Electronic Journal of Mathematical Analysis and Applications
Vol. 10(1) Jan. 2022, pp. 71-83.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

SUM OF THE SQUARES OF TERMS OF GAUSSIAN GENERALIZED TRIBONACCI SEQUENCES: CLOSED FORM FORMULAS OF $\sum_{k=1}^{n} GW_k^2$

Y. SOYKAN, E. TAŞDEMIR AND M. GÖCEN

Abstract. In this paper, closed forms of the sum formulas $\sum_{k=1}^{n} GW_k^2$, $\sum_{k=1}^{n} GW_{k+2}GW_k$ and $\sum_{k=1}^{n} GW_{k+1}GW_k$ for the squares of Gaussian generalized Tribonacci numbers are presented. As special cases, we give sum formulas of Gaussian Tribonacci, Gaussian Tribonacci-Lucas, Gaussian Padovan, Gaussian Perrin, Gaussian Narayana and some other third order linear recurrence sequences. All the summing formulas of well known recurrence sequences are linear except the cases Gaussian Pell-Padovan and Gaussian Padovan-Perrin.

1. Introduction

The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1.$$

The Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field. The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \qquad (1)$$

where W_0, W_1, W_2 are arbitrary complex numbers and r, s, t are real numbers.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1) holds for all integer n.

If we set $r = s = t = 1$ and $W_0 = 0, W_1 = 1, W_2 = 1$ then $\{W_n\}$ is the well-known Tribonacci sequence and if we set $r = s = t = 1$ and $W_0 = 3, W_1 = 1, W_2 = 3$ then $\{W_n\}$ is the well-known Tribonacci-Lucas sequence.

1991 *Mathematics Subject Classification.* 11B37, 11B39, 11B83.

Key words and phrases. Generalized Tribonacci numbers, Gaussian Tribonacci numbers, Gaussian Padovan numbers, Gaussian Perrin numbers, sum formulas, sum of squares.

Submitted Feb. 24, 2021.

In fact, the generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Tribonacci-Lucas, Padovan (Cordonnier), Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas.

We now present some background about Gaussian and Gaussian generalized Tribonacci numbers. In literature, there have been so many studies of the sequences of Gaussian numbers. A Gaussian integer z is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, $a, b \in \mathbb{Z}$. These numbers is denoted by $\mathbb{Z}[i]$. The norm of a Gaussian integer $a + ib$, $a, b \in \mathbb{Z}$ is its Euclidean norm, that is, $N(a + ib) = \sqrt{a^2 + b^2} = \sqrt{(a + ib)(a - ib)}$. For more information about this kind of integers, see the work of Fraleigh [4].

If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers.

The Gaussian generalized Tribonacci sequence $\{GW_n(GW_0, GW_1, GW_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{GW_n\}_{n\geq 0}$) is defined as follows:

$$\begin{aligned} GW_n &= rGW_{n-1} + sGW_{n-2} + tGW_{n-3},\ GW_0 = W_0 + W_{-1}i, \\ GW_1 &= W_1 + W_0 i,\ GW_2 = W_2 + W_1 i,\ n \geq 3 \end{aligned} \tag{2}$$

where r, s, t are real numbers.

The sequence $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = -\frac{s}{t}GW_{-(n-1)} - \frac{r}{t}GW_{-(n-2)} + \frac{1}{t}GW_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (2) holds for all integer n.

Note that for $n \geq 0$

$$GW_n = W_n + iW_{n-1}. \tag{3}$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}$$

In fact, the Gaussian generalized Tribonacci sequence is the generalization of the well-known sequences like Gaussian Tribonacci, Gaussian Tribonacci-Lucas, Gaussian Padovan (Cordonnier), Gaussian Perrin, Gaussian Padovan-Perrin, Gaussian Narayana, Gaussian third order Jacobsthal and Gaussian third order Jacobsthal-Lucas. In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t and initial values.

Table 1 A few special case of Gaussian generalized Tribonacci sequences.

Sequences (Numbers)	Notation
Gaussian Tribonacci	$\{GT_n\} = \{W_n(0, 1, 1+i; 1, 1, 1)\}$
Gaussian Tribonacci-Lucas	$\{GK_n\} = \{W_n(3-i, 1+3i, 3+i; 1, 1, 1)\}$
Gaussian third order Pell	$\{GP_n^{(3)}\} = \{W_n(0, 1, 2+i; 2, 1, 1)\}$
Gaussian third order Pell-Lucas	$\{GQ_n^{(3)}\} = \{W_n(3-i, 2+3i, 6+2i; 2, 1, 1)\}$
Gaussian third order modified Pell	$\{GE_n^{(3)}\} = \{W_n(-i, 1, 1+i; 2, 1, 1)\}$
Gaussian Padovan (Cordonnier)	$\{GP_n\} = \{W_n(1, 1+i, 1+i; 0, 1, 1)\}$
Gaussian Perrin	$\{GE_n\} = \{W_n(3-i, 3i, 2; 0, 1, 1)\}$
Gaussian Padovan-Perrin	$\{GS_n\} = \{W_n(i, 0, 1; 0, 1, 1)\}$
Gaussian Pell-Padovan	$\{GR_n\} = \{W_n(1-i, 1+i, 1+i; 0, 2, 1)\}$
Gaussian Pell-Perrin	$\{GC_n\} = \{W_n(3-4i, 3i, 2; 0, 2, 1)\}$
Gaussian Jacobsthal-Padovan	$\{GQ_n\} = \{W_n(1, 1+i, 1+i; 0, 1, 2)\}$
Gaussian Jacobsthal-Perrin	$\{GD_n\} = \{W_n(3-\frac{1}{2}i, 3i, 2; 0, 1, 2)\}$
Gaussian Narayana	$\{GN_n\} = \{W_n(0, 1, 1+i; 1, 0, 1)\}$
Gaussian third order Jacobsthal	$\{GJ_n^{(3)}\} = \{W_n(0, 1, 1+i; 1, 1, 2)\}$
Gaussian third order Jacobsthal-Lucas	$\{Gj_n^{(3)}\} = \{W_n(2+i, 1+2i, 5+i; 1, 1, 2)\}$

In 1963, Horadam [9] introduced the concept of complex Fibonacci number called as the Gaussian Fibonacci number. Pethe [13] defined the complex Tribonacci numbers at Gaussian integers, see also [5].

