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ON h-TRANSFORMATION OF SOME SPECIAL FINSLER SPACE

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Abstract. The purpose of the present paper is to find the relation between the v-curvature tensor with respect to Cartan's connection of Finsler space $F^n = (M^n, L)$ and $\overline{F^n} = (M^n, \bar{L})$ where $\bar{L}(x, y)$ is obtained from $L(x, y)$ by the transformation $\bar{L}(x, y) = e^\sigma L(x, y) + b_i(x, y)y^i$ and $b_i(x, y)$ is an h- vector in (M^n, L) . we shall also study the properties of Finsler space $\bar{F^n}$ under the condition that F^n is some special Finsler space . In particular of $e^\sigma L(x, y)$ is conformal change then $(v)h$ and $(v)hv$ torsion tensors of $(M^n, \bar{L})$ have been obtained .

1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space , where M^n is an n-dimensional differentiable manifold and $L(x, y)$ is the Finsler fundamental, function. Matsumoto [1] introduced transformation of Finsler metric

$$\bar{L} = e^\sigma L + b_i(x)y^i \tag{1.1}$$

and obtained the relation between the Cartan's connection coefficients of F^n and $\bar{F^n}$ =$(M^n, \bar{L})$. It has been assumed that the function b_i in (1.1) are functions of co- ordinates x^i only. If in (1.1) $\sigma(x)$ vanishes and $L(x, y)$ is a metric function of Riemannian space then $\bar{L}(x, y)$ reduces to the Randers Space which is introduced by G. Randers [3]. If $L(x, y)$ is a metric function of Riemannian space then $\bar{L}(x, y)$ reduces to the β-conformal change. H. Izumi [2] introduced the h-vector $b_i(x, y)$ in the conformal transformation of Finsler space, which is v-covariantly constant with respect to Cartan's connection $C\Gamma$ and satisfies $LC^h_{ij}b_n$ =ρh_{ij} where C^h_{ij} is Cartan's C-tensor, h_{ij} is the angular metric tensor, ρ a function which depends only on co-ordinates and is given by, $\rho = \frac{1}{(n-1)}\ L\ C^i\ b_i$ and $C^i = C^i_{jk}\ g^{jk}$ is the torsion vector. Thus the h-vector b_i is not only a function of co-ordinates but it is a function of directional arguement satisfying $L\ \frac{\partial b_i}{\partial y^i} = \rho\ h_{ij}$. Many authors A.Taleshian et.al.[10] and S.H. Abed [11] studied the properties of such Finsler Spaces obtained by this metric. In this paper we consider the metric function given by equation $\bar{L} = e^\sigma L(x, y) + b_i(x, y)y^i$, which generalizes many Changes in Finsler geometry, called h- conformal transfofmation of Finsler metric. The section second

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of this paper gives the relation between Cartan connection $C\Gamma$ of F^n=(M^n, L) and $\bar{F^n} = (M^n, \bar{L})$. The third section is devoted to find the torsion tensors $\bar{R}_{ijk}$ of $\bar{F^n}$ and we consider the case that this space is of scalar curvature. The fourth section is devoted to find the torsion tensor $\bar{P}_{hjk}$ and to consider the case that this space becomes a Landsberg space.

For an h-vector b_i , we have the following[2].

Lemma 1.1 If b_i is an h-vector then the function ρ and $\overline{l_i}$ = b_i - $\rho e^{\sigma} l_i$ are independent of y^i.

Lemma 1.2 The magnitude b of an h-vector b_i is independent of y^i.

2. Cartan's connection of the space $\bar{F^n}$

Let b_i be a vector field in the Finsler space (M^n, L) , if b_i satisfies the conditions

$$(1) b_{i|j} = 0 \qquad (2) L C^h{}_{ij} b_h = \rho h_{ij} \tag{2.1}$$

then the vector field b_i is called an $h-$vector[2]. Here $i|j$ denote the v-covariant derivative with respect to Cartan's connection $C\Gamma$. C^h_{ij} is the Cartan's C tensor, h_{ij} is the angular metric tensor and ρ be a function given by

$$\rho = (n-1)^{-1} L C^i b_i \tag{2.2}$$

where C^i is the torsion vector $C^i{}_{jk} g^{jk}$. from (2.1) we get

$$\rho_{\dot{j}} b_i = L^{-1} \rho h_{ij} \tag{2.3}$$

Throughout the paper we shall use the notation

$L_i = \dot{\partial}_i L$, $L_{ij} = \dot{\partial}_i \dot{\partial}_j L$ ... .

