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ASYMPTOTIC BEHAVIOR FOR A STRUCTURE WITH INTERFACIAL SLIP AND ONE DISCONTINUOUS LOCAL INTERNAL KELVIN-VOIGT DAMPING

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Abstract. In the present paper, we study the stabilization of laminated beams with one discontinuous local internal viscoelastic damping of Kelvin-Voigt type. First, we prove the strong stability of the system using the Arendt-Batty Theorem. Finally, we present the polynomial stability by using Borichev-Tomilov's result of optimality.

1. Introduction

In this work we study the stability for a structure with interfacial slip and frictional damping, given by,

$$\begin{cases} \rho_1\, u_{tt} + k\,(\psi - u_x)_x - k\, d(x)\,(s_{xt} - u_t) = 0, \\ \rho_2\; (s-\psi)_{tt} - b\,(s-\psi)_{xx} - k\,(\psi - u_x) + \beta\,(s-\psi)_t = 0, \\ \rho_2\, s_{tt} - b\, s_{xx} + 3\,k\,(\psi - u_x) - 3\,k\,[d(x)\,(s_{xt} - u_t)]_x + 4\,\delta\, s_t = 0, \end{cases} \tag{1.1}$$

where $(x,\, t) \in (0,\, L) \times (0,\, +\infty)$. The coefficients ρ_1, ρ_2, k, b, γ and β are positive and denote the density of the beam, the mass moment of inertia, the shear of stiffness, the flexural rigidity, the effective damping of the rotational angle and the adhesive damping parameter respectively. The function $u = u(x,\, t)$ denotes the transverse displacement, $\psi = \psi(x,\, t)$ represents the rotation angle, and $s = s(x,\, t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x respectively.

We consider the following boundary conditions:

$$u(0,\, L) = u(L,\, t) = \psi(0,\, t) = \psi(L,\, t) = s(0,\, t) = s(L,\, t) = 0 \tag{1.2}$$

and initial data

$$\begin{cases} (u(x,\,0),\, \psi(x,\,0),\, s(x,\,0)) = (u_0(x),\, \psi_0(x),\, s_0(x))\,, \\ (u_t(x,\,0),\, \psi_t(x,\,0),\, s_t(x,\,0)) = (u_1(x),\, \psi_1(x),\, s_1(x))\,. \end{cases} \tag{1.3}$$

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We suppose that there exists $0 < \alpha < \beta < L$ and a positive constant d_0 such that

$$d(x) = \begin{cases} d_0 & \text{if} \quad x \in (\alpha,\, \beta), \\ 0 & \text{if} \quad x \in (0,\, \alpha) \cup (\beta,\, L). \end{cases} \tag{1.4}$$

The model (1.1) is closely related with the Timoshenko's theory. Timoshenko's theory started in 1921 [25, 26] and since then, has been extensively studied by several authors over different points of view. Most of them studied the global well-posedness, asymptotic behavior among other properties (see [1, 4, 9, 10, 11] and references therein).

Hansen in 1994 (see [12, 13] for model description) derived from Timoshenko's theory the model (1.1), that describes a structure of two identical beams of uniform thickness with an adhesive layer (of negligible thickness and mass) bonding the two adjoining surfaces in such a way that a slip is allowed while they are continuously in contact with each other. The model (1.1) is called laminated beams and has gained a lot of interest in recent years. We mention for instance [2, 6, 7, 8, 15, 16, 17, 19, 20, 22, 23, 24] and references therein.

In this work, we have obtained the polynomial stability for a structure with interfacial slip with one discontinuous local internal Kelvin-Voigt damping. In order to obtain the results we will use the theorem given by Arendt-Batty [3] together with a result given by Borichev-Tomilov [5]. Indeed, our main results is:

Theorem 1.1. *There is a constant $C > 0$ such that for every $U_0 \in \mathcal{D}(\mathcal{A})$, we have*

$$E(t) \leq \frac{C}{t}\, \|W_0\|_{\mathcal{D}(\mathcal{A})}, \quad t > 0. \tag{1.5}$$

This paper is organized as follows. In Section 2 we recall some auxiliary results. Section 3 by semigroup theory of linear operators we obtain an existence theorem of solutions of system (2.7)-(2.9). In Section 4, we show the strong stability by using Arendt-Batty's theorem. In Section 5, we present the polynomial stability of the corresponding semigroup using Borichev-Tomilov's result of optimality.