There are other several studies dedicated to these sequences of Gaussian numbers. We present some works on Gaussian Generalized Fibonacci Numbers in the following Table 2.

Table 2. A few special study of Gaussian Generalized Fibonacci Numbers.

Name of sequence	Papers which deal with Gaussian Numbers
Gaussian Generalized Fibonacci	[1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 20]
Gaussian Generalized Tribonacci	[15, 19]
Gaussian Generalized Tetranacci	[16, 18]
Gaussian Generalized Pentanacci	[17]

2. Main Result

Let

$$G\Delta = \left(s + rt - t^2 + 1\right)(r + s + t - 1)(r - s + t + 1).$$

Theorem 1. *If* $G\Delta \neq 0$ *then*

(a):

$$\sum_{k=1}^{n} GW_k^2 = \frac{G\Delta_1}{G\Delta}$$

(b):

$$\sum_{i=k}^{n} GW_{k+1}GW_k = \frac{G\Delta_2}{G\Delta}$$

(c):

$$\sum_{k=1}^{n} GW_{k+2}GW_k = \frac{G\Delta_3}{G\Delta},$$

where

$$\begin{aligned}
G\Delta_1 &= -(t^2+rt+s-1)GW_{n+3}^2 \\
&\quad -(r^3t+r^2t^2+r^2s+r^2+t^2+2rst+rt+s-1)GW_{n+2}^2 \\
&\quad -(r^3t+r^2t^2+s^2t^2-rs^2t-s^3+r^2s+4rst \\
&\quad +r^2+s^2+t^2+rt+s-1)GW_{n+1}^2 \\
&\quad +2\left(r+t\right)\left(s+rt\right)GW_{n+3}GW_{n+2} \\
&\quad +2t\left(r+st\right)GW_{n+3}GW_{n+1} \\
&\quad -2t\left(s-1\right)\left(s+rt\right)GW_{n+2}GW_{n+1} \\
&\quad +(2rst+2r^2+t^2+rt+s-1)GW_3^2 \\
&\quad +(r^3t+r^2t^2+r^2s+2rst+r^2+t^2+rt+s-1)GW_2^2 \\
&\quad +(r^3t+r^2t^2+s^2t^2-rs^2t-s^3+r^2s+4rst+r^2 \\
&\quad +s^2+t^2+rt+s-1)GW_1^2 \\
&\quad -2\left(r+st\right)GW_4GW_3-2t(r^2-s^2+rt+s)GW_3GW_2 \\
&\quad +2t\left(s-1\right)\left(s+rt\right)GW_2GW_1
\end{aligned}$$

and

$$\begin{aligned}
G\Delta_2 &= \left(r+st\right)GW_{n+3}^2+\left(s+rt\right)\left(t+rs\right)GW_{n+2}^2 \\
&\quad +t^2\left(r+st\right)GW_{n+1}^2 \\
&\quad -(2rst+r^2+s^2+t^2-1)GW_{n+3}GW_{n+2} \\
&\quad +t(r^2-s^2-t^2+1)GW_{n+3}GW_{n+1} \\
&\quad -(r^3t-rt^3-rs^2t+r^2s-s^3-st^2+2rst+r^2 \\
&\quad +s^2+t^2+rt+s-1)GW_{n+2}GW_{n+1} \\
&\quad +(r^3-rs^2-rt^2-st)GW_3^2-\left(t+rs\right)\left(s+rt\right)GW_2^2 \\
&\quad -t^2\left(r+st\right)GW_1^2-(r^2-s^2-t^2+1)GW_4GW_3 \\
&\quad +(r^2s-st^2-s^3+2rst+r^2+s^2+t^2+s-1)GW_3GW_2 \\
&\quad +(-rt^3+r^3t-rs^2t+r^2s-st^2-s^3+r^2 \\
&\quad +s^2+t^2+2rst+rt+s-1)GW_2GW_1
\end{aligned}$$

and

$$\begin{aligned}
G\Delta_3 &= (r^2-s^2+rt+s)GW_{n+3}^2 \\
&-(rs^2t-rt^3-r^2t^2+r^2s+s^2-s)GW_{n+2}^2 \\
&+t^2(r^2-s^2+rt+s)GW_{n+1}^2 \\
&-(r+t)\left(r^2-s^2+t^2-1\right)GW_{n+3}GW_{n+2} \\
&-(r^2s-st^2-s^3+2rst+r^2+s^2+t^2+s-1)GW_{n+3}GW_{n+1} \\
&+t\,(s-1)\,(r^2-s^2+t^2-1)GW_{n+2}GW_{n+1} \\
&+(rs^2t+r^4-r^2s^2-r^2t^2+2r^2s-rt^3+r^3t+s^2-s)GW_3^2 \\
&+(rs^2t-rt^3-r^2t^2+r^2s+s^2-s)GW_2^2 \\
&-t^2(r^2-s^2+rt+s)GW_1^2 \\
&-(r^3-t^3-rs^2-rt^2+r^2t+s^2t+2rs+r+t)GW_4GW_3 \\
&+(r^3s-st^3+s^3t-rst^2-rs^3+r^2st+rs^2+rt^2+r^2t \\
&-s^2t+r^3+t^3+rs+st-r-t)GW_3GW_2 \\
&+(s+rt-t^2+1)\,(r-s+t+1)\,(r+s+t-1)\,GW_3GW_1 \\
&-t\,(s-1)\,(r^2-s^2+t^2-1)GW_2GW_1
\end{aligned}$$

