 = \{ x \in B \mid \|x\| < H_3 \},
\]

then

\[
\|Ty_1\| \geq \|y_1\|, \quad \text{for} \ y_1 \in P \cap \partial \Omega_3.
\]
Next, since \( f_{i} = 0, 1 \leq i \leq n \), there exist \( \eta_i > 0 \) and \( \overrightarrow{p}_i > 0 \) such that, for each \( 1 \leq i \leq n \),
\[
f_i(x) \leq \eta_i x, \quad x \geq \overrightarrow{p}_i,
\]
where \( \eta_i \leq \eta_1 \) and \( \eta_1 \) is given in (10).

For each \( 1 \leq i \leq n \), set
\[
f_i^*(x) = \sup_{0 \leq s \leq x} f_i(s).
\]
Then, it is straightforward that, for each \( 1 \leq i \leq n \), \( f_i^* \) is a nondecreasing real-valued function, \( f_i \leq f_i^* \) and
\[
\lim_{x \to \infty} \frac{f_i^*(x)}{x} = 0.
\]
It follows that there exists \( H_4 > \max\{2H_3, \overrightarrow{p}_i\} \) such that, for each \( 1 \leq i \leq n \),
\[
f_i^*(x) \leq f_i^*(H_4), \quad 0 < x \leq H_4.
\]
Choose \( y_1 \in P \) with \( ||y_1|| = H_4 \). Then, using the usual bootstrapping argument, we have
\[
Ty_1(t) = \int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)\int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots \Delta s_2 \Delta s_1
\]
\[
\leq \int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1^* \left( \int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots \Delta s_2 \right) \Delta s_1
\]
\[
\leq \int_{t_1}^{\sigma(t_m)} G(\sigma(s_1), s_1)p_1(s_1)f_1^*(H_4) \Delta s_1
\]
\[
\leq \int_{t_1}^{\sigma(t_m)} G(\sigma(s_1), s_1)p_1(s_1)\eta_1 H_4 \Delta s_1
\]
\[
\leq H_4 = ||y_1||.
\]
Hence, \( ||Ty_1|| \leq ||y_1|| \). So, if we set
\[
\Omega_4 = \{ x \in B \mid ||x|| < H_4 \},
\]
then
\[
||Ty_1|| \leq ||y_1||, \quad \text{for } y_1 \in P \cap \partial \Omega_4.
\]  \( \text{(15)} \)
Applying Theorem 5 to (14) and (15), we obtain that \( T \) has a fixed point \( y_1 \in P \cap (\overrightarrow{p}_i \setminus \Omega_4) \), which in turn with \( y_{n+1} = y_1 \), yields an \( n \)-tuple \( (y_1, y_2, \cdots, y_n) \) satisfying (1)-(2). The proof is completed.

4. Examples

Let us consider the examples to illustrate our results.

Example 1. Let \( T = \{ (\frac{1}{2})^p : p \in \mathbb{N}_0 \} \cup [1, 3] \), \( n = 2, m = 3, \alpha = 5, \beta = 2, t_1 = \frac{1}{2}, t_2 = 1 \) and \( t_3 = 2 \). Now, consider the iterative systems of nonlinear dynamic equations on time scales
\[
y_1^{\Delta}(t) + p_1(t)f_1(y_2(t)) = 0, \quad t \in \left[ \frac{1}{2}, \sigma(2) \right]_T,
\]
\[
y_2^{\Delta}(t) + p_2(t)f_2(y_1(t)) = 0, \quad t \in \left[ \frac{1}{2}, \sigma(2) \right]_T.
\]  \( \text{(16)} \)
satisfying three-point boundary conditions

\[
\begin{aligned}
&y_1(\frac{1}{2}) = 0, \quad 5y_1(\sigma(2)) + 2y_1^\Delta(\sigma(2)) = y_1^\Delta(1), \\
y_2(\frac{1}{2}) = 0, \quad 5y_2(\sigma(2)) + 2y_2^\Delta(\sigma(2)) = y_2^\Delta(1),
\end{aligned}
\]

(17)

where \(p_1(t) = p_2(t) = t\), \(f_1(y_2) = y_2^2(1 + e^{-2y_2})\) and \(f_2(y_1) = y_1^2(1 - 3e^{-y_1})\).

Then all the conditions of Theorem 6 are satisfied and hence, the boundary value problem (16)-(17) has at least one positive solution.

**Example 2.** Let \(T = \{\frac{1}{2}p : p \in \mathbb{N}_0\} \cup [1, 3]\), \(n = 2\), \(m = 3\), \(\alpha = 5\), \(\beta = 2\), \(t_1 = \frac{1}{2}\), \(t_2 = 1\) and \(t_3 = 2\). Now, consider the iterative systems of nonlinear dynamic equations on time scales

\[
\begin{aligned}
y_1^\Delta^\Delta(t) + p_1(t)f_1(y_2(t)) &= 0, \quad t \in \left[\frac{1}{2}, \sigma(2)\right], \\
y_2^\Delta^\Delta(t) + p_2(t)f_2(y_1(t)) &= 0, \quad t \in \left[\frac{1}{2}, \sigma(2)\right],
\end{aligned}
\]

(18)

satisfying three-point boundary conditions

\[
\begin{aligned}
y_1(\frac{1}{2}) = 0, \quad 5y_1(\sigma(2)) + 2y_1^\Delta(\sigma(2)) = y_1^\Delta(1), \\
y_2(\frac{1}{2}) = 0, \quad 5y_2(\sigma(2)) + 2y_2^\Delta(\sigma(2)) = y_2^\Delta(1),
\end{aligned}
\]

(19)

where \(p_1(t) = p_2(t) = \frac{1}{2}\), \(f_1(y_2) = y_2^2\) and \(f_2(y_1) = y_1^3\).

Then all the conditions of Theorem 7 are satisfied and hence, the boundary value problem (18)-(19) has at least one positive solution.

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