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A NEW FAMILY OF HORADAM NUMBERS

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Abstract. In this paper, we define a new family of (k,t)-Horadam numbers and obtain Binet formula for this family. We give the relationship between this family and the known generalized t-Horadam number. Then we prove the Cassini and Catalan identities for this family. Furthermore, we investigate the sums, the recurrence relations and generating functions of this family.

1. Introduction

Many researchers have been and still are interested in various positive integers such as Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, etc. For interest application of these numbers in science and nature, one can see [1, 4, 5, 7, 10, 11, 14, 16]. Falcon gave some properties of general k-Fibonacci numbers [6]. Mikkawy presented a new family of k-Fibonacci numbers and several properties of this family [8]. Öcal et.al. introduced some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers [9]. One of the latest works is [12] where they defined a new family of Gauss k-Lucas numbers and the generalized polynomials for these numbers. They obtained some interesting properties of the family and its polynomials.

There are many works related to Horadam numbers and their applications [2, 3, 13, 15].

Now, we introduce the generalized t-Horadam sequence $H_{t,n}$.

Let t be any positive real number and $f(t), g(t)$ be scaler-value polynomials. For $n \in \mathbb{N}$ and $f^2(t)+4g(t)>0$, the generalized t-Horadam number $H_{t,n}$ is defined by

$$H_{t,n} = f(t)H_{t,n-1} + g(t)H_{t,n-2} \tag{1.1}$$

where $H_{t,0}=a$, $H_{t,1}=b$ [15].

Binet formula for this number is

$$H_{t,n} = \frac{c\alpha^n - d\beta^n}{\alpha-\beta}, \tag{1.2}$$

where $c=b-a\beta$, $d=b-a\alpha$, $\alpha = \frac{f(t)+\sqrt{f^2(t)+4g(t)}}{2}$ and $\beta = \frac{f(t)-\sqrt{f^2(t)+4g(t)}}{2}$ [15].

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In this paper, we define the new family of (k,t)-Horadam numbers and obtain Binet formula of the family. We give the relationships between the family and the known generalized t-Horadam numbers. More, we obtain the sums, the recurrence relations and generating functions of this family. Then we prove Cassini and Catalan identities for the family.

2. Main Results

Definition 2.1. *Let be any natural numbers $m(\neq 0), k(\neq 0), r$ such that $n = mk+r$ and $0 \leq r < k$. The family of (k,t)-Horadam numbers is defined by*

$$H_{t,n}^{(k)} = \sum_{i=1}^{r} \binom{r}{i} \left(f(t)H_{t,m}\right)^{k-i} \left(g(t)H_{t,m-1}\right)^{i}. \tag{2.1}$$

The Binet formula for the family of (k,t)-Horadam numbers is

$$H_{t,n}^{(k)} = \left[\frac{c\alpha^m - d\beta^m}{\alpha-\beta}\right]^{k-r} \left[\frac{c\alpha^{m+1} - d\beta^{m+1}}{\alpha-\beta}\right]^{r}. \tag{2.2}$$

Let's give some values for the family of (k,t)-Horadam numbers in Table 2.1.

Table 2.1 Sample values for $H_{t,n}^{(k)}$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$H_{t,0}^{(k)}$	$H_{t,0}$	a^2	a^3	a^4	a^5
$H_{t,1}^{(k)}$	$H_{t,1}$	ab	a^2b	a^3b	a^4b
$H_{t,2}^{(k)}$	$H_{t,2}$	b^2	ab^2	a^2b^2	a^3b^2
$H_{t,3}^{(k)}$	$H_{t,3}$	$b(bf(t)+ag(t))$	b^3	ab^3	a^2b^3
$H_{t,4}^{(k)}$	$H_{t,4}$	$(bf(t)+ag(t))^2$	$b^2(bf(t)+ag(t))$	b^4	ab^4

