RIGHT AND LEFT DISLOCATED $b$-METRIC SPACES AND
FIXED POINT THEOREMS

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Abstract. Using the concept of generalized contraction, some fixed point theorems are investigated in the context of right and left dislocated $b$-metric spaces. We have proved $\varphi$-contraction and Reich type contraction in right and left dislocated $b$-metric spaces.

1. Introduction

One branch of generalizations of celebrated Banach contraction principle is based on the replacement of contraction condition imposed on $T : X \to X$, where $(X, d)$ is a complete metric space. The weaker condition described by Browder [1] as, $d(T x, T y) \leq \varphi d(x, y)$ for all $x, y \in X$, where $\varphi$ is a comparison function introduced by Berinde [2]. Reich [3] generalized the Banach contraction principle by introducing a new type of contraction condition which were given the name of Reich type contraction. In similar direction Istratescu [4] introduced the convex type contraction and generalized Banach contraction principle for such a type of contraction condition.

The notion of $b$-metric space was introduced by Czerwik [5] in connection with some problems concerning with the convergence of non-measurable functions with respect to measure. Fixed point theorems regarding $b$-metric spaces was obtained in [6] and [7]. In 2013, Shukla [8] generalized the notion of $b$-metric spaces and introduced the concept of partial $b$-metric spaces. Rahman and Sarwar [9] further generalized the concept of $b$-metric space and initiated the notion of dislocated quasi-$b$-metric space. Fixed point theorems in dislocated quasi-$b$-metric spaces are established by the researchers in [10] and [11].

Recently in 2017, Mujeeb and Sarwar [12] investigated right and left dislocated $b$-metric spaces and proved some fixed point results in such type of spaces.

In this work, we have proved $\varphi$-contraction and Reich type of contraction in the setting of right and left dislocated $b$-metric space which generalize and extend some existing fixed point results of the literature in these newly discovered spaces.
2. Preliminaries

**Definition 2.1.**[9]. Let $X$ be a non-empty set and $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi-$b$-metric if $\forall \ x, y, z \in X$

\[ (d_1) \ d(x, y) = d(y, x) = 0 \text{ implies that } x = y; \]
\[ (d_2) \ d(x, y) \leq k[d(x, z) + d(y, z)]. \]

The pair $(X, d)$ is called dislocated quasi-$b$-metric space or shortly $(dq \ b$-metric) space.

**Definition 2.2.**[12]. Let $X$ be a non empty set. Let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called right dislocated $b$-metric if $\forall \ x, y, z \in X$ satisfying

\[ rd_1 \ d(x, y) = d(y, x) = 0 \text{ implies that } x = y; \]
\[ rd_2 \ d(x, y) \leq k[d(x, z) + d(y, z)]. \]

And the pair $(X, d)$ is called right dislocated $b$-metric (rd $b$-metric) space.

**Definition 2.3.**[12]. Let $X$ be a non empty set. Let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called left dislocated $b$-metric if $\forall \ x, y, z \in X$ satisfying

\[ ld_1 \ d(x, y) = d(y, x) = 0 \text{ implies that } x = y; \]
\[ ld_2 \ d(x, y) \leq k[d(z, x) + d(z, y)]. \]

And the pair $(X, d)$ is called left dislocated $b$-metric (ld $b$-metric) space.

**Remarks.** For some interesting properties and examples of right and left dislocated $b$-metric space see [12].

**Definition 2.4.**[12]. A sequence $\{x_n\}$ in $X$ is called rd $b$-convergent in $X$ if there exists $x \in X$ such that $\lim_{n \to \infty} d(x, x_n) = 0$. In this case $x$ is called the rd $b$-limit of the sequence $\{x_n\}$.

Unlike $b$-metric space rd $b$-metric space need not be left and right convergent. But in case of rd $b$-metric space it is rd $b$-convergent only.