The quantities and operations refereing to $\bar{F^n}$ are indicated by putting bar, thus from (1.1) we get

$$\begin{aligned}
&(a) \bar{L}_i = e^{\sigma} L_i + b_i \\
&(b) \bar{L}_{ij} = (e^{\sigma} + \rho) L_{ij} \\
&(c) \bar{L}_{ijk} = (e^{\sigma} + \rho) L_{ijk} \\
&(d) \bar{L}_{ijkh} = (e^{\sigma} + \rho) L_{ijkh}
\end{aligned} \tag{2.4}$$

and so on . If l_i, h_{ij}, g_{ij} and C_{ijk} denote the normalized element of support, the angular metric tensor, the fundamental metric tensor and Cartan's C-tensor of F^n respectively, then these quantities in $\bar{F^n}$ are obtained by (2.4) as [9]

$$\overline{l_i} = e^{\sigma} l_i + b_i \tag{2.5}$$

$$\bar{h}_{ij} = \tau(e^{\sigma} + \rho) h_{ij} \tag{2.6}$$

$$\bar{g}_{ij} = \tau(e^{\sigma} + \rho) g_{ij} + [e^{2\sigma} - \tau(e^{\sigma} + \rho)] l_i l_j + e^{\sigma} b_i l_j + e^{\sigma} l_i b_j + b_i b_j \tag{2.7}$$

$$\overline{C}_{ijk} = \tau(e^{\sigma} + \rho) C_{ijk} + (2L)^{-1} (e^{\sigma} + \rho) V_{ijk}(h_{ij} m_k) \tag{2.8}$$

where $\tau = \frac{\bar{L}}{L}$, m_i = b_i - $\beta L^{-1} l_i$ and V_{ijk} {} denotes the cyclic interchange of indices i, j, k and summation . From (2.6) and (2.8) we get the following,

Lemma 2.1 If F^n is C-reducible Finsler space then $\bar{F^n}$ is also a C-reducible Finsler space. From (2.7), the relation between contravariant components of the fundamental tensor is given by

$$\begin{aligned}
\bar{g}^{ij} = (\tau(e^{\sigma} + \rho)^{-1} g^{ij} - \tau^{-3} (e^{\sigma} + e)^{-1} (e^{2\sigma}(1 - b^2) \\
- \tau(e^{\sigma} + e)) l^i l^j - \tau^{-2} (e^{\sigma} + \rho)^{-1} (l^i b^j + l^j b^i)
\end{aligned} \tag{2.9}$$

where b is the magnitude of the vector $b^i = g^{ij}b_j$. From (2.8) and (2.9), we get

$$\bar{C}^h_{ij} = C^h_{ij} + (2\bar{L})^{-1}(h_{ij}m^h + h^h_j m_i + h^h_i m_j)$$
$$- \bar{L}^{-1}[\rho + L(2\bar{L})^{-1}(b^2 - \beta^2 L^{-2}))h_ij + L\overline{L}^{-1} m_i m_j]l^n \qquad (2.10)$$

Now we shall be concerned with Cartan's connection of F^n and $\bar{F}^n$, this connection is denoted by $C\Gamma$= (F^i_{jk}, N^i_k, C^i_{jk}). Here N^i_k=F^i_{0k} (= $Y^j F^i_{jk}$) and C^h_{ij} = $g^{hk}C_{ijk}$. Since for a Cartan's connection $L_{ij}|r = 0$, we obtain

$$\partial_k L_{ij} = L_{ijr}N^r_k + L_{rj}F^r_{ik} + L_{ir}F^r_{jk}. \qquad (2.11)$$

Differentiation of equation (2.4b) leads to

$$\partial_k \overline{L}_{ij} = (e^\sigma + \rho)\partial_k L_{ij} + \rho_k L_{ij} \qquad (2.12)$$

where we put $\rho_k = \partial_k \rho = \rho_{|k}$. If we put

$$D^i_{jk} = \overline{F}^i_{jk} - F^i_{jk} \qquad (2.13)$$

then the difference D^i_{jk} is obviously a tensor of (1.2) type. In virtue of (2.11) equation (2.12) is written in the tensorial form as,

$$(e^\sigma + \rho)(L_{ijr}D^r_{0k} + L_{rj}D^r_{ik} + L_{ir}D^r_{jk} = \rho_k L_{ij} \qquad (2.14)$$

In order to find the difference tensor D^i_{jk}, we construct supplementary equation to (2.14) from (2.4a) we obtain

$$\rho_j \bar{L}_i = e^\sigma \partial_j L_i + \partial_j b_i \qquad (2.15)$$

From $L_{i|j} = 0$ equation (2.15) is written in the form

$$\bar{L}_{ir}\bar{N}^r_j + \bar{L}_r\bar{F}^r_{ij} = (e^\sigma + \rho)L_{ir}N^r_j + (L_r + b_r)F^r_{ij} + b_{i|j}$$

By means of (2.4) and (2.13) this equation may be written in the tensorial form as,

$$(e^\sigma + \rho)L_{ir}D^r_{0j} + (l_r + b_r)D^r_{ij} = bi|j \qquad (2.16)$$

To find the difference tensor D^i_{jk} we have the following[4],

Lemma 2.2 The system of algebraic equation

$$(1)L_{ir}A^r = B_i \qquad (2)(l_r + b_r)A^r = B$$

has a unique solution A^r for given B and B_i such that $B_i l^i = 0$, The solution is given by

$$A^i = LB^i + \tau^{-1}(B - LB_\beta)l^i$$

.

where subscript β denote the contraction by b^i

Now we give the following result.