Results;

$$\begin{aligned} H_{t,0}^{(k)} &= a^k, \\ H_{t,1}^{(k)} &= a^{k-1}b, \\ H_{t,k}^{(k)} &= b^k, \\ H_{t,n}^{(1)} &= H_{t,n}, \\ H_{t,kn}^{(k)} &= (H_{t,n})^k, \\ H_{t,kn+1}^{(k)} &= (H_{t,n})^{k-1}H_{t,n+1}. \end{aligned} \tag{2.3}$$

From Equations (1.2) and (2.2), we can give this family via the generalized t-Horadam numbers.

$$H_{t,n}^{(k)} = (H_{t,m})^{k-r}(H_{t,m+1})^r, n = mk+r. \tag{2.4}$$

Additionally, we can express the family of (k,t)-Horadam numbers by matrix methods. Indeed, it is clear that

$$H_{t,n}^{k-1}h_n = \begin{bmatrix} H_{t,kn+1}^{(k)} & H_{t,kn}^{(k)} \\ H_{t,kn}^{(k)} & H_{t,kn-1}^{(k)} \end{bmatrix}$$

where $h_n = \begin{bmatrix} H_{t,n+1} & H_{t,n} \\ H_{t,n} & H_{t,n-1} \end{bmatrix}, n \geq 1.$

Particular cases of the family of (k,t)-Horadam numbers are as follows

- If $f(t)=s$, $g(t)=1$, $a=0$, $b=1$ in (2.1), then the family of generalized s-Fibonacci numbers $F_{s,n}^{(k)}$ is obtained [1],
- If $f(t)=s$, $g(t)=1$, $a=2$, $b=1$ in (2.1), then the family of generalized s-Lucas numbers $L_{s,n}^{(k)}$ is obtained [1],
- If $f(t)=g(t)=1$, $a=1$, $b=1$ in (2.1), then the family of generalized Fibonacci numbers $F_n^{(k)}$ is obtained [8],
- If $f(t)=2$, $g(t)=1$, $a=0$, $b=1$ in (2.1), then the family of generalized Pell numbers $P_n^{(k)}$ is obtained,
- If $f(t)=1$, $g(t)=2$, $a=0$, $b=1$ in (2.1), then the family of generalized Jacobsthal numbers $J_n^{(k)}$ is obtained [4],
- If $f(t)=1$, $g(t)=2$, $a=2$, $b=1$ in (2.1), then the family of generalized Jacobsthal-Lucas numbers $j_n^{(k)}$ is obtained [4],
- If $f(t)=p$, $g(t)=q$, $a=0$, $b=1$ in (2.1), then the family of generalized k-Fibonacci numbers $U_n^{(k)}$ is obtained [5],
- If $f(t)=p$, $g(t)=q$, $a=2$, $b=p$ in (2.1), then the family of generalized k-Lucas numbers $V_n^{(k)}$ is obtained [5],
- If $f(t)=g(t)=1$ in (2.1), then the family of generalized k-Fibonacci numbers $G_n^{(k)}$ is obtained [16].

Theorem 2.1. *Let $G^{(2)}(x,t)$ denote the generating function of $H_{t,n}^{(2)}$. For $k=2$ and $n=2m+r$, we have the following recurrence relation and the generating function of the family for (k,t)-Horadam numbers:*

i) $H_{t,n}^{(2)}=f(t)H_{t,n-1}^{(2)}+f(t)g(t)H_{t,n-3}^{(2)}+g^2(t)H_{t,n-4}^{(2)}$,

ii) $G^{(2)}(x,t)=\dfrac{a^2+(ab-a^2f(t))x+(b^2-abf(t))x^2+(abg(t)-a^2f(t)g(t))x^3}{1-f(t)x-f(t)g(t)x^3-g^2(t)x^4}$.

Proof. *i)* There are two cases of subscript n. From (1.1) and (2.4), we obtain as follow.

