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ON STEKLOV BOUNDARY VALUE PROBLEMS FOR $p(x)$-LAPLACIAN EQUATIONS

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Abstract. Under suitable assumptions on the potential of the nonlinearity, we study the existence of solutions for a Steklov problem involving the $p(x)-$Laplacian. Our approach is based on variational methods.

1. Introduction

In recent years there has been an increasing interest in the study of various mathematical problems with variable exponent (see for example [2, 6, 7, 17]). The nonlinear problems involving the $p(x)$-Laplace operator are extremely attractive because they can be used to model dynamical phenomena which arise from the study of electrorheological fluids or elastic mechanics [20]. Problems with variable exponent growth conditions also appear in the modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids. The detailed application background of the $p(x)$-Laplacian can be found in [1, 3, 7, 11, 12, 13, 16].

In this study, we provide existence results for the following class of Steklov boundary value problems for some $p(x)$-Laplacian

$$\begin{cases} -\Delta_{p(x)}u + a(x)\,|u|^{p(x)-2}\,u = f(x,u), & \text{in } \Omega; \\ |\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial \eta} = g(x,u), & \text{on } \partial\Omega. \end{cases} \tag{P}$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$ -Laplacian operator. Denote $C_+\left(\overline{\Omega}\right) := \left\{ p \in C\left(\overline{\Omega}\right) : 1 < \min_{x\in\overline{\Omega}} p\left(x\right) := p^- \leq \max_{x\in\overline{\Omega}} p\left(x\right) := p^+ < \infty \right\}$ and let $p(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \geq N$ for $x \in \overline{\Omega}$, $\Omega \subset \mathbb{R}^N$, for $N \geq 2$, is a bounded domain with $\partial\Omega \in C^{0,1}$, and $\frac{\partial}{\partial\eta} = \eta \cdot \nabla$ is a normal derivative on $\partial\Omega$ and $a,\ f, g$ satisfy the following conditions:

(f_1) $a \in L^\infty(\Omega)$ with $a^- = ess\inf_{x\in\Omega} a(x) > 0$.

(f_2) $f \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ and there exist constant $b_0 > 0$ such that:

$$|f(x,s)| \leq a_0(x) + b_0|s|^{\alpha(x)} \text{ for all } (x,s) \in \overline{\Omega}\times\mathbb{R},$$

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where $a_0 \in L^{\frac{\alpha(x)}{\alpha(x)-1}}$ and $\alpha \in C\left(\overline{\Omega}\right)$, $1 < \alpha(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \geq N$.

(f_3) $g \in C(\partial\Omega \times \mathbb{R}, \mathbb{R})$ and there exist constant $b_1 > 0$ such that

$$|g(x,s)| \leq a_1(x) + b_1|s|^{\beta(x)} \text{ for all } (x,s) \in \partial\Omega \times \mathbb{R},$$

where $a_1 \in L^{\frac{\beta(x)}{\beta(x)-1}}$ and $\beta \in C(\partial\Omega)$,$1 < \beta(x) < p^{\partial}(x) = \frac{(N-1)p(x)}{N-p(x)}$ if $p(x) < N$ and $p^{\partial}(x) = \infty$ if $p(x) \geq N$.

(f_4) There exist constants λ, $\mu \in \mathbb{R}$ such that

$$\limsup_{|u|\to+\infty} \frac{p^+ G(x,u)}{|u|^{p^-}} \leq \lambda$$

uniformly for $x \in \partial\Omega$ and

$$\limsup_{|u|\to+\infty} \frac{p^+ F(x,u)}{|u|^{p^-}} \leq \mu$$

uniformly for $x \in \overline{\Omega}$, with

$$\lambda_1\left(p^- + \mu_1\right)\lambda + \lambda_1\mu < 1 \text{ when } \lambda_1\left(p^- + \mu_1\right)\lambda < 1 \text{ and } \lambda_1\mu < 1, \tag{1.1}$$

where $G(x,u) = \int\limits_0^u g(x,s)ds,\ F(x,u) = \int\limits_0^u f(x,s)ds$ and $\mu_1 > 0,\ \lambda_1 > 0$ constants .