2. Preliminary

Throughout this paper, C is a generic constant, not necessarily the same at each occasion (it will change from line to line) and depends on the indicated quantities. In the following lemma we prove the dissipative properties of the system (1.1) in the sense that its energy is non-decreasing with respect to time, that is,

Lemma 2.1. *(Energy of the system) For every solution $(u,\ \psi,\ s)$ of the system (1.1), the total energy $\mathcal{E} : \mathbb{R}^+ \to \mathbb{R}^+$ defined at time t by*

$$\mathcal{E}(t) = \frac{1}{2}\Big[3\,\rho_1\,\|u_t\|^2_{L^2(0,\,L)} + 3\,k\,\|\psi - u_x\|^2_{L^2(0,\,L)} + \rho_2\,\|s_t\|^2_{L^2(0,\,L)} + b\,\|s_x\|^2_{L^2(0,\,L)} + 3\,\rho_2\,\|(s-\psi)_t\|^2_{L^2(0,\,L)} + 3\,b\,\|(s-\psi)_x\|^2_{L^2(0,\,L)}\Big] \tag{2.1}$$

satisfies

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) &= -\,3\,k\int_0^L d(x)\,|s_{xt}-u_t|^2\;dx - 3\beta\int_0^L |(s-\psi)_t|^2\;dx - 4\,\delta\int_0^L |s_t|^2\\ &= -3\,k\,d_0\int_\alpha^\beta |s_{xt}-u_t|^2\;dx - \;3\beta\int_0^L |(s-\psi)_t|^2\;dx - 4\,\delta\int_0^L |s_t|^2\;dx.\end{aligned} \tag{2.2}$$

Proof. Let $(u,\,u_t,\,\psi,\,\psi_t,\,s,\,s_t)$ be a regular solution of system (1.1). Multiplying $(1.1)_1$ by $3\,u_t$ we have

$$\frac{1}{2}\,\frac{d}{dt}\,3\,\rho_1\int_0^L u_t^2\;dx + 3\,k\int_0^L (\psi-u_x)_x\;u_t\;dx - 3\,k\int_0^L d(x)\,(s_{xt}-u_t)\,u_t\;dx = 0.$$

Using integration by parts and (1.2) it follows that

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,3\,\rho_1\int_0^L u_t^2\;dx &- 3\,k\int_0^L (\psi-u_x)\;u_{xt}\;dx\\ &- 3\,k\int_0^L d(x)\,s_{xt}\,u_t\;dx + 3\,k\int_0^L d(x)\,u_t^2\;dx = 0.\end{aligned}$$

The second term in the last equation can be written as

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,3\,\rho_1\int_0^L u_t^2\;dx &- 3\,k\int_0^L (\psi-u_x)\;(u_x-\psi+\psi)_t\;dx\\ &- \;3\,k\int_0^L d(x)\,s_{xt}\,u_t\;dx + 3\,k\int_0^L d(x)\,u_t^2\;dx = 0.\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\,3\,\rho_1\int_0^L u_t^2\;dx + 3\,k\int_0^L (\psi-u_x)\;(\psi-u_x)_t\;dx\\ &\quad - 3\,k\int_0^L (\psi-u_x)\;\psi_t\;dx - 3\,k\int_0^L d(x)\,s_{xt}\,u_t\;dx + 3\,k\int_0^L d(x)\,u_t^2\;dx = 0.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[3\,\rho_1\|u_t\|^2_{L^2(0,\,L)} + 3\,k\,\|\psi-u_x\|^2_{L^2(0,\,L)}\right]\\ &-\;3\,k\int_0^L (\psi-u_x)\;\psi_t\;dx - 3\,k\int_0^L d(x)\,s_{xt}\,u_t\;dx + 3\,k\int_0^L d(x)\,u_t^2\;dx = 0.\end{aligned} \tag{2.3}$$

Now, multiplying $(1.1)_3$ by s_t and integrating over $(0,\,L)$ yields

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\,\rho_2\int_0^L s_t^2\;dx - b\int_0^L s_{xx}\,s_t\;dx\\ &+\;3\,k\int_0^L (\psi-u_x)\,s_t\;dx - 3\,k\int_0^L [d(x)\,(s_{xt}-u_t)]_x\,s_t + 4\,\delta\int_0^L |s_t|^2\;dx = 0.\end{aligned}$$