Theorem 2.1 The Cartan's connection of $\bar{F}^n$ is completely determined by equation (2.14) and (2.16) in terms of F^n. It is obvious that (2.16) is equivalent to the two equations ,

$$(e^\sigma + \rho)(L_{ir}D^r_{0j} + L_{jr}D^r_{0i}) + 2(l_r + b_r)D^r_{ij} = 2E_{ij} \qquad (2.17)$$
$$(e^\sigma + \rho)(L_{ir}D^r_{0j} - L_{jr}D^r_{0i}) = 2F_{ij} \qquad (2.18)$$

Where we put,

$$2E_{ij} = b_{i|j} + b_{j|i}, 2F_{ij} = b_{i|j} - b_{j|i} \qquad (2.19)$$

on the other hand (2.14) is equivalent to

$$2(e^\sigma + \rho)L_{jr}D^r_{ik} + (e^\sigma + \rho)(L_{ijr}D^r_{0k} + L_{jkr}D^r_{0i}$$

$$- L_{kir}D^r_{oj}) = \rho_k L_{ij} + \rho_i L_{jk} - \rho_j L_{ki} \tag{2.20}$$

contracting (2.17) with y^j, we get

$$(e^\sigma + \rho)L_{ir}D^r_{00} + 2(l_r + b_r)D^r_{0i} = 2E_{i0}. \tag{2.21}$$

Similarly from (2.18) and (2.20), we obtain

$$(e^\sigma + \rho)L_{ir}D^r_{00} = 2F_{i0} \tag{2.22}$$

$$(e^\sigma + \rho)(L_{ir}D^r_{0j} + L_{jr}D^r_{0i} + L_{ijr}D^r_{00}) = \rho_0 L_{ij} \tag{2.23}$$

contracting of (2.21) with y^i gives

$$(l_r + b_r)D^r_{00} = E_{00} \tag{2.24}$$

Now first consider (2.22) and (2.24) and apply lemma (2.1) to obtain,

$$D^i_{00} = (e^\sigma + \rho)^{-1}2LF^i_0 + \tau^{-1}(E_{00} - 2L(e^\sigma + \rho)^{-1}F_{\beta 0})l^i \tag{2.25}$$

where we put $F^i_0 = g^{ij}F_{j0}$

Secondly we add (2.18) and (2.23) to obtain

$$L_{ir}D^r_{0j} = G_{ij} \tag{2.26}$$

where we put

$$G_{ij} = (2(e^\sigma + \rho))^{-1}(2F_{ij} + \rho_0 L_{ij} - (e^\sigma + \rho)L_{ijr}D^r_{00}). \tag{2.27}$$

The equation (2.21) is written in the form

$$(l_r + b_r)D^r_{0j} = G_J \tag{2.28}$$

where we put

$$G_j = E_{j0} - 2^{-1}(e^\sigma + \rho)L_{jr}D^r_{00}. \tag{2.29}$$

Substituting from (2.25) in (2.27), we obtain

$$G_{ij} = (e^\sigma + \rho)^{-1}[F_{ij} - LL_{ijr}F^r_0 + L_{ij}((e^\sigma + \rho)E_{00} - 2LF_{\beta 0} + \overline{L}\rho_0)(2\overline{L})^{-1}] \tag{2.30}$$

By virtue of (2.22),G_j are written as

$$G_j = E_{j0} - F_{j0} \tag{2.31}$$

Thus we have obtained the system of equation's (2.26) and (2.28),and applying lemma (2.2) to these equation's we obtain

$$D^i_{0j} = LG^i_j + \tau^{-1}(G_j - LG_{\beta j})l^i \tag{2.32}$$

where we put $G^i_j = g^{ir}\ G_{rj}$

Finally from (2.20) and (2.17), we get

$$L_{ir}D^r_{jk} = H_{ijk} \qquad\qquad (l_r + b_r)D^r_{jk} = H_{jk} \tag{2.33}$$

where we put

$$H_{jk} = E_{jk} - \frac{(e^\sigma + \rho)}{2}(L_{jr}D^r_{0k} + L_{kr}D^r_{0j})$$

$$H_{ijk} = (2(e^\sigma + \rho))^{-1}(\rho_k L_{ij} + e_j L_{ik} - \rho_i L_{kj}) - \frac{1}{2}(L_{ijr}D^r_{0k} + L_{ikr}D^r_{0j} - L_{kjr}D^r_{0i})$$