(f_5) There exist $R_1 > 0,\ \theta > p^+$ such that for all $|s| \geq R_1$ and $x \in \partial\Omega,\ 0 < \theta F(x,s) \leq f(x,s)s$.

(f_6) There exist $R_2 > 0, \theta > p^+$ such that for all $|s| \geq R_2$ and $x \in \Omega$, $0 < \theta G(x,s) \leq g(x,s)s$.

In this section we recall some results on variable exponent Lebesgue-Sobolev spaces. The reader is refereed to [8, 10] and references therein for more details.

For any $p \in C_+\left(\overline{\Omega}\right)$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable: } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

The modular of $L^{p(x)}(\Omega)$ which is the mapping $\sigma_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ is defined by

$$\sigma_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx$$

for all $u \in L^{p(x)}(\Omega)$ with the norm

$$|u|_{L^{p(x)}(\Omega)} := |u|_{p(x)} = \inf\left\{ \rho > 0 : \sigma_{p(x)}\left(\frac{u}{\rho}\right) \leq 1 \right\},$$

and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} := \|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Since a verifies (f_1), the following norms given by

$$\|u\|_a = \inf\left\{ \rho > 0 : \int_\Omega \left(\left|\frac{\nabla u}{\rho}\right|^{p(x)} + a(x)\left|\frac{u}{\rho}\right|^{p(x)} \right) dx \leq 1 \right\}.$$

Then, it is easy to see that $\|u\|_a$ is a norm on $W^{1,p(x)}(\Omega)$ equivalent to $\|u\|_{1,p(x)}$. In particular (see [7])

$$\frac{[a^-]_{\frac{1}{p}}}{1+[a^-]_{\frac{1}{p}}}\|u\|_{1,p(x)} \leq \|u\|_a \leq (1+|a|_\infty)^{\frac{1}{p^-}}\|u\|_{1,p(x)}$$

for each $u \in W^{1,p(x)}(\Omega)$, where, for $h > 0$ and $t \in C(\overline{\Omega})$ with $t^- > 1$, we put

$$[h]_t := \min\left\{h^{t^-}, h^{t^+}\right\}.$$

Proposition 1 ([9,10,11]). *Let $p : \mathbb{R} \to \mathbb{R}^+$ be Lipschitz continuous and satisfy $1 < p^- \leq p^+ < N$ and $q : \mathbb{R} \to \mathbb{R}^+$ be a measurable function. If $p(x) \leq q(x) \leq p^*(x)$, $x \in \Omega$ then there is a continuous embedding $W^{1,q(x)}(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$.*

Proposition 2 ([8,9,10]). *Let $\mu(u) = \int_\Omega \left(|\nabla u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right)dx$. For $u \in W^{1,p(x)}(\Omega)$ we have*

1. $\|u\|_a < 1 (= 1; > 1) \Leftrightarrow \mu(u) < 1 (= 1; > 1)$;
2. *If* $\|u\|_a < 1$ *then* $\|u\|_a^{p^+} \leq \mu(u) \leq \|u\|_a^{p^-}$;
3. *If* $\|u\|_a > 1$ *then* $\|u\|_a^{p^-} \leq \mu(u) \leq \|u\|_a^{p^+}$.

Proposition 3 ([8,9,10]). *If $p^- > 1$ and $p^+ < \infty$ then, the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

For $p(x) \equiv p = 2$, Auchmuty [5] proved that the Steklov eigenfunctions form a complete orthonormal system for the space $[H_0^1(\Omega)]^\perp$ in $H^1(\Omega)$ with respect to the specific inner products. Some previous studies have treated the nonlinear Steklov problem, but only [4] considered $p = 2$ and [21] dealt with $p > 1$. The inhomogeneous Steklov problems involving the p-Laplacian has been the object of study in, for example, [19] , in which the authors have studied this class of inhomogeneous Steklov problems in the cases of $p(x) \equiv p = 2$ and of $p(x) \equiv p > 1$, respectively.

Existence and multiplicity of solutions for a Steklov problem involving the $p(x)$-Laplacian are provided in Afrouzi, Hadjian, Heidarkhani [1] and Allaoui, El Amrouss, Ourraoui [3]. Their approach is based on variational methods.

