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THE BIFURCATION ANALYSIS OF THE SCHRÖDINGER EQUATION WITH POWER LAW NONLINEARITY

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ABSTRACT. In this paper, the Schrödinger equation with power law nonlinearity will be considered. The bifurcation analysis will be applied to extract travlling wave solutions.

1. Introduction

The schrödinger equation is one of the important partial differential equations and plays a vital role in various areas of physical, biological and engineering sciences. It appears in the study of nonlinear optics, plasma physics, mathematical bioscience, quantum mechanics and several other disciplines. In recent years, some methods were introduced in order to find the explicit and approximate solution of this equation in linear or nonlinear case [1-11]. One of the considerable cases of the nonlinear schrödinger equations is power law nonlinearity which was studied by Wazwaz in [7].
In this paper, we will consider the nonlinear schrödinger equation with power law nonlinearity with following form

$$iw_t + w_{xx} + a|w|^{2n}w = 0, \tag{1}$$

where a is a real parameter and $w = w(x,t)$ is a complex-valued function of two real variables x, t.
The integrability aspects and the bifurcation analysis will be the main focus of this paper. The rest of the paper is structured as follows: In section 2 the bifurcation analysis will be carried out for this paper. In section 3 we will obtain travelling wave solutions. Finally, section 4 is devoted to our conclusions.

2. Bifurcation analysis

In this section, we investigate the bifurcation analysis of the schrödinger equation with power law nonlinearity.

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To facilitate further on our analysis, we assume that Eq. (1) has travelling wave solutions in the form

$$w(x,t) = \varphi(\xi)e^{i(\alpha x+\beta t)},\ \xi = k(x-2\alpha t) \tag{2}$$

where k, α and β are real constants.

Substituting (2) into (1), we have

$$-(\beta+\alpha^2)\phi + k^2\varphi'' + a\varphi^{2n+1} = 0. \tag{3}$$

For simplicity, we assume

$$A = \frac{(\beta+\alpha^2)}{k^2}, \tag{4}$$

$$B = \frac{a}{k^2}, \tag{5}$$

thus (3) leads to ordinary differential equation (ODE)

$$\varphi'' - A\varphi + B\varphi^{2n+1} = 0. \tag{6}$$

Let $x = \varphi(\xi), y = \varphi'(\xi)$. Then Eq. (6) reduced to the following planar dynamic system:

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = A\varphi - B\varphi^{2n+1}, \end{cases} \tag{7}$$

which admits the following Hamiltonian function:

$$H(\varphi, y) = y^2 + \frac{B}{n+1}\varphi^{2n+2} - A\varphi^2. \tag{8}$$

Now, we investigate the bifurcation phase portraits of system (7) in the parameter space (A, B). Assume

$$f(\varphi) = A\varphi - B\varphi^{2n+1}, \tag{9}$$

obviously, when $AB > 0$, $f(\varphi)$ has three zero points, φ_-, φ_0 and φ_+, which are given as follows:

$$\varphi_- = -(\frac{A}{B})^{\frac{1}{2n}},\ \varphi_0 = 0,\ \varphi_+ = (\frac{A}{B})^{\frac{1}{2n}} \tag{10}$$

when $AB \leq 0$, $f(\varphi)$ has only one zero point

$$\phi_0 = 0.$$

Suppose that $(\varphi, 0)$ is a singular point of system (7), then at the singular point $(\varphi, 0)$ the characteristic values of the linearized system of system (7) is

$$\lambda_{1,2} = \pm\sqrt{f'(\varphi)}. \tag{11}$$

According to the qualitative theory of dynamical systems, we conclude that

(1) If $f'(\varphi) < 0$, then $(\varphi, 0)$ is a center point.

(2) If $f'(\varphi) > 0$, then $(\varphi, 0)$ is a saddle point.

(3) If $f'(\varphi) = 0$, then $(\varphi, 0)$ is a degenerate saddle point.

Thus, we obtain the phase portraits of system (7) in figure 1 (when $AB > 0$) and (when $AB < 0$).

Let

$$H(\varphi, y) = h, \tag{12}$$

where h is Hamiltonian.

Next, we consider the relations between the orbits of (7) and the Hamiltonian h. Set

$$h^* = \mid H(\varphi_+, 0) \mid = \mid H(\varphi_-, 0) \mid . \tag{13}$$

According to figure 1, we get the following propositions.