Now applying lemma (2.1) to equation (2.33), we get

$$D^i_{jk} = LH^i_{jk} + \tau^{-1}(H_{jk} - LH_{\beta jk})l^i \tag{2.34}$$

where we put $H^i_{jk} = g^{hi}\ H_{hjk}$. By virtue of (2.32) H_{ijk} and H_{jk} are written in terms of known quantites,

$$H_{ijk} = \frac{1}{2}L(L_{kjr}G^r_i - L_{ijr}G^r_k - L_{ikr}G^r_j) + L_{ij}A_k + L_{ik}A_j - L_{jk}A_i \tag{2.35}$$

$$H_{jk} = E_{jk} - (e^\sigma + \rho)\frac{L}{2}(L_{jr}G^r_k + L_{kr}G^r_j) \tag{2.36}$$

where

$$A_i = (2(e^\sigma + \rho))^{-1}\rho_i + (2\tau)^{-1}(G_i - LG_{\beta i})$$

3. The h-torsion tensor $\bar{R}_{hjk}$ of $\bar{F}^n$

Let F^n be a locally Minkowski space whose fundamental function L is expressed by $L(y) = (g_{ij}y^iy^j)^{\frac{1}{2}}(y^i = dx^i)$ in terms of an adoptable co-ordinate system x^i . The connection parameter CΓ of the certain connection of F^n is given by

$$F^i_{jk} = 0, N^i_j = F^i_{0j} = 0, C^i_{jk} = g^{ir}C_{rjk} \tag{3.1}$$

Thus the h-covariant differentiation $X_{i|j}$ of a covariant vector field X_i may be written as $X_{i|j} = \partial_j X_i$. In view of (2.13), (2.32) and (3.1), the connection parameter $\bar{N}^i_j$ of $\bar{F}^n$ may be written as

$$\overline{N}^i_j = LG^i_j + \tau^{-1}(G_j - LG_{\beta j})l^i \tag{3.2}$$

The value of G_{ij} in (2.30) may be written as

$$G_{ij} = (e^\sigma + e)^{-1}\{A_{ij} + L^{-1}(F_{j0}(l_i + F_{i0}l_j) + L_j) + Gh_{ij}\} \tag{3.3}$$

where

$$G = (2L\bar{L})^{-1}((e^\sigma + \rho)E_{00} - 2LF_{\beta 0} + \bar{L}\rho_0) \tag{3.4}$$

and

$$A_{ij} = F_{ij} - 2C_{ijr}F^r_0 \tag{3.5}$$

The h-torsion tensor $\bar{R}_{hjk}$ of $(M^n, \bar{L}$ is defined

$$\bar{R}_{hjk} = V_{(j,k)}\{\bar{h}_{hi}(\partial_k\bar{N}^r_j - \bar{N}^r_k\dot{\partial}_r\bar{N}^i_j)\} \tag{3.6}$$

The symbol $V_{(j,k)}$ denotes the interchange of (j, k) and substraction. In view of (2.6), we have

$$\bar{R}_{hjk} = V_{(j,k)}\{(e^\sigma + \rho)\bar{L}L_{hi}(\partial_k\bar{N}^i_j - \bar{N}^r_j\dot{\partial}_r\bar{N}^i_j)\} \tag{3.7}$$

By virtue of (3.1) and (2.13) equation (2.26) may be written as $L_{hi}\bar{N}^i_j = G_{hi}$, by which we write $L_{hi}\partial_h\bar{N}^i_j = G_{h|j}$ and $V_{(j,k)}\{L_{hi}\bar{N}^r_k\dot{\partial}_r\bar{N}^i_j\} = V_{(j,k)}(LG^r_k\partial_rG_{hj})$ Thus (3.7) may be written as

$$\bar{R}_{hjk} = (e^\sigma + \rho)V_{(j,k)}\{\bar{L}(G_{hj|k} - LG^r_k\partial_rG_{hj})\} \tag{3.8}$$

By virtue of equation (3.3), we have

$$\begin{aligned} G_{kj|h} = {} & (e^\sigma + \rho)^{-1}[A_{hj|k} + L^{-1}(l_hF_{j0|k} + l_jF_{h0|k}) + G|kh_{hj}] \\ & - (e^\sigma + \rho)^{-2}\rho_k(A_{hj} + \overline{L}(l_hF_{j0} + l_jF_{h0}) + G_hh_j) \end{aligned} \tag{3.9}$$

$$\begin{aligned} \dot{\partial}_rG_{hj} = {} & (e^\sigma + \rho)^{-1}[-2(F_{m0}\dot{\partial}_rC^m_{hj} + C^m_{hj}F_{mr}) + \dot{\partial}_rGh_{hj} + (G \\ & - \rho_0(2L)^{-1})(2C_{hjr} - L^{-1}(l_hh_{jr} + l_jh_{hr})) + L^{-2}(h_{hr} - l_hl_r)F_{j0} \end{aligned}$$

$$+ (h_{jr} - l_j l_r)F_{h0}) + L^{-1}((l_h F_{jr} + l_j F_{hr} + 2^{-1}(\rho_j h_{hr} - \rho_h h_{jr})). \tag{3.10}$$