**Definition 2.5.**[12]. A sequence $\{x_n\}$ in $X$ is called ld $b$-convergent in $X$ if there exists $x \in X$ such that $\lim_{n \to \infty} d(x_n, x) = 0$. In this case $x$ is called the ld $b$-limit of the sequence $\{x_n\}$.

In case of ld $b$-metric space a convergent sequence need only to be ld $b$-convergent.

**Remarks.** Since the notion of ld $b$-metric space is look like a dual notion of rd $b$-metric space. Therefore, we state the following definitions and some basic properties for right dislocated $b$-metric spaces only which may be easily carried out for left dislocated $b$-metric spaces.

The following definitions can be found in [12].

**Definition 2.6.** A sequence $\{x_n\}$ in rd or ld $b$-metric space is called Cauchy sequence if for $\epsilon > 0$ there exist $n_0 \in N$, such that for $m > n \geq n_0$, we have $d(x_n, x_m) < \epsilon$.

**Definition 2.7.** A rd or ld $b$-metric space $(X, d)$ is said to be complete if every Cauchy sequence in $X$ converges to a point in $X$.

**Definition 2.8.** Let $(X, d)$ be a rd or ld $b$-metric space. A mapping $T : X \to X$ is called contraction if $k \geq 1$ there exists a constant $\alpha \in [0, 1)$ with $k\alpha < 1$ and for all $x, y \in X$ satisfying

\[ d(Tx, Ty) \leq \alpha d(x, y). \]

The following result may be seen in [12].

**Lemma 1.** Every subsequence of rd or ld $b$-convergent sequence to $x_0$ is rd $b$-convergent to $x_0$. 
Lemma 2. Limit of convergent sequence in \(rd\) or \(ld\) \(b\)-metric space is unique.

Lemma 3. Let \((X,d)\) be a \(rd\) or \(ld\) \(b\)-metric space and \(\{x_n\}\) be a sequence in \(rd\) \(b\)-metric space such that
\[
d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n)
\]
for \(n = 1, 2, 3, \ldots\) and \(0 \leq \alpha k < 1\) where \(\alpha \in [0,1)\) and \(k\) is defined in \(rd\) \(b\)-metric space. Then \(\{x_n\}\) is a Cauchy sequence in \(X\).

Lemma 4. Let \((X,d)\) be a \(rd\) or \(ld\) \(b\)-metric space. If \(T : X \to X\) is a contraction. Then \(T\) is \(rd\) \(b\)-continuous.

Theorem 1. Let \((X,d)\) be a complete \(rd\) or \(ld\) \(b\)-metric space. If \(T : X \to X\) is a contraction. Then \(T\) has a unique fixed point.

Theorem 2. Every \(\varphi\)-contraction \(T : X \to X\) where \((X,d)\) is a complete metric space, is a Picard’s operator.

Definition 2.9. A map \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) is called comparison function if it satisfies:

1. \(\varphi\) is monotonic increasing;
2. The sequence \(\{\varphi^n(t)\}_{n=0}^{\infty}\) converge to zero for all \(t \in \mathbb{R}_+\) where \(\varphi^n\) stand for \(n\)th iterate of \(\varphi\).
   If \(\varphi\) satisfies:
3. \(\sum_{k=0}^{\infty} \varphi^k(t)\) converge for all \(t \in \mathbb{R}_+\).

Then \(\varphi\) is called \(c\)-comparison function.

Thus every comparison function is \(c\)-comparison function. A prototype example for comparison function is
\[
\varphi(t) = \alpha t \quad t \in \mathbb{R}_+ \quad 0 \leq \alpha < 1.
\]

Some more examples and properties of comparison and \(c\)-comparison function can be found in [2].

3. Main Results

Theorem 1. Let \((X,d)\) be a complete right (left) dislocated \(b\)-metric space and \(T : X \to X\) be a continuous function for \(k \geq 1\) satisfying
\[
d(Tx, Ty) \leq \varphi d(x, y)
\]
for all \(x, y \in X\) where \(\varphi\) is a comparison function. Then \(T\) has a unique fixed point in \(X\).