Proof. First, we obtain $\sum_{k=1}^{n} GW_k^2$. Using the recurrence relation

$$GW_n = rGW_{n-1} + sGW_{n-2} + tGW_{n-3}$$

i.e.

$$GW_{n+3} = rGW_{n+2} + sGW_{n+1} + tGW_n$$

or

$$tGW_n = GW_{n+3} - rGW_{n+2} - sGW_{n+1}$$

we obtain

$$\begin{aligned}
t^2GW_n^2 &= GW_{n+3}^2 + r^2GW_{n+2}^2 + s^2GW_{n+1}^2 - 2rGW_{n+3}GW_{n+2} \\
&\quad -2sGW_{n+3}GW_{n+1} + 2rsGW_{n+2}GW_{n+1} \\
t^2GW_{n-1}^2 &= GW_{n+2}^2 + r^2GW_{n+1}^2 + s^2GW_n^2 - 2rGW_{n+2}GW_{n+1} \\
&\quad -2sGW_{n+2}GW_n + 2rsGW_{n+1}GW_n \\
&\quad \vdots \\
t^2GW_2^2 &= GW_5^2 + r^2GW_4^2 + s^2GW_3^2 - 2rGW_5GW_4 \\
&\quad -2sGW_5GW_3 + 2rsGW_4GW_3 \\
t^2GW_1^2 &= GW_4^2 + r^2GW_3^2 + s^2GW_2^2 - 2rGW_4GW_3 \\
&\quad -2sGW_4GW_2 + 2rsGW_3GW_2.
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned} t^2\sum_{k=1}^{n}GW_k^2 &= \sum_{k=4}^{n+3}GW_k^2+r^2\sum_{k=3}^{n+2}GW_k^2+s^2\sum_{k=2}^{n+1}GW_k^2 \\ &\quad -2r\sum_{k=3}^{n+2}GW_{k+1}GW_k-2s\sum_{k=2}^{n+1}GW_{k+2}GW_k \\ &\quad +2rs\sum_{k=2}^{n+1}GW_{k+1}GW_k. \end{aligned} \tag{4}$$

Note that if we replace the followings into (4),

$$\begin{aligned} \sum_{k=4}^{n+3}GW_k^2 &= -GW_1^2-GW_2^2-GW_3^2+GW_{n+1}^2+GW_{n+2}^2+GW_{n+3}^2 \\ &\quad +\sum_{k=1}^{n}GW_k^2, \\ \sum_{k=3}^{n+2}GW_k^2 &= -GW_1^2-GW_2^2+GW_{n+1}^2+GW_{n+2}^2+\sum_{k=1}^{n}GW_k^2, \\ \sum_{k=2}^{n+1}GW_k^2 &= -GW_1^2+GW_{n+1}^2+\sum_{k=1}^{n}GW_k^2, \\ \sum_{k=3}^{n+2}GW_{k+1}GW_k &= -GW_2GW_1-GW_3GW_2+GW_{n+2}GW_{n+1} \\ &\quad +GW_{n+3}GW_{n+2}+\sum_{k=1}^{n}GW_{k+1}GW_k, \\ \sum_{k=2}^{n+1}GW_{k+1}GW_k &= -GW_2GW_1+GW_{n+2}GW_{n+1}+\sum_{k=1}^{n}GW_{k+1}GW_k, \\ \sum_{k=2}^{n+1}GW_{k+2}GW_k &= -GW_3GW_1+GW_{n+3}GW_{n+1}+\sum_{k=1}^{n}GW_{k+2}GW_k. \end{aligned}$$

we get

$$
\begin{aligned}
t^2\sum_{k=1}^{n} GW_k^2 \quad = \quad & (-r^2GW_1^2 - r^2GW_2^2 + r^2GW_{n+1}^2 + r^2GW_{n+2}^2 \qquad (5)\\
& -s^2GW_1^2 + s^2GW_{n+1}^2 - GW_1^2 - GW_2^2 - GW_3^2 + GW_{n+1}^2\\
& +GW_{n+2}^2 + GW_{n+3}^2 + (1+r^2+s^2)\sum_{k=1}^{n} GW_k^2)\\
& +(2rGW_1GW_2 - 2rGW_{n+1}GW_{n+2} - 2rGW_{n+2}GW_{n+3}\\
& +2rGW_2GW_3 + 2rsGW_{n+1}GW_{n+2} - 2rsGW_1GW_2\\
& +(-2r+2rs)\sum_{k=1}^{n} GW_kGW_{k+1})\\
& -2s(-GW_3GW_1 + GW_{n+3}GW_{n+1} + \sum_{k=1}^{n} GW_{k+2}GW_k).
\end{aligned}
$$

Next we obtain $\sum_{k=1}^{n} GW_{k+1}GW_k$. Multiplying the both side of the recurrence relation

$$tGW_n = GW_{n+3} - rGW_{n+2} - sGW_{n+1}$$

by GW_{n+1} we get

$$tGW_{n+1}GW_n = GW_{n+3}GW_{n+1} - rGW_{n+2}GW_{n+1} - sGW_{n+1}^2.$$

Then using last recurrence relation, we obtain

$$
\begin{aligned}
tGW_{n+1}GW_n &= GW_{n+3}GW_{n+1} - rGW_{n+2}GW_{n+1} - sGW_{n+1}^2\\
tGW_nGW_{n-1} &= GW_{n+2}GW_n - rGW_{n+1}GW_n - sGW_n^2\\
&\vdots\\
tGW_3GW_2 &= GW_5GW_3 - rGW_4GW_3 - sGW_3^2\\
tGW_2GW_1 &= GW_4GW_2 - rGW_3GW_2 - sGW_2^2.
\end{aligned}
$$