Proposition 1. Suppose that $A > 0$ and $B > 0$, we have
(i) when $h > 0$, system (7) has a periodic orbit Γ_1;
(ii) when $h = 0$, system (7) has two homoclinic orbits Γ_2 and Γ_2^*;
(iii) when $-h^* < h < 0$, system (7) has two periodic orbits Γ_3 and Γ_3^*;
(iv) when $h \leq -h^*$, system (7) does not have any closed orbit.

Proposition 2. Suppose that $A < 0$ and $B < 0$, we have
(i) when $h = 0$, system (7) has two periodic orbits Γ_4 and Γ_4^*;
(ii) when $0 < h < h^*$, system (7) has three periodic orbits Γ_5, Γ_5^* and Γ_5^*;
(iii) when $h = h^*$, system (7) has two heteroclinic orbits Γ_6 and Γ_6^*;
(iv) when $h < 0$ or $h > h^*$, system (7) does not have any closed orbit.

From the qualitative theory of dynamical systems, we know that a smooth solitary wave solution of a partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. A smooth kink wave solution or an unbounded wave solution corresponds to a smooth heteroclinic orbit of a traveling wave equation. Similarly, a periodic orbit of a traveling wave equation corresponds to a periodic traveling wave solution of a partial differential system. According to above analysis, we have the following propositions [12, 13].

Proposition 3. Suppose that $A > 0$ and $B > 0$, we have
(i) when $h > 0$, (7) has two periodic wave solutions (corresponding to the periodic orbit Γ_1 in Figure 1);
(ii) when $h = 0$, (7) has two solitary wave solutions (corresponding to the homoclinic orbits Γ_2 and Γ_2^* in Figure 1);
(iii) when $-h^* < h < 0$, (7) has two periodic wave solutions (corresponding to the periodic orbits Γ_2 and Γ_2^* in Figure 1).

Proposition 4. Suppose that $A < 0$ and $B < 0$, we have
(i) when $h = 0$, (7) has four periodic singular wave solutions (corresponding to the periodic orbits Γ_4 and Γ_4^* in Figure 1);
(ii) when $0 < h < h^*$, (7) has four periodic singular wave solutions and a periodic wave solution (corresponding to the periodic orbits Γ_5, Γ_5^* and Γ_5^* in Figure 1)
(iii) when $h = h^*$, (7) has two kink profile solitary wave solutions and two singular solitary wave solutions (corresponding to the heteroclinic orbits Γ_6 and Γ_6^* in Figure 1).

3. Traveling wave solutions

First, we will obtain the explicit expressions of traveling wave solutions for the (1) when $A > 0$ and $B > 0$. From the phase portrait, we see that are two symmetric homoclinic orbits Γ_2 and Γ_2^* connected at the saddle point $(0, 0)$. In (φ, y) -plane,

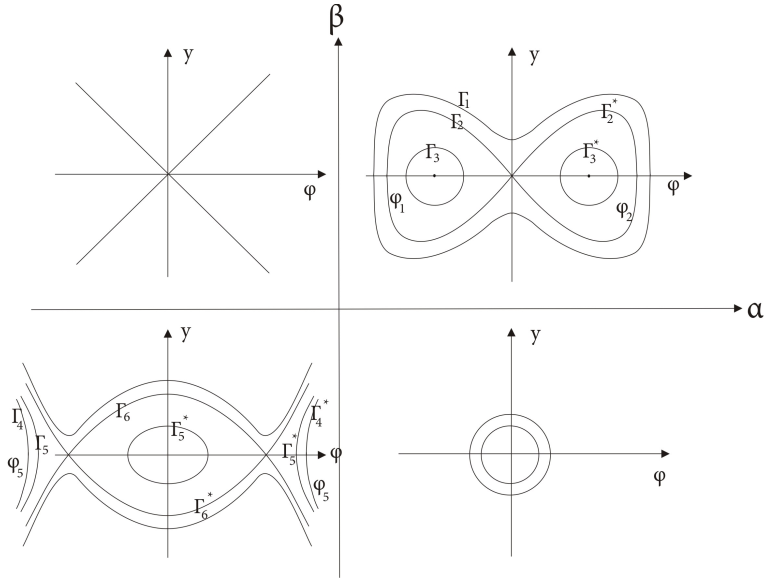

FIGURE 1. The bifurcation phase portraits of system (7)

the expressions of the homoclinic orbits are given as

$$y = \pm\sqrt{\frac{B}{n+1}}\varphi\sqrt{-\varphi^{2n} + \frac{(n+1)A}{B}}. \tag{14}$$