For $n=2m$,

$$\begin{aligned} H_{t,2m}^{(2)} &= (H_{t,m})^2 \\ &= H_{t,m}\left(f(t)H_{t,m-1}+g(t)H_{t,m-2}\right) \\ &= f(t)H_{t,m-1}H_{t,m}+g(t)H_{t,m-2}\left(f(t)H_{t,m-1}+g(t)H_{t,m-2}\right) \\ &= f(t)H_{t,2m-1}^{(2)}+f(t)g(t)H_{t,2m-3}^{(2)}+g^2(t)H_{t,2m-4}^{(2)}. \end{aligned}$$

For $n=2m+1$,

$$\begin{aligned} H_{t,2m+1}^{(2)} &= H_{t,m}H_{t,m+1} \\ &= H_{t,m}\left(f(t)H_{t,m}+g(t)H_{t,m-1}\right) \\ &= f(t)H_{t,m}^2+g(t)H_{t,m-1}\left(f(t)H_{t,m-1}+g(t)H_{t,m-2}\right) \\ &= f(t)H_{t,2m}^{(2)}+f(t)g(t)H_{t,2m-2}^{(2)}+g^2(t)H_{t,2m-3}^{(2)}. \end{aligned}$$

ii) Let $G^{(2)}(x,t)$ be generating function for the $H_{t,n}^{(2)}$,

$$G^{(2)}(x,t)=\sum_{n=0}^{\infty}H_{t,n}^{(2)}x^n. \tag{2.5}$$

If $G^{(2)}(x,t)$ given in (2.5) multiply with $f(t)x$, $f(t)g(t)x^3$ and $g^2(t)x^4$, respectively, then we get

$$\left.\begin{array}{c} f(t)xG^{(2)}(x,t)=f(t)\sum_{n=1}^{\infty}H_{t,n-1}^{(2)}x^n \\ f(t)g(t)x^3G^{(2)}(x,t)=f(t)g(t)\sum_{n=3}^{\infty}H_{t,n-3}^{(2)}x^n \\ g^2(t)x^4G^{(2)}(x,t)=g^2(t)\sum_{n=4}^{\infty}H_{t,n-4}^{(2)}x^n \end{array}\right\}. \tag{2.6}$$

Consequently, by subtracting (2.6) from (2.5), it is obtained the following equation

$$G^{(2)}(x,t)=\frac{H_{t,0}^{(2)}+H_{t,1}^{(2)}x+H_{t,2}^{(2)}x^2+H_{t,3}^{(2)}x^3-f(t)x\left(H_{t,0}^{(2)}+H_{t,1}^{(2)}x+H_{t,2}^{(2)}x^2\right)-f(t)g(t)x^3H_{t,0}^{(2)}}{1-f(t)x-f(t)g(t)x^3-g^2(t)x^4},$$

which completes the proof.

□

Theorem 2.2. *Let $G^{(k)}(x,t)$ denote the generating function of $H_{t,n}^{(k)}$. For $n=mk+1$, we have the following recurrence relation and the generating function for the family of (k,t)-Horadam numbers*

i) $H_{t,n}^{(k)}=f(t)H_{t,n-1}^{(k)}+g(t)H_{t,n-2}^{(k)}$,

ii) $G^{(k)}(x,t)=\frac{a^k+a^{k-1}x(b-af(t))}{1-f(t)x-g(t)x^2}$.

Proof. The proof is done similarly to that of Theorem 2.1. □

Theorem 2.3. *For k and m, we have*

$$(H_{t,m+1})^k-(H_{t,m})^k=H_{t,(m+1)k}^{(k)}-H_{t,mk}^{(k)}.$$

Proof. From (2.4), we obtain

$$\begin{aligned} H_{t,(m+1)k}^{(k)}-H_{t,mk}^{(k)} &= (H_{t,m})^{k-k}(H_{t,m+1})^k-(H_{t,m})^{k-0}(H_{t,m+1})^0 \\ &= (H_{t,m+1})^k-(H_{t,m})^k. \end{aligned}$$

□

Let's suppose that $H_{t,-n}^{(k)}=0$ for $k=1,2,\ldots$.