If we add the equations by side by, we get

$$t\sum_{k=1}^{n} GW_{k+1}GW_k = \sum_{k=2}^{n+1} GW_{k+2}GW_k - r\sum_{k=2}^{n+1} GW_{k+1}GW_k - s\sum_{k=2}^{n+1} GW_k^2.$$

Now it follows that

$$
\begin{aligned}
& t\sum_{k=1}^{n} GW_{k+1}GW_k \qquad (6)\\
= \quad & (-GW_3GW_1 + GW_{n+3}GW_{n+1} + \sum_{k=1}^{n} GW_{k+2}GW_k)\\
& -r(-GW_2GW_1 + GW_{n+2}GW_{n+1} + \sum_{k=1}^{n} GW_{k+1}GW_k)\\
& -s(-GW_1^2 + GW_{n+1}^2 + \sum_{k=1}^{n} GW_k^2).
\end{aligned}
$$

Now, we obtain $\sum_{k=2}^{n} GW_{k+2}GW_k$. Multiplying the both side of the recurrence relation

$$tGW_n = GW_{n+3} - rGW_{n+2} - sGW_{n+1}$$

by GW_{n+2} we get

$$tGW_{n+2}GW_n = GW_{n+3}GW_{n+2} - rGW_{n+2}GW_{n+2} - sGW_{n+2}GW_{n+1}.$$

Then using last recurrence relation, we obtain

$$\begin{aligned} tGW_{n+2}GW_n &= GW_{n+3}GW_{n+2} - rGW_{n+2}^2 - sGW_{n+2}GW_{n+1} \\ tGW_{n+1}GW_{n-1} &= GW_{n+2}GW_{n+1} - rGW_{n+1}^2 - sGW_{n+1}GW_n \\ &\vdots \\ tGW_5GW_3 &= GW_6GW_5 - rGW_5^2 - sGW_5GW_4 \\ tGW_4GW_2 &= GW_5GW_4 - rGW_4^2 - sGW_4GW_3. \end{aligned}$$

If we add the equations by side by, we get

$$t\sum_{k=2}^{n} GW_{k+2}GW_k = \sum_{k=4}^{n+2} GW_{k+1}GW_k - r\sum_{k=4}^{n+2} GW_k^2 - s\sum_{k=3}^{n+1} GW_{k+1}GW_k.$$

Now it follows that

$$\begin{aligned} & t(-GW_3GW_1 + \sum_{k=1}^{n} GW_{k+2}GW_k) \qquad (7) \\ = \; & (-GW_4GW_3 - GW_3GW_2 - GW_2GW_1 + GW_{n+3}GW_{n+2} + GW_{n+2}GW_{n+1} \\ & + \sum_{k=1}^{n} GW_{k+1}GW_k) - r(-GW_1^2 - GW_2^2 - GW_3^2 + GW_{n+1}^2 + GW_{n+2}^2 \\ & + \sum_{k=1}^{n} GW_k^2) - s(-GW_3GW_2 - GW_2GW_1 + GW_{n+2}GW_{n+1} \\ & + \sum_{k=1}^{n} GW_{k+1}GW_k). \end{aligned}$$

Solving the system (5)-(6)-(7), the results in (a), (b) and (c) follow.

3. Specific Cases

In this section, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n} GW_i^2$, $\sum_{k=1}^{n} GW_{i+1}GW_i$ and $\sum_{k=1}^{n} GW_{i+2}GW_i$ for the specific case of sequence $\{GW_n\}$.

Taking $r = s = t = 1$ in Theorem 1, we obtain the following Proposition.

Proposition 2. *If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:*
$\sum_{k=1}^{n} GW_k^2 = \frac{1}{4}(-GW_{n+3}^2 - 4GW_{n+2}^2 - 5GW_{n+1}^2 + 4GW_{n+2}GW_{n+3}$
$+2GW_{n+1}GW_{n+3} + 3GW_3^2 + 4GW_2^2 + 5GW_1^2 - 2GW_4GW_3 - 2GW_2GW_3)$,
$\sum_{k=1}^{n} GW_{k+1}GW_k = \frac{1}{4}(GW_{n+3}^2 + 2GW_{n+2}^2 + GW_{n+1}^2 - 2GW_{n+2}GW_{n+3}$
$-2GW_{n+1}GW_{n+2} - GW_3^2 - 2GW_2^2 - GW_1^2 + 2GW_2GW_3 + 2GW_1GW_2)$,
$\sum_{k=1}^{n} GW_{k+2}GW_k = \frac{1}{4}(GW_{n+3}^2 + GW_{n+1}^2 - 2GW_{n+1}GW_{n+3} + GW_3^2 - GW_1^2 -$
$2GW_3GW_4 + 2GW_2GW_3 + 4GW_1GW_3)$.

From the above Proposition, we have the following Corollary which gives sum formulas of Gaussian Tribonacci numbers (take $GW_n = GT_n$ with $GT_0 = 0, GT_1 = 1, GT_2 = 1 + i$).