Substituting (14) into $\frac{d\varphi}{d\xi} = y$ and integrating them along orbits Γ_2 and Γ_2^*, we have

$$\pm\int_{\varphi_1}^{\varphi} \frac{1}{s\sqrt{-s^{2n} + (n+1)\frac{B}{A}}} ds = \sqrt{\frac{A}{n+1}} \int_0^{\xi} ds, \tag{15}$$

$$\pm\int_{\varphi_2}^{\varphi} \frac{1}{s\sqrt{-s^{2n} + (n+1)\frac{B}{A}}} ds = \sqrt{\frac{A}{n+1}} \int_0^{\xi} ds, \tag{16}$$

Completing the above integrals we obtain

$$\varphi = (\sqrt{\frac{(n+1)A}{B}} sechn\sqrt{A}\xi)^{\frac{1}{n}} \tag{17}$$

$$\varphi = -(\sqrt{\frac{(n+1)A}{B}} sechn\sqrt{A}\xi)^{\frac{1}{n}}. \tag{18}$$

Using the notations of (2), we get the following singular solitary wave solutions:

$$w_1(x,t) = (\sqrt{\frac{(n+1)A}{B}} sechn\sqrt{A}k(x - 2\alpha t))^{\frac{1}{n}} e^{i(\alpha x + \beta t)}, \tag{19}$$

$$w_2(x,t) = -(\sqrt{\frac{(n+1)A}{B}} sechn\sqrt{A}k(x - 2\alpha t))^{\frac{1}{n}} e^{i(\alpha x + \beta t)}, \tag{20}$$

where the parameters A and B are given by (4) and (5), respectively.

Second, we will obtain the explicit expressions of traveling wave solutions for (1) when $A < 0$ and $B < 0$. From the phase portrait, we note that there are two special orbits Γ_4 and Γ_4^*, which have the same Hamiltonian with that of center point $(0,0)$. In (φ, y)-plane, the expressions of the orbits are given as

$$y = \pm\sqrt{-\frac{B}{n+1}}\varphi\sqrt{\varphi^{2n} - \frac{(n+1)A}{B}}. \tag{21}$$

Substituting (21) into $\frac{d\varphi}{d\xi} = y$ and integrating them along orbits Γ_4 and Γ_4^*, it follows that

$$\pm\int_{\varphi_3}^{\varphi} \frac{1}{s\sqrt{s^{2n} - (n+1)\frac{B}{A}}} ds = \sqrt{-\frac{A}{n+1}}\int_0^{\xi} ds, \tag{22}$$

$$\pm\int_{\varphi_4}^{\varphi} \frac{1}{s\sqrt{s^{2n} - (n+1)\frac{B}{A}}} ds = \sqrt{-\frac{A}{n+1}}\int_0^{\xi} ds, \tag{23}$$

$$\pm\int_{\varphi}^{\infty} \frac{1}{s\sqrt{s^{2n} - (n+1)\frac{B}{A}}} ds = \sqrt{-\frac{A}{n+1}}\int_0^{\xi} ds, \tag{24}$$

Completing integrals (22)-(24), we obtain

$$\varphi = \pm(\sqrt{\frac{(n+1)A}{B}}\csc n\sqrt{-A}\xi)^{\frac{1}{n}} \tag{25}$$

$$\varphi = \pm(\sqrt{\frac{(n+1)A}{B}}\sec n\sqrt{-A}\xi)^{\frac{1}{n}}. \tag{26}$$

From the notations of (2), we get the following periodic singular wave solutions:

$$w_1 = \pm(\sqrt{\frac{(n+1)A}{B}}\csc n\sqrt{-A}k(x - 2\alpha t))^{\frac{1}{n}} e^{i(\alpha x + \beta t)}, \tag{27}$$

$$w_2 = \pm(\sqrt{\frac{(n+1)A}{B}}\sec n\sqrt{-A}k(x - 2\alpha t))^{\frac{1}{n}} e^{i(\alpha x + \beta t)}, \tag{28}$$

where the parameters A and B are given by (4) and (5), respectively.

4. Conclusion

In this work the bifurcation analysis was used to present an analytic study for nonlinear schrodinger equation with power law nonlinearity. Subsequently, the bifurcation analysis of the dynamical system was carried out. This bifurcation analysis additionally obtained the phase portraits of the dynamical system. Furthermore, several nonlinear wave solutions were extracted from this analysis.

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