Integrating by parts and using (1.2) it follows that

$$\begin{aligned}&\frac{1}{2}\,\frac{d}{dt}\,\rho_2\int_0^L s_t^2\;dx + b\int_0^L s_x\,s_{xt}\;dx\\ &\quad+\;3\,k\int_0^L (\psi-u_x)\,s_t\;dx + 3\,k\int_0^L d(x)\,(s_{xt}-u_t)\,s_{xt}\;dx + 4\,\delta\int_0^L |s_t|^2\;dx = 0.\end{aligned}$$

Then

$$\frac{1}{2}\,\frac{d}{dt}\left[\rho_2\,\|s_t\|^2_{L^2(0,\,L)}+b\|s_x\|^2_{L^2(0,\,L)}\right]+3\,k\int_0^L(\psi-u_x)\,s_t\;dx$$
$$+\,3\,k\int_0^L d(x)\,s_{xt}^2\;dx-3\,k\int_0^L d(x)\,u_t\,s_{xt}\;dx+4\,\delta\int_0^L|s_t|^2\;dx=0. \quad (2.4)$$

Finally, multiplying $(1.1)_2$ by $3\,(s-\psi)_t$ and integrating over $(0,\,L)$ we have

$$3\rho_2\int_0^L(s-\psi)_{tt}\,(s-\psi)_t\;dx-3\,b\int_0^L(s-\psi)_{xx}\;(s-\psi)_t\;dx$$
$$-\;3\,k\int_0^L(\psi-u_x)\,(s-\psi)_t\;dx+3\beta\int_0^L|(s-\psi)_t|^2\,dx=0.$$

Integrating by parts and using (1.2) it follows that

$$\frac{1}{2}\,\frac{d}{dt}\,3\,\rho_2\int_0^L|\,(s-\psi)_t\,|^2\;dx+\frac{1}{2}\,\frac{d}{dt}\,3\,b\int_0^L|\,(s-\psi)_x\,|^2\;dx$$
$$-\;3\,k\int_0^L(\psi-u_x)\,(s-\psi)_t\;dx+3\beta\int_0^L|(s-\psi)_t|^2\,dx=0.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\left[3\,\rho_2\,\|(s-\psi)_t\|^2_{L^2(0,\,L)}+3\,b\,\|(s-\psi)_x\|^2_{L^2(0,\,L)}\right]$$
$$-\;3\,k\int_0^L(\psi-u_x)\,(s-\psi)_t\;dx+3\beta\int_0^L|(s-\psi)_t|^2\,dx=0. \quad (2.5)$$

Adding (2.3), (2.4) and (2.5) the Lemma 2.1 follows. □

For convenience, from now and on, we introduce a new variable z given by

$$z=s-\psi\quad\text{(the effective rotation angle)}\iff\quad\psi=s-z, \quad (2.6)$$

then (2.5) becomes

$$\frac{1}{2}\frac{d}{dt}\left[3\,\rho_2\,\|z_t\|^2_{L^2(0,\,L)}+3\,b\,\|z_x\|^2_{L^2(0,\,L)}\right]$$
$$-\;3\,k\int_0^L(s-z-u_x)\,z_t\;dx+3\beta\int_0^L|z_t|^2\,dx=0.$$

Performing the change of variable (2.6), system (1.1) leads to

$$\begin{cases}\rho_1\,u_{tt}+k\,(s-z-u_x)_x-k\,d(x)\,(s_{xt}-u_t)=0,\\ \rho_2\,z_{tt}-b\,z_{xx}-k\,(s-z-u_x)+\beta\,z_t=0,\\ \rho_2\,s_{tt}-b\,s_{xx}+3\,k\,(s-z-u_x)-3\,k\,[d(x)\,(s_{xt}-u_t)]_x+4\,\gamma\,s+4\,\delta\,s_t=0,\end{cases} \quad (2.7)$$

with the boundary conditions

$$u(0,\,L)=u(L,\,t)=z(0,\,t)=z(L,\,t)=s(0,\,t)=s(L,\,t)=0 \quad (2.8)$$

and initial data

$$\begin{cases}(u(x,\,0),\,z(x,\,0),\,s(x,\,0))=(u_0(x),\,z_0(x),\,s_0(x))\,,\\ (u_t(x,\,0),\,z_t(x,\,0),\,s_t(x,\,0))=(u_1(x),\,z_1(x),\,s_1(x))\,.\end{cases} \quad (2.9)$$

We present the following definition.