Corollary 3. *For $n \geq 1$, Gaussian Tribonacci numbers have the following properties:*
$\sum_{k=1}^{n} GT_k^2 = \frac{1}{4}(-GT_{n+3}^2 - 4GT_{n+2}^2 - 5GT_{n+1}^2 + 4GT_{n+2}GT_{n+3} + 2GT_{n+1}GT_{n+3} - 2i)$,
$\sum_{k=1}^{n} GT_{k+1}GT_k = \frac{1}{4}(GT_{n+3}^2 + 2GT_{n+2}^2 + GT_{n+1}^2 - 2GT_{n+2}GT_{n+3} - 2GT_{n+1}GT_{n+2})$,
$\sum_{k=1}^{n} GT_{k+2}GT_k = \frac{1}{4}(GT_{n+3}^2 + GT_{n+1}^2 - 2GT_{n+1}GT_{n+3} - 2i)$.

Taking $GW_n = GK_n$ with $GK_0 = 3 - i, GK_1 = 1 + 3i, GK_2 = 3 + i$ in the above Proposition, we have the following Corollary which presents sum formulas of Gaussian Tribonacci-Lucas numbers.

Corollary 4. *For $n \geq 1$, Gaussian Tribonacci-Lucas numbers have the following properties:*
$\sum_{k=1}^{n} GK_k^2 = \frac{1}{4}(-GK_{n+3}^2 - 4GK_{n+2}^2 - 5GK_{n+1}^2 + 4GK_{n+2}GK_{n+3}$
$+2GK_{n+1}GK_{n+3} - 36 - 16i)$,
$\sum_{k=1}^{n} GK_{k+1}GK_k = \frac{1}{4}(GK_{n+3}^2 + 2GK_{n+2}^2 + GK_{n+1}^2 - 2GK_{n+2}GK_{n+3}$
$-2GK_{n+1}GK_{n+2} - 12 - 8i)$,
$\sum_{k=1}^{n} GK_{k+2}GK_k = \frac{1}{4}(GK_{n+3}^2 + GK_{n+1}^2 - 2GK_{n+1}GK_{n+3} - 36)$.

Taking $r = 2, s = 1, t = 1$ in Theorem1, we obtain the following Proposition.

Proposition 5. *If $r = 2, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*
$\sum_{k=1}^{n} GW_k^2 = \frac{1}{9}(-GW_{n+3}^2 - 9GW_{n+2}^2 - 10GW_{n+1}^2 + 6GW_{n+2}GW_{n+3}$
$+2GW_{n+1}GW_{n+3} + 5GW_3^2 + 9GW_2^2 + 10GW_1^2 - 2GW_4GW_3 - 4GW_2GW_3)$,
$\sum_{k=1}^{n} GW_{k+1}GW_k = \frac{1}{9}(GW_{n+3}^2 + 3GW_{n+2}^2 + GW_{n+1}^2 - 3GW_{n+2}GW_{n+3}$
$+GW_{n+1}GW_{n+3} - 6GW_{n+1}GW_{n+2} + GW_3^2 - 3GW_2^2 - GW_1^2 - GW_4GW_3$
$+4GW_2GW_3 + 6GW_1GW_2)$,
$\sum_{k=1}^{n} GW_{k+2}GW_k = \frac{1}{9}(2GW_{n+3}^2 + 2GW_{n+1}^2 - 3GW_{n+2}GW_{n+3} - 4GW_{n+1}GW_{n+3}$
$+8GW_3^2 - 2GW_1^2 - 5GW_3GW_4 + 8GW_2GW_3 + 9GW_1GW_3)$.

From the last Proposition, we have the following Corollary which gives sum formulas of Gaussian third-order Pell numbers (take $GW_n = GP_n^{(3)}$ with $GP_0^{(3)} = 0, GP_1^{(3)} = 1, GP_2^{(3)} = 2 + i$).

Corollary 6. *For $n \geq 1$, Gaussian third-order Pell numbers have the following properties:*
$\sum_{k=1}^{n} GP_k^{(3)2} = \frac{1}{9}(-GP_{n+3}^{(3)2} - 9GP_{n+2}^{(3)2} - 10GP_{n+1}^{(3)2} + 6GP_{n+2}^{(3)}GP_{n+3}^{(3)} + 2GP_{n+1}^{(3)}GP_{n+3}^{(3)} - 2i)$,
$\sum_{k=1}^{n} GP_{k+1}^{(3)}GP_k^{(3)} = \frac{1}{9}(GP_{n+3}^{(3)2} + 3GP_{n+2}^{(3)2} + GP_{n+1}^{(3)2} - 3GP_{n+2}^{(3)}GP_{n+3}^{(3)} + GP_{n+1}^{(3)}GP_{n+3}^{(3)} - 6GP_{n+1}^{(3)}GP_{n+2}^{(3)} - i)$,
$\sum_{k=1}^{n} GP_{k+2}^{(3)}GP_k^{(3)} = \frac{1}{9}(2GP_{n+3}^{(3)2} + 2GP_{n+1}^{(3)2} - 3GP_{n+2}^{(3)}GP_{n+3}^{(3)} - 4GP_{n+1}^{(3)}GP_{n+3}^{(3)} + (170 + 135i))$.

Taking $GW_n = GQ_n^{(3)}$ with $GQ_0^{(3)} = 3 - i, GQ_1^{(3)} = 2 + 3i, GQ_2^{(3)} = 6 + 2i$ in the last Proposition, we have the following Corollary which presents sum formulas of Gaussian third-order Pell-Lucas numbers.