Proof. Let \(x_0\) be arbitrary in \(X\) we define a sequence \(\{x_n\}\) in \(X\) as following
\[
x_0, x_1 = Tx_0, x_2 = Tx_1, \ldots, x_{n+1} = Tx_n \quad \text{for all} \quad n \in \mathbb{N}.
\]
Now to show that \(\{x_n\}\) is a Cauchy sequence in \(X\) consider
\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).
\]
Using (2) we have
\[
d(x_n, x_{n+1}) \leq \varphi d(x_{n-1}, x_n).
\]
Similarly one can show that
\[
d(x_{n-1}, x_n) \leq \varphi d(x_{n-2}, x_{n-1}).
\]
Putting (3) in (4) we have
\[
d(x_n, x_{n+1}) \leq \varphi^2 d(x_{n-2}, x_{n-1}).
\]
Proof. Let \( X \) is a unique fixed point in \( T \) and

By the definition of the sequence we get

\[
d(x_n, x_{n+1}) \leq k \cdot d(x_n, x_{n+1}) + k^2 \cdot d(x_{n+1}, x_{n+2}) + k^3 \cdot d(x_{n+2}, x_{n+3}) + \ldots. \tag{5}
\]

Using \((5)\) we have

\[
d(x_n, x_m) \leq k \cdot \varphi^n d(x_0, x_1) + k^2 \cdot \varphi^{n+1} d(x_0, x_1) + k^3 \cdot \varphi^{n+2} d(x_0, x_1) + \ldots.
\]

Since \( \varphi \) is a comparison function so taking \( n, m \to \infty \) we get

\[
\lim_{n, m \to \infty} d(x_n, x_m) = 0.
\]

Which show that \( \{x_n\} \) is a Cauchy sequence in complete right (left) dislocated \( b \)-metric space \( X \). So there exists \( z \in X \) such that \( x_n \to z \) as \( n \to \infty \).

Now to show that \( z \) is the fixed point of \( T \). Since \( x_n \to z \) as \( n \to \infty \) using the continuity of \( T \) we have

\[
\lim_{n \to \infty} T x_n = T z
\]

which implies that

\[
\lim_{n \to \infty} x_{n+1} = T z.
\]

Thus \( T z = z \). So \( z \) is the fixed point of \( T \).

Uniqueness: Suppose that \( T \) has two fixed points \( z \) and \( w \) for \( z \neq w \). Consider

\[
d(z, w) = d(T z, T w).
\]

Using \((2)\) we have

\[
d(z, w) \leq \varphi d(z, w).
\]

Since \( \varphi \) is a comparison function so the above inequality is possible only if \( d(z, w) = 0 \) similarly one can show that \( d(w, z) = 0 \). So by \((d_1)\) \( z = w \). Hence \( T \) has a unique fixed point in \( X \).

Remark. Theorem 1 generalize Banach contraction principle and the result established by Matkowski \([13]\) in right (left) dislocated \( b \)-metric spaces.

Theorem 2. Let \((X, d)\) be a complete right or (left) dislocated \( b \)-metric space and \( T : X \to X \) is a continuous self-mapping satisfying

\[
d(T x, T y) \leq \alpha \cdot d(x, y) + \beta \cdot d(x, T x) + \gamma \cdot d(y, T y) \tag{6}
\]

for all \( x, y \in X \) and \( \alpha, \beta, \gamma \geq 0 \) with \( k \alpha + k \beta + \gamma < 1 \) where \( k \geq 1 \). Then \( T \) has a unique fixed point in \( X \).