In 2015 Godoi, Miyagaki, Rodrigues [14] provided existence results for the following class of Steklov-Neumann boundary value problems for some quasilinear elliptic equations

$$\begin{cases} -\operatorname{div}|\nabla u|^{p-2}\nabla u + c(x)|u|^{p-2}u = f(x,u), & \text{in } \Omega; \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \eta} = g(x,u), & \text{on } \partial\Omega. \end{cases} \qquad (P^*)$$

Here the functions $c : \Omega \to \mathbb{R}$ and $f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:
$(P1)$ $c \in L^\infty(\Omega), c(x) \geq 0$, for almost everywhere $x \in \Omega$ and $\int_\Omega c(x)dx > 0$.
$(P2)$ $f, g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.
$(P3)$ The constants $a_1, a_2 > 0$ exist such that

$$|g(x,u)| \leq a_1 + a_2|u|^s, \forall (x,u) \in \partial\Omega \times \mathbb{R},$$

with $0 < s < p_*^1(N) - 1$, where $p_*^1(N) = \frac{(N-1)p}{N-p}$ if $p < N$ and $p_*^1(N) = \infty$ if $p \geq N$.
$(P4)$ The constants $b_1, b_2 > 0$ exist such that

$$|f(x,u)| \leq b_1 + b_2 |u|^t, \forall (x,u) \in \overline{\Omega} \times \mathbb{R},$$

with $0 < t < p_*(N) - 1$, where $p_*(N) = \frac{Np}{N-p}$ if $p < N$ and $p_*(N) = \infty$ if $p \geq N$.
$(P5)$ The constant $\pi \in \mathbb{R}$ exist such that

$$\limsup_{|u| \to +\infty} \frac{pG(x,u)}{|u|^p} \leq \pi < \pi_1$$

uniformly for $x \in \partial\Omega$ and the constant $\eta \in \mathbb{R}$ exist such that

$$\limsup_{|u| \to +\infty} \frac{pF(x,u)}{|u|^p} \leq \eta < \eta_1$$

uniformly for $x \in \Omega$, with

$$\pi_1 \eta + \eta_1 \pi < \eta_1 \pi_1.$$

If conditions $(P1) - (P5)$ are satisfied, problem (P^*) has at least one weak solution $u \in W^{1,p}(\Omega)$.

Be noted, firstly in article [18], problem (P^*) was addressed in condition $p = 2$. After, authors in article [14] generalized problem (P^*) to p-Laplacian.

In [14] and [18], authors used inequalities $\|u\|_c^p \geq \eta_1 \|u\|_{p,\partial}^p$ and $\|u\|_c^p \geq \pi_1 \|u\|_p^p$ to prove the coercivity of functional, where η_1 the first Steklov eigenvalue and π_1 the first Neumann eigenvalue, where

$$\|u\|_c = \left(\int_\Omega \left(|\nabla u|^p + c(x) |u|^p \right) dx \right)^{\frac{1}{p}}$$

and

$$\|u\|_p = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}},$$

$$\|u\|_{p,\partial} = \left(\int_{\partial\Omega} |u|^p \, dx \right)^{\frac{1}{p}}$$

are norms in $L^p(\Omega)$ and $L^p(\partial\Omega)$, respectively.

We note that we deal with the problem (P) consist of $p(x)$-Laplacian, naturally, the solution of the problem have been made in the variable exponent Lebesgue-Sobolev spaces. Therefore, there exist constants η_1 and π_1 (see [14]). Thus, in this paper, we will discuss the inequalities

$$\|u\|_{1,p^-}^{p^-} \leq \lambda_1 \|u\|_a^{p^-}, u \in W^{1,p(x)}(\Omega), \tag{1.2}$$

where $\lambda_1 > 0$ and

$$\int_{\partial\Omega} |f| \, d\sigma \leq \int_\Omega |\nabla f| \, dx + \mu_1 \int_\Omega |f| \, dx, \ f \in W^{1,1}(\Omega), \tag{1.3}$$

where $\mu_1 > 0$ (see detail [15]).