Corollary 7. *For $n \geq 1$, Gaussian third-order Pell-Lucas numbers have the following properties:*
$\sum_{k=1}^{n} GQ_k^{(3)2} = \frac{1}{9}(-GQ_{n+3}^{(3)2} - 9GQ_{n+2}^{(3)2} - 10GQ_{n+1}^{(3)2} + 6GQ_{n+2}^{(3)}GQ_{n+3}^{(3)} + 2GQ_{n+1}^{(3)}GQ_{n+3}^{(3)} - (81+6i))$,
$\sum_{k=1}^{n} GQ_{k+1}^{(3)}GQ_k^{(3)} = \frac{1}{9}(GQ_{n+3}^{(3)2} + 3GQ_{n+2}^{(3)2} + GQ_{n+1}^{(3)2} - 3GQ_{n+2}^{(3)}GQ_{n+3}^{(3)} + GQ_{n+1}^{(3)}GQ_{n+3}^{(3)} - 6GQ_{n+1}^{(3)}GQ_{n+2}^{(3)} - (54+9i))$,
$\sum_{k=1}^{n} GQ_{k+2}^{(3)}GQ_k^{(3)} = \frac{1}{9}(2GQ_{n+3}^{(3)2} + 2GQ_{n+1}^{(3)2} - 3GQ_{n+2}^{(3)}GQ_{n+3}^{(3)} - 4GQ_{n+1}^{(3)}GQ_{n+3}^{(3)} + (-162+30i))$.

From the last Proposition, we have the following Corollary which gives sum formulas of Gaussian third-order modified Pell numbers (take $GW_n = GE_n^{(3)}$ with $GE_0^{(3)} = -i, GE_1^{(3)} = 1, GE_2^{(3)} = 1+i$).

Corollary 8. *For $n \geq 1$, Gaussian modified third-order modified Pell numbers have the following properties:*
$\sum_{k=1}^{n} GE_k^{(3)2} = \frac{1}{9}(-GE_{n+3}^{(3)2} - 9GE_{n+2}^{(3)2} - 10GE_{n+1}^{(3)2} + 6GE_{n+2}^{(3)}GE_{n+3}^{(3)} + 2GE_{n+1}^{(3)}GE_{n+3}^{(3)} - 2i)$,
$\sum_{k=1}^{n} GE_{k+1}^{(3)}GE_k^{(3)} = \frac{1}{9}(GE_{n+3}^{(3)2} + 3GE_{n+2}^{(3)2} + GE_{n+1}^{(3)2} - 3GE_{n+2}^{(3)}GE_{n+3}^{(3)} + GE_{n+1}^{(3)}GE_{n+3}^{(3)} - 6GE_{n+1}^{(3)}GE_{n+2}^{(3)} + 5i)$,
$\sum_{k=1}^{n} GE_{k+2}^{(3)}GE_k^{(3)} = \frac{1}{9}(2GE_{n+3}^{(3)2} + 2GE_{n+1}^{(3)2} - 3GE_{n+2}^{(3)}GE_{n+3}^{(3)} - 4GE_{n+1}^{(3)}GE_{n+3}^{(3)} + 4i)$.

Taking $r = 0, s = 1, t = 1$ in Theorem1, we obtain the following Proposition.

Proposition 9. *If $r = 0, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*
$\sum_{k=1}^{n} GW_k^2 = -2GW_{n+1}^2 - GW_{n+3}^2 - GW_{n+2}^2 + 2GW_{n+2}GW_{n+3} + 2GW_{n+1}GW_{n+3} + GW_3^2 + GW_2^2 + 2GW_1^2 - 2GW_4GW_3$,
$\sum_{k=1}^{n} GW_{k+1}GW_k = GW_{n+3}^2 + GW_{n+2}^2 + GW_{n+1}^2 - GW_{n+2}GW_{n+3} - GW_{n+1}GW_{n+3} - GW_3^2 - GW_2^2 - GW_1^2 + GW_4GW_3$,
$\sum_{k=1}^{n} GW_{k+2}GW_k = GW_{n+2}GW_{n+3} - GW_3GW_4 + GW_1GW_3$.

From the last Proposition, we have the following Corollary which gives sum formulas of Gaussian Padovan numbers (take $GW_n = GP_n$ with $GP_0 = 1, GP_1 = 1+i, GP_2 = 1+i$).

Corollary 10. *For $n \geq 1$, Gaussian Padovan numbers have the following properties:*
$\sum_{k=1}^{n} GP_k^2 = -GP_{n+3}^2 - GP_{n+2}^2 - 2GP_{n+1}^2 + 2GP_{n+2}GP_{n+3} + 2GP_{n+1}GP_{n+3} - (1+2i)$,
$\sum_{k=1}^{n} GP_{k+1}GP_k = GP_{n+3}^2 + GP_{n+2}^2 + GP_{n+1}^2 - GP_{n+2}GP_{n+3} - GP_{n+1}GP_{n+3} - (1+2i)$,
$\sum_{k=1}^{n} GP_{k+2}GP_k = GP_{n+2}GP_{n+3} - (1+3i)$.

Taking $GW_n = GE_n$ with $GE_0 = 3-i, GE_1 = 3i, GE_2 = 2$ in the last Proposition, we have the following Corollary which presents sum formulas of Gaussian Perrin numbers.

Corollary 11. *For $n \geq 1$, Gaussian Perrin numbers have the following properties:*

$$\begin{aligned}
\sum_{k=1}^{n} GE_k^2 &= -GE_{n+3}^2 - GE_{n+2}^2 - 2GE_{n+1}^2 + 2GE_{n+2}GE_{n+3} \\
&\quad +2GE_{n+1}GE_{n+3} - (9+14i), \\
\sum_{k=1}^{n} GE_{k+1}GE_k &= GE_{n+3}^2 + GE_{n+2}^2 + GE_{n+1}^2 - GE_{n+2}GE_{n+3} \\
&\quad -GE_{n+1}GE_{n+3} + i, \\
\sum_{k=1}^{n} GE_{k+2}GE_k &= GE_{n+2}GE_{n+3} - (6+4i).
\end{aligned}$$

From the last Proposition, we have the following Corollary which gives sum formulas of Gaussian Padovan-Perrin numbers (take $GW_n = GS_n$ with $GS_0 = i, GS_1 = 0, GS_2 = 1$).