Proof. Let \( x_0 \) be arbitrary in \( X \) we define a sequence \( \{x_n\} \) in \( X \) as following

\[
x_0, x_1 = T x_0, x_2 = T x_1, \ldots, x_{n+1} = T x_n.
\]

Now to show that \( \{x_n\} \) is a Cauchy sequence consider

\[
d(x_n, x_{n+1}) = d(T x_{n-1}, T x_n).
\]

Using \((6)\) we have

\[
d(x_n, x_{n+1}) = d(T x_{n-1}, T x_n) \leq \alpha \cdot d(x_{n-1}, x_n) + \beta \cdot d(x_{n-1}, T x_{n-1}) + \gamma \cdot d(x_n, T x_n).
\]

By the definition of the sequence we get

\[
d(x_n, x_{n+1}) \leq \alpha \cdot d(x_{n-1}, x_n) + \beta \cdot d(x_{n-1}, x_n) + \gamma \cdot d(x_n, x_{n+1}).
\]
Simplification yields
\[ d(x_n, x_{n+1}) \leq \frac{\alpha + \beta}{1 - \gamma} \cdot d(x_{n-1}, x_n). \]

Let
\[ h = \frac{\alpha + \beta}{1 - \gamma} < \frac{1}{k}. \]

So the above inequality become
\[ d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n). \]

Also
\[ d(x_{n-1}, x_n) \leq h \cdot d(x_{n-2}, x_{n-1}). \]

Thus
\[ d(x_n, x_{n+1}) \leq h^2 \cdot d(x_{n-2}, x_{n-1}). \]

Similarly proceeding we get
\[ d(x_n, x_{n+1}) \leq h^n \cdot d(x_0, x_1). \]

Since \( h < \frac{1}{k} \). Taking limit \( n \to \infty \), so \( h^n \to 0 \) and
\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \]

So by Lemma 3 \( \{x_n\} \) is a Cauchy sequence in complete right or (left) dislocated \( b \)-metric space so there must exist \( u \in X \) such that
\[ \lim_{n \to \infty} (x_n, u) = 0. \]

Now to show that \( u \) is the fixed point of \( T \). Since \( x_n \to u \) as \( n \to \infty \) using the continuity of \( T \) we have
\[ \lim_{n \to \infty} T x_n = T u \]
which implies that
\[ \lim_{n \to \infty} x_{n+1} = T u. \]

Thus \( Tu = u \). So \( u \) is the fixed point of \( T \).

**Uniqueness:** Let \( T \) have two fixed points i.e \( u, v \) with \( u \neq v \) then we have
\[ d(u, v) = d(T u, T v) \leq \alpha \cdot d(u, v) + \beta \cdot d(u, T u) + \gamma \cdot d(v, T v) \]
\[ d(u, v) = d(T u, T v) \leq \alpha \cdot d(u, v) + \beta \cdot d(u, u) + \gamma \cdot d(v, v). \]

Putting \( u = v \) in (6) one can easily show that \( d(u, u) = d(v, v) = 0 \). Thus the above equation become
\[ d(u, v) \leq \alpha \cdot d(u, v). \]

The above inequality is possible only if \( d(u, v) = 0 \) similarly one can show that \( d(v, u) = 0 \). So by \( (d_1) \) we get that \( u = v \). Thus fixed point of \( T \) is unique.

**Corollary.** Let \((X, d)\) be a complete right or (left) dislocated \( b \)-metric space and \( T : X \to X \) is a continuous self-mapping satisfying
\[ d(T x, T y) \leq \alpha \cdot d(x, y) + \beta \cdot d(x, T x) \]
for all \( x, y \in X \) and \( \alpha, \beta \geq 0 \) with \( k\alpha + k\beta < 1 \) where \( k \geq 1 \). Then \( T \) has a unique fixed point in \( X \).
Corollary. Let $(X,d)$ be a complete right or (left) dislocated $b$-metric space and $T : X \to X$ is a continuous self-mapping satisfying

$$d(Tx, Ty) \leq \alpha \cdot d(x, y)$$

for all $x, y \in X$ and $\alpha \geq 0$ with $0 \leq k\alpha < 1$ where $k \geq 1$. Then $T$ has a unique fixed point in $X$.

Remarks. Theorem 2 generalize Reich type contraction and extend Banach contraction principle and convex type contraction in complete right or (left) dislocated $b$-metric spaces.

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References


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