Theorem 2.4. *For the $n,s\geq 0$, $n+s\geq 1$, we have*

$$H_{t,2(n+s-1)}^{(2)}-H_{t,n+s}H_{t,n+s-2} = (-g(t))^{n+s-1}(a^2g(t)+abf(t)-b^2).$$

Proof. From (2.3) and Theorem 5 in [15], we get

$$\begin{aligned} H_{t,2(n+s-1)}^{(2)}-H_{t,n+s}H_{t,n+s-2} &= H_{t,n+s-1}^2-H_{t,n+s}H_{t,n+s-2}, \\ &= (-g(t))^{n+s-1}(a^2g(t)+abf(t)-b^2). \end{aligned}$$

□

Theorem 2.5. *For $1\leq s<k$, $n,k\geq 2$ and s,n,k integer numbers, Cassini Identity for $H_{t,n}^{(k)}$ is as follows:*

$$H_{t,kn+s}^{(k)}H_{t,kn+s-2}^{(k)}-\left(H_{t,kn+s-1}^{(k)}\right)^2=\begin{cases} H_{t,n}^{2k-2}(-g(t))^{n-1}(a^2g(t)+abf(t)-b^2), & s=1 \\ 0, & s\neq 1 \end{cases}.$$

Proof. For $s=1$, we obtain the following equations from (2.4) and Theorem 5 in [15]

$$\begin{aligned} H^{(k)}_{t,kn+1}H^{(k)}_{t,kn-1}-\left(H^{(k)}_{t,kn}\right)^2 &= \left(H^{k-1}_{t,n}H_{t,n+1}\right)\left(H_{t,n-1}H^{k-1}_{t,n}\right)-\left(H^{k}_{t,n}\right)^2 \\ &= H^{2k-2}_{t,n}\left(H_{t,n+1}H_{t,n-1}-(H_{t,n})^2\right) \\ &= H^{2k-2}_{t,n}(-g(t))^{n-1}(a^2g(t)+abf(t)-b^2). \end{aligned}$$

For $s\neq 1$, we have

$$\begin{aligned} H^{(k)}_{t,kn+s}H^{(k)}_{t,kn+s-2}-\left(H^{(k)}_{t,kn+s-1}\right)^2 &= \left(H^{k-s}_{t,n}H^{s}_{t,n+1}\right)(H^{k-s+2}_{t,n}H^{s-2}_{t,n+1})-\left(H^{k-s+1}_{t,n}H^{s-1}_{t,n+1}\right)^2 \\ &= H^{2k-2s+2}_{t,n}(H^{2s-2}_{t,n+1}-H^{2s-2}_{t,n+1}) \\ &= 0. \end{aligned}$$

□

Theorem 2.6. *For $0\leq 3s<k$, $n,k\geq 2$ and s,n,k integer numbers, Catalan Identity for $H^{(k)}_{t,n}$ is as follows:*

$$H^{(k)}_{t,kn+3s}H^{(k)}_{t,kn+s}-\left(H^{(k)}_{t,kn+2s}\right)^2=0$$

Proof. From (2.4), we obtain

$$\begin{aligned} H^{(k)}_{t,kn+3s}H^{(k)}_{t,kn+s}-\left(H^{(k)}_{t,kn+2s}\right)^2 &= \left(H^{k-3s}_{t,n}H^{3s}_{t,n+1}\right)\left(H^{k-s}_{t,n}H^{s}_{t,n+1}\right)-\left(H^{k-2s}_{t,n}H^{2s}_{t,n+1}\right)^2 \\ &= 0. \end{aligned}$$

□

Proposition 2.1. *We obtain the following interesting properties for $H^{(2)}_{t,n}$*

i) $H^{(2)}_{t,2n+2}-f^2(t)H^{(2)}_{t,2n}=2f(t)g(t)H^{(2)}_{t,2n-1}+g^2(t)H^{(2)}_{t,2n-2}$,

ii) $f(t)H^{(2)}_{t,2n-1}+g(t)H^{(2)}_{t,2n-2}=H^{(2)}_{t,2n}+(-g(t))^{n-1}(a^2g(t)+abf(t)-b^2)$.