Definition 2.2. [18] *Assume that $\mathcal{A}$ is the generator of C_0-semigroup of contractions $\left(e^{t\,\mathcal{A}}\right)_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. The C_0-semigroup $\left(e^{t\,\mathcal{A}}\right)_{t\geq 0}$ is said to be*
a) Strongly stable if

$$\lim_{t\to+\infty}\|e^{t\,\mathcal{A}}\,x_0\|_{\mathcal{H}}=0,\quad \forall\,x_0\in\mathcal{H}. \tag{2.10}$$

b) Exponentially (or uniformly) stable if there exists two positive constant M and ε such that

$$\|e^{t\,\mathcal{A}}\,x_0\|_{\mathcal{H}}\leq M\,e^{-\,\varepsilon\,t}\,\|x_0\|_{\mathcal{H}},\quad \forall\,t>0,\quad \forall\,x_0\in\mathcal{H}. \tag{2.11}$$

c) Polynomially stable if there exists two positive constant C and α such that

$$\|e^{t\,\mathcal{A}}\,x_0\|_{\mathcal{H}}\leq C\,t^{-\,\alpha}\,\|\mathcal{A}x_0\|_{\mathcal{H}},\quad \forall\,t>0,\quad \forall\,x_0\in\mathcal{D}(\mathcal{A}). \tag{2.12}$$

The following results will use some time from now on and are fundamental to the proof of our theorems of stability.

Theorem 2.3. [3] *Assume that $\mathcal{A}$ is the generator of a C_0-semigroup of contractions $(e^{t\,\mathcal{A}})_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A})\cap i\,\mathbb{R}$ is countable, where $\sigma(\mathcal{A})$ denote the spectrum of $\mathcal{A}$, then the C_0-semigroup $(e^{t\,\mathcal{A}})_{t\geq 0}$ is strongly stable.*

Theorem 2.4. [5] *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(e^{t\,\mathcal{A}})_{t\geq 0}$ on $\mathcal{H}$. If $i\,\mathbb{R}\subset\varrho(\mathcal{H})$, then for a fixed $\ell>0$ the following conditions are equivalent*

$$\sup_{\lambda\in\mathbb{R}}\left\|(i\,\lambda\,I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}=\mathcal{O}(|\lambda|^{\ell}),$$

$$\left\|e^{t\,\mathcal{A}}\,U_0\right\|^2_{\mathcal{H}}\leq\frac{C}{t^{2/\ell}}\,\|U_0\|^2_{\mathcal{D}(\mathcal{A})},\quad \forall\,t>0,\quad U_0\in\mathcal{D}(\mathcal{A}),\quad \text{for some}\quad C>0.$$

3. Setting of the Semigroup

In this section, we use results of the semigroup theory of linear operators to obtain an existence theorem of solutions of system (2.7)-(2.9) (see [21]). We introduce the Hilbert space

$$\mathcal{H}=[H^1_0(0,\,L)\times L^2(0,\,L)]^3,$$

equipped with the inner product ($u_t=U$, $\psi_t=\Psi$, $s_t=S$ with $z=s-\psi$ and $Z=S-\Psi$)

$$\begin{aligned}\langle W,\,\widetilde{W}\rangle_{\mathcal{H}}=&3\rho_1\int_0^L U\,\widetilde{U}\;dx+3\rho_2\int_0^L Z\,\widetilde{Z}\;dx+\rho_2\int_0^L S\,\widetilde{S}\;dx\\&+\;b\int_0^L s_x\,\widetilde{s}_x\;dx+3k\int_0^L(s-z-u_x)\,(\widetilde{s}-\widetilde{z}-\widetilde{u}_x)\;dx+3b\int_0^L z_x\,\widetilde{z}_x\;dx,\end{aligned} \tag{3.1}$$

where $W=(u,\,u_t=U;\,z,\,z_t=Z;\,s,\,s_t=S)\in\mathcal{H}$, $\widetilde{W}=(\widetilde{u},\,\widetilde{U};\,\widetilde{z},\,\widetilde{Z};\,\widetilde{s},\,\widetilde{S})$ and norm