Since our approach is variational, we define the Euler–Lagrange functional associated with the problem (P), $I : (W^{1,p(x)}(\Omega), \|u\|_{1,p(x)}) \to \mathbb{R}$ by

$$I(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + a(x) |u|^{p(x)}}{p(x)} dx - \int_\Omega F(x,u) dx - \int_{\partial\Omega} G(x,u) d\sigma. \tag{1.4}$$

We find that I belongs to $C^1(W^{1,p(x)}(\Omega),\mathbb{R})$ and its Gateaux derivative is given by

$$\langle I'(u),\upsilon\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u\nabla\upsilon + a(x)|u|^{p(x)-2}u\upsilon\right)dx$$
$$-\int_\Omega f(x,u)\upsilon dx - \int_{\partial\Omega} g(x,u)\upsilon d\sigma$$

for all $u,\upsilon \in W^{1,p(x)}(\Omega)$.Therefore, the critical points of I are the exact weak solutions to problem (P).

Definition 4. Let $(E,\|u\|_E)$ be a Banach space, and $I: E\to\mathbb{R}$ a C^1- functional. We say that I satisfies Palais-Smale condition, denoted (PS), if any sequence (u_n) in E, such that $I(u_n)$ is bounded in E and $I'(u_n)\to 0$, admits a convergent subsequence in E.

The following classic abstract result can be found in [22].

Proposition 5. *Let E be a Banach space. If $J\in C^1(E,\mathbb{R})$ is bounded from below and it satisfies the (PS) condition, then $c=\inf_E J$ is a critical value of J.*

2. Main results

Thus, we establish our main result.

Theorem 6. *If (f_1)-(f_6) hold. Then, problem (P) has at least a nontrivial weak solution $u\in W^{1,p(x)}(\Omega)$.*

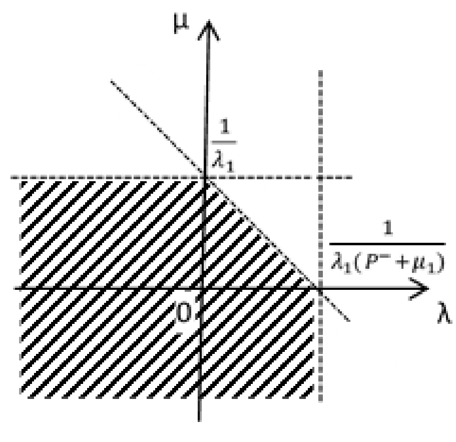

Figure 1

Figure 1 shows the cartesian plane $\lambda o\mu$ of the region described by $\lambda_1(p^-+\mu_1)\lambda + \lambda_1\mu < 1$ with $\lambda_1(p^-+\mu_1)\lambda<1$ and $\lambda_1\mu<1$.
Proof. Using this fact, we prove the following claim.

Claim 1. *The functional I is coercive on $(W^{1,p(x)}(\Omega),\|u\|_a)$, i.e.,*

$$I(u)\to+\infty \text{ as } \|u\|_a\to+\infty.$$

First in inequality (1.3)

$$f = |u|^{p^-},$$

we take, then

$$\nabla f = \nabla\left(|u|^{p^-}\right) = p^- |u|^{p^- -1} \nabla u \; sign\,(u)$$

and we apply Young's

$$\nabla\left(|u|^{p^-}\right) \le \left(p^- - 1\right)|u|^{p^-} + |\nabla u|^{p^-}$$

we get these inequations.

By the continuity of F, G and (f_4), we have

$$G(x,u) \le \frac{\lambda}{p^+}|u|^{p^-} + C, \; C > 0 \tag{2.1}$$

and

$$F(x,u) \le \frac{\mu}{p^+}|u|^{p^-} + C, \; C > 0 \tag{2.2}$$

for all $x \in \overline{\Omega}$ and $u \in \mathbb{R}$. From (1.3) and (2.1) we get

$$\begin{aligned} &\frac{\lambda}{p^+}\int_{\partial\Omega} G(x,u)d\sigma \\ &\le \frac{\lambda}{p^+}\left(\int_\Omega |\nabla u|^{p^-}dx + \left(p^- + \mu_1 - 1\right)\int_\Omega |u|^{p^-}dx\right) + C\,|\partial\Omega|\,. \end{aligned} \tag{2.3}$$