Corollary 12. *For $n \geq 1$, Gaussian Padovan-Perrin numbers have the following properties:*

$$\begin{aligned}
\sum_{k=1}^{n} GS_k^2 &= -GS_{n+3}^2 - GS_{n+2}^2 - 2GS_{n+1}^2 + 2GS_{n+2}GS_{n+3} \\
&\quad +2GS_{n+1}GS_{n+3} - 2i, \\
\sum_{k=1}^{n} GS_{k+1}GS_k &= GS_{n+3}^2 + GS_{n+2}^2 + GS_{n+1}^2 - GS_{n+2}GS_{n+3} \\
&\quad -GS_{n+1}GS_{n+3} + i, \\
\sum_{k=1}^{n} GS_{k+2}GS_k &= GS_{n+2}GS_{n+3} - i.
\end{aligned}$$

Taking $r = 0, s = 1, t = 2$ in Theorem 1, we obtain the following Proposition.

Proposition 13. *If $r = 0, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:*

$\sum_{k=1}^{n} GW_k^2 = \frac{1}{2}(GW_{n+3}^2 + GW_{n+2}^2 + 2GW_{n+1}^2 - GW_{n+2}GW_{n+3} - 2GW_{n+1}GW_{n+3} - GW_3^2 - GW_2^2 - 2GW_1^2 + GW_4GW_3)$,

$\sum_{k=1}^{n} GW_{k+1}GW_k = \frac{1}{4}(-GW_{n+3}^2 - GW_{n+2}^2 - 4GW_{n+1}^2 + 2GW_{n+2}GW_{n+3} + 4GW_{n+1}GW_{n+3} + GW_3^2 + GW_2^2 + 4GW_1^2 - 2GW_4GW_3)$,

$\sum_{k=1}^{n} GW_{k+2}GW_k = \frac{1}{2}(GW_{n+2}GW_{n+3} + 2GW_1GW_3 - GW_3GW_4)$.

From the last Proposition, we have the following Corollary which gives sum formulas of Gaussian Jacobsthal-Padovan numbers (take $GW_n = GQ_n$ with $GQ_0 = 1, GQ_1 = 1+i, GQ_2 = 1+i$).

Corollary 14. *For $n \geq 1$, Gaussian Jacobsthal-Padovan numbers have the following properties:*

$\sum_{k=1}^{n} GQ_k^2 = \frac{1}{2}(GQ_{n+3}^2 + GQ_{n+2}^2 + 2GQ_{n+1}^2 - GQ_{n+2}GQ_{n+3} - 2GQ_{n+1}GQ_{n+3} - 2)$,

$\sum_{k=1}^{n} GQ_{k+1}GQ_k = \frac{1}{4}(-GQ_{n+3}^2 - GQ_{n+2}^2 - 4GQ_{n+1}^2 + 2GQ_{n+2}GQ_{n+3} + 4GQ_{n+1}GQ_{n+3} - (4+8i))$,

$\sum_{k=1}^{n} GQ_{k+2}GQ_k = \frac{1}{2}(GQ_{n+2}GQ_{n+3} - (2+4i))$.

Taking $GW_n = GD_n$ with $GD_0 = 3 - \frac{1}{2}i, GD_1 = 3i, GD_2 = 2$ in the last Proposition, we have the following Corollary which presents sum formulas of Gaussian Jacobsthal-Perrin numbers.

Corollary 15. *For $n \geq 1$, Gaussian Jacobsthal-Perrin numbers have the following properties:*
$\sum_{k=1}^{n} QD_k^2 = \frac{1}{2}(QD_{n+3}^2 + QD_{n+2}^2 + 2QD_{n+1}^2 - QD_{n+2}QD_{n+3} - 2QD_{n+1}QD_{n+3} + (-18+16i))$,
$\sum_{k=1}^{n} QD_{k+1}QD_k = \frac{1}{4}(-QD_{n+3}^2 - QD_{n+2}^2 - 4QD_{n+1}^2 + 2QD_{n+2}QD_{n+3} + 4QD_{n+1}QD_{n+3} - 56i)$,
$\sum_{k=1}^{n} QD_{k+2}QD_k = \frac{1}{2}(QD_{n+2}QD_{n+3} - (12+4i))$.

Taking $r = 1, s = 0, t = 1$ in Theorem 1, we obtain the following Proposition.

Proposition 16. *If $r = 1, s = 0, t = 1$ then for $n \geq 1$ we have the following formulas:*
$\sum_{k=1}^{n} GW_k^2 = \frac{1}{3}(-GW_{n+3}^2 - 4GW_{n+2}^2 - 4GW_{n+1}^2 + 4GW_{n+2}GW_{n+3} + 2GW_{n+1}GW_{n+3} + 2GW_{n+1}GW_{n+2} + 3GW_3^2 + 4GW_2^2 + 4GW_1^2 - 2GW_4GW_3 - 4GW_2GW_3 - 2GW_1GW_2)$,
$\sum_{k=1}^{n} GW_{k+1}GW_k = \frac{1}{3}(GW_{n+3}^2 + GW_{n+2}^2 + GW_{n+1}^2 - GW_{n+2}GW_{n+3} + GW_{n+1}GW_{n+3} - 2GW_{n+1}GW_{n+2} - GW_2^2 - GW_1^2 - GW_3GW_4 + GW_3GW_2 + 2GW_1GW_2)$,
$\sum_{k=1}^{n} GW_{k+2}GW_k = \frac{1}{3}(2GW_{n+3}^2 + 2GW_{n+2}^2 + 2GW_{n+1}^2 - 2GW_{n+2}GW_{n+3} - GW_{n+1}GW_{n+2} - GW_{n+1}GW_{n+3} - 2GW_2^2 - 2GW_1^2 - 2GW_3GW_4 + 2GW_3GW_2 + 3GW_3GW_1 + GW_1GW_2)$.