From equation (3.3) and (3.10), we get

$$(e^\sigma + \rho)^2 V_{(j,k)}\{G^r_k \dot{\partial}_r G_{hj}\} = V_{(j,k)}\{-[A^r_j \partial_r G + G\dot{\partial}_j G + L^{-1} l_j (F^r_0 \dot{\partial}_r G + G^2)$$
$$- L^{-2} G(F_{j0} - 2^{-1}\rho_0 l_j + 2^{-1} L\rho_j)]h_{hk} + 2A^r_j (F_{s0}\dot{\partial}_r C^s_{hk} + C^s_{hk} F_{sr} + (2L)^{-1}\rho_0 C_{hkr})$$
$$+ 2GF_{s0}(\dot{\partial}_j C^s_{hk} + 2C^s_{jr} C^r_{hk}) - L^{-2}(A_{hj} F_{k0} - F_{h0} F_{jk} - F^r_0 F_{jr} l_h l_k) - L^{-1}[A^r_j F_{hr} l_k$$
$$+ 2F^r_0((F_{s0}\dot{\partial}_r C^s_{hj} + C^s_{hj}(F_{sr})l_k + 2F^r_0 C^s_{rj} F_{sk} l_h)] - L^{-2}\rho_0 C_{hjr} F^r_o l_k + 2^{-1} L^{-2}\rho_0 (l_h A_{jk}$$
$$+ l_j A_{hk} + L^{-1} l_h l_j A_{k0}) + 2^{-1} L^{-1}(\rho_j A_{hk} - \rho_h A_{jk}) + 2^{-1} L^{-2}\rho_j (l_h A_{k0} + l_h F_{ho})\}. \tag{3.11}$$

on substituting (3.9) and (3.11) in (3.8) and we get

Theorem 3.1 The h-torsion tensor $\bar{R}_{hjk}$ of the Finsler space $\bar{F}^n$ is written in the form

$$\bar{R}_{hjk} = (e^\sigma + \rho)^{-1} V_{(j,k)}\{\bar{L}L\ G'_j\ h_{hk} + L^2 K_{hjk} + (l_h k_{jk} + l_j K_{kh}) - l_h l_j k_{0k}\} \tag{3.12}$$

where

$$G'_j = A^r_j \dot{\partial}_r G + G\dot{\partial}_j G - L^{-1}(G_{|j}(e^\sigma + rho) - (F^r_r \dot{\partial}_r G + G^2)l_j)$$
$$-L^{-2} G F_{j0} + 2^{-1} L^{-2} G(L\rho_j - \rho_0 l_j).$$

$$K_{jk} = K_{jok} - \tau(A^i_k F_{ji} - 2GC^s_{jk} F_{s0} + L^{-1}(2F_{j0}F_{k0} + \rho_0 A_{jk}$$
$$+\rho_0 C_{jkr} F^r_0 + (2L)^{-1}(\rho_k F_{j0} + \rho_j F_{k0}))$$

$$K_{hjk} = \tau[L^{-1}(e^\sigma + rho)A_{hjk} - 2A^r_j(F_{s0}\dot{\partial}_r C^s_{hk} + C^s_{hk} F_{sr}) - 2GF_{s0}(\dot{\partial}_j C^t_{hk} + 2C^s_{jr} C^r_{hk})$$
$$+L^{-2}(A_{hj} F_{k0} - F_{h0} F_{jk}) + \rho_0 L^{-1} C_{hjr} A^r_h + (2L)^{-1}(\rho A_{hk} + \rho_h A_{jk})].$$

If the Finsler space $\bar{F}^n$ is of scalar curvature $\bar{R}$ then we have the equation $\bar{R}_{i0j} = \bar{R}\ \bar{L}^2\ \bar{h}_{ij}$ [4]. If the scalar $\bar{R}$ is constant then $\bar{F}^n$ is said to be of constant curvature. From equation (3.12) the contracted h-torsion tensor $\bar{R}_{i0j}$ of $\bar{F^n}$ is given by

$$\bar{R}_{i0j} = (e^\sigma + \rho)^{-1}(\bar{L}LG'_0 h_{ij} + L^2 W_{ij} - L(l_i W_{j0} + l_j W_{i0}) + W_{00} l_i l_j) \tag{3.13}$$

where we put $W_{ij} = K_{i0j} - K_{ij0} + K_{ij}$ and W_{ij} is symmetric in the indices i and j. Equation $\bar{R}_{i0j} = \bar{R}\bar{L}^2\bar{h}_{ij}$ may be written as $\bar{R}_{i0j} = \tau(e^\sigma + \rho)\bar{R}\bar{L}^2 h_{ij}$.Thus from equation (3.13) we get the following :

Theorem 3.2 Let $\bar{F}^n$ be a Finsler space with the metric $\bar{L} = e^\sigma L + \beta$ where $L = (g_{ij}(y)y^i y^j)^{1/2}$, $\beta = b_i(x,y)y^i$ and b_i is an h vector in (M^n, L). If $\bar{F}^n$ is of scalar curvature $\bar{R}$ then the matrix $[\lambda h_{ij} - W_{ij}]$ is of rank less than three where $\lambda = \tau((e^\sigma + \rho)^2 \tau^2 \bar{R} - G'_0)$.