For $\|u\|_a > 1$ and (1.4) apply the inequalities (2.1)-(2.3) and (1.2) then we get

$$\begin{aligned} I(u) &\ge \frac{1}{p^+}\int_\Omega \left(|\nabla u|^{p(x)} + a(x)\,|u|^{p(x)}\right)dx \\ &- \frac{\lambda}{p^+}\left(\int_\Omega |\nabla u|^{p^-}dx + \left(p^- + \mu_1 - 1\right)\int_\Omega |u|^{p^-}dx\right) \\ &- \frac{\mu}{p^+}\int_\Omega |u|^{p^-}dx - C\left(|\partial\Omega| + |\Omega|\right) \\ &= \frac{1}{p^+}\|u\|_a^{p^-} - \frac{\lambda\left(p^- + \mu_1 - 1\right) + \mu}{p^+}\int_\Omega |u|^{p^-}dx - \frac{\lambda}{p^+}\int_\Omega |\nabla u|^{p^-}dx. \\ &\ge \frac{1}{p^+}\|u\|_a^{p^-} - \max\left\{\frac{\lambda\left(p^- + \mu_1 - 1\right) + \mu}{p^+}, \frac{\lambda}{p^+}\right\} \times \\ &\times \left(\int_\Omega |u|^{p^-}dx + \int_\Omega |\nabla u|^{p^-}dx\right) - C\left(|\partial\Omega| + |\Omega|\right) \\ &\ge \frac{1}{p^+}\|u\|_a^{p^-} - \frac{\left(p^- + \mu_1\right)\lambda + \mu}{p^+}\left(\int_\Omega |\nabla u|^{p^-}dx + \int_\Omega |u|^{p^-}dx\right) - C\left(|\partial\Omega| + |\Omega|\right) \\ &= \frac{1}{p^+}\|u\|_a^{p^-} - \frac{\left(p^- + \mu_1\right)\lambda + \mu}{p^+}\|u\|_{1,p^-}^{p^-} - C\left(|\partial\Omega| + |\Omega|\right) \\ &\ge \frac{1}{p^+}\left\{1 - \lambda_1\left[\left(p^- + \mu_1\right)\lambda - \mu\right]\right\}\|u\|_a^{p^-} - C\left(|\partial\Omega| + |\Omega|\right). \end{aligned}$$

By condition (1.1) we have

$$\lambda_1\left(p^- + \mu_1\right)\lambda + \lambda_1\mu < 1.$$

Hence, the functional I is coercive.

Claim 2. *The functional I is bounded from below.*

This is an immediate consequence of Claim 1.

Claim 3. *I verifies (PS), the Palais -Smale condition.*

Proof. Now, to verify the (PS)-condition it is sufficient to prove that any (PS)-sequence is bounded. To this end, suppose that $\{u_n\} \subset W^{1,p(x)}(\Omega)$ is a (PS)-sequence; i.e., there is $M > 0$ such that

$$\sup |I(u_n)| \leq M, \ I'(u_n) \to 0 \text{ as } n \to \infty.$$

Let us show that $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Using hypothesis (f_5),(f_6), since $I(u_n)$ is bounded, we have for n large enough:

$$\begin{aligned}
M+1 &\geq \langle I(u_n), u_n\rangle - \frac{1}{\theta}\langle I'(u_n), u_n\rangle + \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + a(x)\,|u_n|^{p(x)}\right)dx - \int_\Omega F(x,u_n)dx \\
&- \int_{\partial\Omega} G(x,u_n)d\sigma - \frac{1}{\theta}\int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)\,|u_n|^{p(x)}\right)dx \\
&+ \frac{1}{\theta}\int_\Omega f(x,u_n)u_n dx + \frac{1}{\theta}\int_{\partial\Omega} g(x,u_n)u_n d\sigma + \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\int_\Omega \left(|\nabla u_n|^{p(x)} + a(x)\,|u_n|^{p(x)}\right)dx \\
&- \int_\Omega \left(F(x,u_n) - \frac{1}{\theta}f(x,u_n)u_n\right)dx \\
&- \int_{\partial\Omega} \left(G(x,u_n) - \frac{1}{\theta}g(x,u_n)u_n\right)dx + \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\|u_n\|_a^{p^-} - C_1 - C_2 - \frac{1}{\theta}\|I'(u_n)\|_{\left(W^{1,p(x)}(\Omega)\right)^*}\|u_n\|_a \\
&\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\|u_n\|_a^{p^-} - \frac{C_3}{\theta}\|u_n\|_a - C_1 - C_2,
\end{aligned}$$