Proof. *i)* From (2.3) and (1.1), we get

$$\begin{aligned} H^{(2)}_{t,2n+2}-f^2(t)H^{(2)}_{t,2n} &= H^2_{t,n+1}-(f(t)H_{t,n})^2 \\ &= (H_{t,n+1}-f(t)H_{t,n})(H_{t,n+1}+f(t)H_{t,n}) \\ &= g(t)H_{t,n-1}(2f(t)H_{t,n}+g(t)H_{t,n-1}) \\ &= 2f(t)g(t)H_{t,n-1}H_{t,n}+g^2(t)H^2_{t,n-1} \\ &= 2f(t)g(t)H^{(2)}_{t,2n-1}+g^2(t)H^{(2)}_{t,2n-2} \end{aligned}$$

ii) From (2.3) and (1.1), we obtain

$$\begin{aligned} f(t)H^{(2)}_{t,2n-1}+g(t)H^{(2)}_{t,2n-2} &= f(t)H_{t,n-1}H_{t,n}+g(t)(H_{t,n-1})^2 \\ &= H_{t,n-1}(f(t)H_{t,n}+g(t)H_{t,n-1}) \\ &= H_{t,n-1}H_{t,n+1}. \end{aligned}$$

Then, from (2.3) and Theorem 5 in [15], we get

$$\begin{aligned} f(t)H^{(2)}_{t,2n-1}+g(t)H^{(2)}_{t,2n-2} &= (H_{t,n})^2+(-g(t))^{n-1}(a^2g(t)+abf(t)-b^2) \\ &= H^{(2)}_{t,2n}+(-g(t))^{n-1}(a^2g(t)+abf(t)-b^2). \end{aligned}$$

□

Theorem 2.7. *Sums for the family of (k,t)-Horadam numbers are*

i) $\sum_{i=0}^{k-1}(f(t))^{-i}H_{t,mk+i}^{(k)}=\frac{H_{t,m}}{H_{t,m-1}}\left(\frac{H_{t,(m+1)k}^{(k)}-(f(t))^{k}H_{t,mk}^{(k)}}{f^{k-1}(t)g(t)}\right)$,

ii) $g^{k-1}(t)\sum_{i=0}^{k-1}\left(\frac{f(t)}{g(t)}\right)^{i}\binom{k-1}{i}H_{t,mk+i}^{(k)}=H_{t,m}H_{t,(m+2)(k-1)}^{(k-1)}$,

iii) $\sum_{i=0}^{k-1}(-1)^{i}\left(f(t)\right)^{k-i-1}\binom{k-1}{i}H_{t,mk+i}^{(k)}=\left(g(t)\right)^{k-1}H_{t,m}H_{t,(m-1)(k-1)}^{(k-1)}$.

Proof. *i)* From (2.4), we can write

$$\begin{aligned}\sum_{i=0}^{k-1}(f(t))^{-i}H_{t,mk+i}^{(k)} &= \sum_{i=0}^{k-1}\left(\frac{H_{t,m+1}}{f(t)H_{t,m}}\right)^{i}H_{t,m}^{k}\\ &= H_{t,m}^{k}\left(\frac{\left(\frac{H_{t,m+1}}{f(t)H_{t,m}}\right)^{k}-1}{\frac{H_{t,m+1}}{f(t)H_{t,m}}-1}\right)\\ &= \frac{H_{t,m}}{f^{k-1}(t)g(t)H_{t,m-1}}\left(H_{t,m+1}^{k}-f^{k}(t)H_{t,m}^{k}\right)\\ &= \frac{H_{t,m}}{H_{t,m-1}}\left(\frac{H_{t,(m+1)k}^{(k)}-f^{k}(t)H_{t,mk}^{(k)}}{f^{k-1}(t)g(t)}\right).\end{aligned}$$

Proofs of *ii)* and *iii)* are similar to that of *i)*. □

Conclusion 2.1. *In this paper, we defined a new family of (k,t)-Horadam numbers. We gave some relations between this family and the known generalized t-Horadam numbers. Then we proved Cassini and Catalan identities for this family. Furthermore, we found the recurrence relations, generating functions and summations of this family. Note that, if we take $f(t), g(t)$, a and b be as integers, our evaluations cover all the results in* [4], [8], [16].

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