Now we consider the case $F_{ij} = 0$. In this case $A_{ij} = 0, K_{ijk} = 0, K_{ij} = 0$ and hence $W_{ij} = 0$ holds good. Therefore the tensor $\bar{R}_{i0j}$ of $\bar{F}^n$ is reduced to the form $\bar{R}_{i0j} = (e^\sigma + \rho)^{-1}\bar{L}LG'_0 h_{ij}$. Consequently we have the following

Theorem 3.3 Let $\bar{F}^n$ be an above Finsler space. If the condition $F_{ij} = 0$ is satisfied, then $\bar{F}^n$ is of scaler curvature $\bar{R} = ((e^\sigma + \rho)\tau)^{-2} G'_0$. Now we get the following,

Theorem 3.4 In the above theorem if the scalar $\bar{R}$ is constant, then $\bar{R} = 0$ and the space $\bar{F}^n$ is a locally Minkowskian space.

Proof. From equation (2.3) and $F_{ij} = 0$ we get

$$2\dot{\partial}_r F_{ij} = L^{-1}(\rho_j h_{ir} - \rho_i h_{jr}) = 0 \tag{3.14}$$

which after contraction with y^i gives $\rho_0 = 0$. Thus contracting equation (3.14) withg^{jr} we get $\rho_j = 0$.Therefore the scalar $\bar{R}$ is written in the form

$$\bar{R} = ((e^\sigma + \rho)\tau)^{-2}(G^2 b - L^{-1}(e^\sigma + \rho)G_{|0} \tag{3.15}$$

From equation (3.15) and $G = (e^\sigma + \rho)E_{00}(2L\bar{L})^{-4}$ it follows that the condition $\bar{R} = constant$ is written in the form

$$\begin{aligned}&[2\beta E_{00|0} - 3E_{00}^2 + 4(L^4 + 6L^2\beta^2 + \beta^4)C] \\ &\quad + 2L[E_{00|0} + 8\beta(L^2 + \beta^2)C] = 0\end{aligned} \tag{3.16}$$

from above equation we see that first bracket is a fourth degree polynomial and second bracket is third degree polynomial in y^i .Therefore we write

$$2\beta E_{00|0} - 3E_{00}^2 + 4(L^4 + 6L^2\beta^2 + \beta^4)C = 0 \tag{3.17}$$

$$E_{00|0} + 8\beta(L^2 + \beta^2)C = 0 \tag{3.18}$$

From equation (3.17) and (3.18) we get

$$3E_{00}^2 = 4C(L^2 - \beta)(L^2 + 3\beta^2) \tag{3.19}$$

If $C \neq 0$ then in view of $F_{ij} = 0$ and $b_{0|0} = 0$, the h-covariant derivative of (3.19) gives

$$3E_{00|0} = 8C\beta(L^2 - 3\beta^2) \tag{3.20}$$

Elimination of $E_{00|0}$ from (3.18) and (3.20) gives $L^2\beta C = 0$ from which we get $\beta = 0$ as $L^2C \neq 0$. Since $\dot{\partial}_j\beta = b_j$, therefore $b_i = 0$ gives $E_{ij} = 0$. Hence Equation (3.19) gives $C = 0$. This contradicts our assumption $C \neq o$. Hence the scalar $\bar{R} = C = 0$ and from equation (3.19) we get $E_{00} = 0$. Since $F_{ij} = 0$ gives $\rho_i = 0$, therefore $E_{00} = 0$ implies $F_{ij} = 0$ that is $b_{i|j} = \partial_j b_i = 0$. Thus b_i does not contain x^i. Hence $\bar{F}^n$ is Locally Minkowskian space.

4. The hv-torsion tensor $\bar{P}_{ijk}$ of $\bar{F}^n$

The hv- torsion tensor $\bar{P}_{hjk}$ of $\bar{F}^n$ is defined as

$$\bar{P}_{hjk} = \bar{C}_{hjk|0} = y^r\partial_r\bar{C}_{hjk} - \dot{\partial}_r\bar{C}_{hjk}\ \bar{N}_0^r - V_{(hjk)}\{\bar{C}_{hjr}\ \bar{F}_{ko}^r\} \tag{4.1}$$

where $V_{(ijk)}$ denotes the cyclic interchange of indices ijk and summation. In view of (2.8) and P_{hjk}=$C_{hjk|0}$ =0, we obtain

$$\begin{aligned}y^r\partial_r\bar{C}_{hjk} = \bar{C}_{hjk|0} = 2(\bar{L}G + F\beta_0)C_{hjk} + V_{(hjk)}\{(2L)^{-1}(\rho_0 m_k \\ + (e^\sigma + \rho)(b_{k|0} - L\tau^{-1}G_o l_k)h_{hj}\}\end{aligned} \tag{4.2}$$

$$\begin{aligned}\dot{\partial}_r\bar{C}_{hjk} = \tau(e^\sigma + \rho)\dot{\partial}_r C_{hjk} + L^{-1}(e^\sigma + \rho)C_{hjk}m_r + V_{(hjk)}\{(e^\sigma + \rho)L^{-1}C_{hjr}m_k \\ + (2L^2)^{-1}(e^\sigma + \rho)(n_{kr} + (\rho - \beta L^{-1})h_{kr}) + 2L^2)^{-1}(e^\sigma + \rho)h_{hr}n_{jk}\}\end{aligned} \tag{4.3}$$

where we put $n_{ij} = l_i m_j + l_j m_i$, therefore from (3.2), (3.3) and (4.3), we get

$$\dot{\partial}_r\bar{C}_{hjk}\bar{N}_0^r = 2\bar{L}\dot{\partial}_r C_{hjk}F_0^r - (2\bar{L}G - \bar{L}\rho_0 - 2F\beta_0)C_{hjk} + V_{(hjk)}\{2F_{r0}C_{hj}^r m_k$$

$$- L^{-1}F_{h0}n_{jk} - h_{hj}(L^{-1}F_{\beta 0}l_k - L^{-1})(\rho - \beta L^{-1})F_{k0} + (G - (2L)^{-1}\rho_0)m_k)\} \tag{4.4}$$