where C_1, C_2 and C_3 are positive constants. Since $\theta > p^+$, from the above inequality we know that $\{u_n\}$ is bounded in $(W^{1,p(x)}(\Omega), \|u\|_a)$.

Next, we show the strong convergence of (u_n) in $W^{1,p(x)}(\Omega)$. Let $(u_n) \subset W^{1,p(x)}(\Omega)$ be (PS) sequence of I in $W^{1,p(x)}(\Omega)$, that is $I(u_n)$ is bounded and $I'(u_n) \to 0$. By the coercivity of I, the sequence (u_n) is bounded in $W^{1,p(x)}(\Omega)$. As $W^{1,p(x)}(\Omega)$ is reflexive, for a subsequence still denoted (u_n), we have

$$u_n \rightharpoonup u \text{ weakly in } W^{1,p(x)}(\Omega) \text{ as } n \to \infty.$$

Since $\alpha(x) < p^*(x)$ and $\beta(x) < p^\partial(x)$ (see $(f_3),(f_4)$), then $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{\alpha(x)}(\Omega)$ (compact) and $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{\beta(x)}(\partial\Omega)$ (compact) (see [9, 10, 11, 12]). Furthermore, we have

$$u_n \to u \text{ strongly in } L^{\alpha(x)}(\Omega) \text{ as } n \to \infty,$$

and

$$u_n \to u \text{ strongly in } L^{\beta(x)}(\partial\Omega) \text{ as } n \to \infty.$$

Therefore

$$\langle I(u_n), u_n - u\rangle \to 0,$$

$$\int_\Omega f(x, u_n)u_n (u_n - u)\, dx \to 0$$

and

$$\int_{\partial\Omega} g(x, u_n)u_n (u_n - u)\, d\sigma \to 0.$$

Thus,

$$\begin{aligned} &\langle A(u_n), u_n - u\rangle \\ : \quad &= \int_\Omega \Big(|\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) + a(x) |u_n|^{p(x)-2} u_n (u_n - u)\Big)\, dx \to 0. \end{aligned}$$

According to the fact that the operator A satisfies condition (S^+)(see [12]), we deduce that $u_n \to u$ strongly in $W^{1,p(x)}(\Omega)$, this completes the proof.

Let (u_n) be a sequence in $(W^{1,p(x)}(\Omega), \|u\|_a)$, where $(I(u_n))$ is bounded in $\mathbb{R}$ and $I'(u_n) \to 0$ in $(\left(W^{1,p(x)}(\Omega)\right)^*, \|u\|_a^*)$ as $m \to \infty$. Since the operators $L_1, L_2 :$ $(W^{1,p(x)}(\Omega)), \|u\|_a) \to \mathbb{R}$, given by

$$L_1 = \int_\Omega F(x, u)dx \text{ and } L_2 = \int_{\partial\Omega} G(x, u)d\sigma$$

are weakly continuous and their derivatives L_1' and L_2' are compacts (see [3]), it is sufficient to show that (u_n) is bounded in $(W^{1,p(x)}(\Omega), \|u\|_a)$. If this is not the case, then a subsequence (u_{n_k}) of (u_n) exists such that $\|u_{n_k}\|_{1,p(x)} \to +\infty$, as $k \to +\infty$. Therefore, by the coercivity, $I, I(u_{n_k}) \to +\infty$, as $k \to +\infty$, which is a contradiction because $(I(u_n))$ is bounded in $\mathbb{R}$.

Now, we can conclude the proof of Theorem 2.1 by applying Proposition 1.4. Hence, I has at least one critical point $u \in W^{1,p(x)}(\Omega)$, i.e., $I'(u) = 0$. Then, u is weak solution of problem (P). Thus, the proof of Theorem 2.1 is complete.

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