From the last Proposition, we have the following Corollary which gives sum formulas of Gaussian Narayana numbers (take $GW_n = GN_n$ with $GN_0 = 0, GN_1 = 1, GN_2 = 1+i$).

Corollary 17. *For $n \geq 1$, Gaussian Narayana numbers have the following properties:*
$\sum_{k=1}^{n} GN_k^2 = \frac{1}{3}(-GN_{n+3}^2 - 4GN_{n+2}^2 - 4GN_{n+1}^2 + 4GN_{n+2}GN_{n+3} + 2GN_{n+1}GN_{n+3} + 2GN_{n+1}GN_{n+2} - 2i)$,
$\sum_{k=1}^{n} GN_{k+1}GN_k = \frac{1}{3}(GN_{n+3}^2 + GN_{n+2}^2 + GN_{n+1}^2 - GN_{n+2}GN_{n+3} + GN_{n+1}GN_{n+3} - 2GN_{n+1}GN_{n+2} - i)$,
$\sum_{k=1}^{n} GN_{k+2}GN_k = \frac{1}{3}(2GN_{n+3}^2 + 2GN_{n+2}^2 + 2GN_{n+1}^2 - 2GN_{n+2}GN_{n+3} - GN_{n+1}GN_{n+2} - GN_{n+1}GN_{n+3} - 2i)$.

References

[1] M. Asci, and E. Gurel, Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials, Notes on Number Theory and Discrete Mathematics, Vol.19(1), 25-36, 2013.
[2] G. Berzsenyi, Gaussian Fibonacci Numbers, Fibonacci Quart., Vol.15(3), 233-236, 1977.
[3] P. Catarino, and H. Campos, A note on Gaussian Modified Pell numbers, Journal of Information & Optimization Sciences, Vol. 39(6), 1363-1371, 2018.
[4] J. B. Fraleigh, A First Course In Abstract Algebra, (2nd ed.), Addison-Wesley, Reading, ISBN 0-201-01984-1, 1976.
[5] E, Gurel, k-Order Gaussian Fibonacci and k-Order Gaussian Lucas Recurrence Relations, Ph.D Thesis, Pamukkale University Institute of Science Mathematics, Denizli, Turkey (2015).
[6] S. Halici, and S. Öz, On some Gaussian Pell and Pell-Lucas numbers, Ordu University Science and Technology Journal, Vol.6(1), 8-18, 2016.
[7] S. Halici, and S. Öz, On Gaussian Pell Polynomials and Their Some Properties, Palestine Journal of Mathematics, Vol 7(1), 251-256, 2018.
[8] C. J. Harman, Complex Fibonacci Numbers, Fibonacci Quart., Vol.19(1), 82-86, 1981.
[9] A. F. Horadam, Complex Fibonacci Numbers and Fibonacci quaternions, Amer. Math. Monthly 70, 289-291, 1963.

[10] J. H. Jordan, Gaussian Fibonacci and Lucas Numbers, Fibonacci Quart., Vol.3, 315-318, 1965.
[11] S. Pethe, and A. F. Horadam, Generalised Gaussian Fibonacci numbers, Bull. Austral. Math. Soc., Vol.33, 37-48, 1986.
[12] S. Pethe, and A. F. Horadam, Generalised Gaussian Lucas Primordial numbers, Fibonacci Quart., Vol.26(1), 20-30, 1988.
[13] S. Pethe, Some Identities for Tribonacci Sequences, The Fibonacci Quart., Vol.26(2), 144-151, 1988.
[14] Y. Soykan, On Summing Formulas For Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers, Advances in Research, Vol.20(2), 1-15, 2019.
[15] Y. Soykan, E. Taşdemir, İ. Okumuş, and M. Göcen, Gaussian Generalized Tribonacci Numbers, Journal of Progressive Research in Mathematics(JPRM), Vol.14(2), 2373-2387, 2018.
[16] Y. Soykan, Gaussian Generalized Tetranacci Numbers, Journal of Advances in Mathematics and Computer Science, Vol.31(3), 1-21, Article no.JAMCS.48063, 2019.
[17] Y. Soykan, On Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers, Preprints 2019, 2019060110 (doi: 10.20944/preprints201906.0110.v1).
[18] D. Taşcı, and H. Acar, Gaussian Tetranacci Numbers, Communications in Mathematics ans Applications, Vol.8(3), 379-386, 2017.
[19] D. Taşcı, Gaussian Padovan and Gaussian Pell-Padovan Sequences, Commun. Fac. Sci. Ank. Ser. A1 Math. Stat., Vol.67(2), 82-88, 2018.
[20] T. Yagmur, and N. Karaaslan, Gaussian Modified Pell Sequence and Gaussian Modified Pell Polynomial Sequence, Aksaray University Journal of Science and Engineering, Vol.2(1), 63-72, 2018.

Yüksel Soykan, Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey
E-mail address: yuksel_soykan@hotmail.com

Erkan Taşdemir, Pinarhisar Vocational School, Kirklareli University, 39300, Kirklareli, Turkey
E-mail address: erkantasdemir@hotmail.com

Melih Göcen, Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey
E-mail address: gocenm@hotmail.com