By virtue of equation (3.2), (3.3) and (2.8), we have

$$V_{(hjk)}\{\bar{C}_{hjr}\bar{F}_{k0}^r\} = 3\bar{L}GC_{hjk} + V_{(hjk)}\{\bar{L}C_{hj}^r(A_{rk} + L^{-1}F_{r0}l_k)$$

$$- 2C_{hj}^r F_{r0}m_k + L^{-1}F_{h0}n_{jk} + \frac{1}{2}h_{ij}(A_{\beta k} + L^{-1}F_{\beta 0}l_k + L^{-2}\beta F_{k0} + 3Gm_k)\} \tag{4.5}$$

from equation (4.2), (4.4) and (4.5) equation (4.1) gives the following
Theorem 4.1 The hv-torsion tensor $\bar{P}_{hjk}$ of a Finsler space $\bar{F}^n$ is written as

$$\bar{P}_{hjk} = -2\tau T_{hjkr} F_0^r + (\bar{L}G - \tau\rho_0)C_{hjk} + V_{(hjk)}\{\tau C_{hj}^r (F_{r0} l_k + L F_{kr}) + h_{hj} P_k\}$$

. where

$$2P_k = -A_{\beta k} + L^{-1}[(e^\sigma + \rho)E_{k0} + (\tau - \rho)F_{k0} - (F_{\beta 0} + 2\bar{L}G - \tau\rho_0) l_k G_{mk}$$

$$T_{hjkr} = LC_{hjk|r} + C_{hjr} l_r V_{(hjk)}\{C_{rjk} l_h\}$$

If the condition $F_{ij} = 0$ is satisfied then the hv-torsion tensor $\bar{P}_{hjk}$ of $\bar{F}^n$ is given by

$$\bar{P}_{hjk} = (\bar{L}G - \tau\rho_0)C_{hjk} + V_{(hjk)}\{h_{hj} P_k\} \tag{4.6}$$

where

$$G = (2L\bar{L})^{-1}\{(e^\sigma + \rho)E_{00} + \bar{L}\rho_0\}$$

$$2P_k = L^{-1}[(e^\sigma + \rho)F_{k0} - (2\bar{L}G - \tau\rho_0) l_k] - G m_k.$$

Now we shall treat a Landsberg space of $\bar{F}^n$. Such a space is by definition, a Finsler space with vanishing of hv-torsion tensor $\bar{P}_{hjk}$. On the other hand a Finsler space $\bar{F}^n$ with $\bar{C}_{hij|k} = 0$ is called a Berwald space.
Theorem 4.2 Let $\bar{F}^n (n \geq 3)$ be a Finsler space with the metric $\bar{L} = e^\sigma L + \beta$ where $L = (g_{ij}(y) y^i y^j)^{1/2}$, $\beta = b_i(x, y) y^i$ and b_i is an h vector in (M^n, L). In the case $F_{ij} = 0$ if $\bar{F}^n$ is a Landsberg space then $\bar{F}^n$ is a Berwald space.
Proof. The condition $(\bar{L}G - \tau\rho_0) = 0$ implies that $E_{00} = 0$ i.e. $E_{ij} = 0$ and hence $F_{ij} = 0$ it follows that b_i is independent of x^i. Thus $\bar{F}^n$ is locally Minkowskian. In the case $(\bar{L}G - \tau\rho_0) \neq 0$,from equation (4.6) it follows that $\bar{P}_{hjk}$ is equivalent to

$$C_{hjk} = (\bar{L}G - \tau\rho_0)^{-1} V_{(hjk)}\{h_{hj} P_k\}$$

Hence F^n is C-reducible, then $\bar{F}^n$ is also C-reducible. Then $\bar{F}^n$ is a Berwald space [5].

5. Some curvature Properties of Finsler space $\bar{F}^n$

The v-curvature tensor S_{ijkl} of F^n with respect to Cartan connection $C\Gamma$ is written in the form

$$S_{ijkl} = C_{ilm} C_{jk}^m - C_{ikm} C_{jl}^m \tag{5.1}$$

A Finsler space F^n $(n \geq 4)$ is called S3 -like[6] if the v-curvature tensor S_{ijkl} is of the form

$$S_{ijkl} = \frac{S}{(n-1)(n-2)}(h_{ik} h_{jl} - h_{il} h_{jk}) \tag{5.2}$$

where the scalar S is function of co-ordinates only.

A non-Riemannian Finsler space F^n $(n \geq 5)$ is called S4-like [7] if S_{ijkl} iis of the form of

$$L^2 S_{ijkl} = h_{ik} M_{jl} + h_{jl} M_{ik} - h_{il} M_{jk} - h_{jk} M_{il} \tag{5.3}$$

whereM_{ij} is symmetric and indicatory tensor.

A non-Riemannian Finsler space F^n of dimension $n \geq 3$ is called C-reducible [5] if the (h)hv-torsion tensorC_{ijk} is written in the form of

$$C_{ijk} = \frac{1}{n+1}(C_i h_{jk} + C_j h_{ki} + C_k h_{ij}) \tag{5.4}$$

where $C_i = C^j_{ij}$

A Finsler space $F^n (n \geq 2)$ with $C^2 = g^{ij} C_i C_j \neq 0$ is called $C2$-like[8], if (h)hv-torsion tensor C_{ijk} is written in the form of

$$C_{ijk} = \frac{1}{C^2} C_i C_j C_k \tag{5.5}$$

A Finsler space $F^n (n \geq 3)$ with $C^2 \neq 0$ 0 is called semi C-reducible[7], if the (h)hv-torsion tensor C_{ijk} is of the form of

$$C_{ijk} = \frac{P}{(n+1)}(h_{ij} C_k + h_{jk} C_i + h_{ki} C_j) + \frac{q}{C^2} C_i C_j C_k \tag{5.6}$$

where p and $q = (1 - p)$ does not vanish p is called the characteristic scalar of the F^n .

From equation (5.1) and (2.8) and (2.10), the v-curvature tensor of $\bar{F}^n$ is given by

$$\bar{S}_{ijkl} = \tau(e^\sigma + \rho) S_{ijkl} + h_{jk} d_{il} + h_{il} d_{jk} - h_{jl} d_{ik} - h_{ik} d_{jl} \tag{5.7}$$

where $d_{il} = (e^\sigma + \rho)\{\frac{m^2}{8LL} h_{il} + \frac{\rho}{2L^2} h_{il} + \frac{1}{4LL} m_i m_l\}$ and $m^2 = m_i m^i$, $m^i = g^{ij} m_j$

Let us suppose that F^n be an S3-like Finsler space, then from equation (5.2), (2.6) and (5.7), we have

$$\bar{S}_{ijkl} = \bar{h}_{il} P_{jk} + \bar{h}_{jk} P_{il} - \bar{h}_{jl} P_{ik} - \bar{h}_{ik} P_{jl}$$

where

$$P_{ij} = \{\tau(e^\sigma + \rho)\}^{-1} - \frac{S}{(n-1)(n-2)} h_{ij}$$

Which shows that $\bar{F^n}$ is an S4 - like Finsler space .

Let us suppose that F^n is an S4-like Finsler space then from the equation (5.3), (2.6) and (5.7), we obtain

$$\bar{S}_{ijkl} = \bar{h}_{jk} B_{il} + \bar{h}_{il} B_{jk} - \bar{h}_{jl} B_{ik} - \bar{h}_{ik} B_{jl}$$

where

$$B_{ij} = \{(\tau(e^\sigma + \rho))^{-1} d_{ij} - L^{-2} M_{ij}\}$$

Which show that $\bar{F^n}$ is an S4 - like Finsler space.

Next let us suppose that $S_{ijkl} = 0$ then from equation (2.6) and (5.7),we get

$$\bar{S}_{ijkl} = \bar{h}_{jk} E_{il} + \bar{h}_{il} E_{jk} - \bar{h}_{jl} E_{ik} - \bar{h}_{ik} E_{jl}$$

Where

$$E_{jk} = \{\tau(e^\sigma + \rho)\}^{-1} d_{jk}$$

which shows that $\bar{F^n}$ is an S4-like Finsler space Summarizing all these results we get the following.

Theorem 5.1 If F^n is any one of the following Finsler spaces

(a) S3-like Finsler space

(b) S4-like Finsler space

(c) A Finsler space with vanishing v-curvature tensor, then F^n is an S4 - like Finsler space.

The v-curvature tensor S_{ijkl} of a C-redusible Finsler space has been obtained by Matsumoto[5] is of the following form

$$S_{ijkl} = (n+1)^{-2}(h_{il}C_{jk}C_{il} - h_{ik}c_{jl} - h_{jl}C_{ik}) \tag{5.8}$$

where (a) $C_{ij} = 2^{-1}C^2h_{ij} + C_iC_j$ Since C_{ij} is a symmetric and indicatory tensor therefore (5.8) shows that F^n is an S4-like Finsler space.

Thus in view of theorem (5.1) we have the following result.

Theorem 5.2 If F^n is a C-reducible Finsler space, then $\bar{F}^n$ is an S4-like Finsler space.

From equation (5.1) and (5.5) it can be shown that the v-curvature tensor S_{ijkl} of a C2-like Finsler space vanishes. Therefore in view of theorem (5.1) we have the following.

Theorem 5.3 If F^n is a C2-like Finsler space, then $\bar{F}^n$ is an S4-like Finsler space.

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