$$\begin{aligned}\|W\|^2_{\mathcal{H}}=&3\rho_1\,\|u_t\|^2_{L^2(0,\,L)}+3\rho_2\,\|z_t\|^2_{L^2(0,\,L)}+\rho_2\,\|s_t\|^2_{L^2(0,\,L)}\\&+\;b\|s_x\|^2_{L^2(0,\,L)}+3k\,\|s-z-u_x\|^2_{L^2(0,\,L)}+3b\,\|z_x\|^2_{L^2(0,\,L)}.\end{aligned} \tag{3.2}$$

Now, we rewrite the system (2.7) as an abstract problem on the Hilbert space $\mathcal{H}$ given by

$$\frac{d}{dt}W(t) = \mathcal{A}W(t), \tag{3.3}$$
$$W(0) = W_0 = (u_0,\, u_1,\, z_0,\, z_1,\, s_0\,,\, s_1), \quad \forall\, t > 0. \tag{3.4}$$

The operator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$$

is given by

$$\mathcal{A}\begin{pmatrix} u \\ U \\ z \\ Z \\ s \\ S \end{pmatrix} = \begin{pmatrix} U \\ \frac{1}{\rho_1}\left[-k\,(s - z - u_x)_x + k\,d(x)\,(S_x - U)\right] \\ Z \\ \frac{1}{\rho_2}\left[b\,z_{xx} + k\,(s - z - u_x) - \beta\, Z\right] \\ S \\ \frac{1}{\rho_2}\left(b\,s_{xx} - 3\,k\,(s - z - u_x) + 3\,k\,[d(x)\,(S_x - U)]_x - 4\,\delta\, S\right) \end{pmatrix} \tag{3.5}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \big\{W \in \mathcal{H} :\ u,\ z,\ s \in H^2(0,\,L) \cap H_0^1(0,\,L),\ U,\ Z,\ S \in H_0^1(0,\,L),$$
$$[d(x)\,(s_x - u)]_x \in L^2(0,\,L)\big\}\,.$$

To prove that $\mathcal{A}$ is the infinitesimal generator of a C_0-contraction semigroup, we consider the two following lemmas.

Lemma 3.1. *The operator $\mathcal{A}$ is a dissipative.*

Proof.

$$
\begin{aligned}
&\langle \mathcal{A}W, W\rangle_{\mathcal{H}} \\
&\quad = 3\rho_1 \int_0^L \frac{1}{\rho_1}\left[-k\,(s-z-u_x)_x + k\,d(x)\,(S_x-U)\right]\overline{U}\;dx \\
&\qquad + 3\rho_2 \int_0^L \frac{1}{\rho_2}\left[b\,z_{xx} + k\,(s-z-u_x) - \beta Z\right]\overline{Z}\;dx \\
&\qquad + \rho_2 \int_0^L \frac{1}{\rho_2}\left[b\,s_{xx} - 3k\,(s-z-u_x) + 3\,k\,[d(x)\,(S_x-U)]_x - 4\,\delta\,S\right]\overline{S}\;dx \\
&\qquad + 3k\int_0^L (S-Z-U_x)\,(\overline{s}-\overline{z}-\overline{u}_x)\;dx + 3\,k\int_0^L Z_x\,\overline{z}_x\;dx + b\int_0^L S_x\,\overline{s}_x\;dx \\
&\quad = -\,3k\int_0^L (s-z-u_x)_x\,\overline{U}\;dx + 3k\int_0^L d(x)\,S_x\,\overline{U}\;dx - 3k\int_0^L d(x)\,|U|^2\;dx \\
&\qquad + 3b\int_0^L z_{xx}\,\overline{Z}\;dx + 3k\int_0^L (s-z-u_x)\,\overline{Z}\;dx - \beta\int_0^L |Z|^2\;dx - 4\delta\int_0^L |S|^2 \\
&\qquad + b\int_0^L s_{xx}\,\overline{S}\;dx - 3k\int_0^L (s-z-u_x)\,\overline{S}\;dx + 3k\int_0^L [d(x)\,(S_x-U)]_x\,\overline{S}\;dx \\
&\qquad + b\int_0^L S_x\,\overline{s}_x\;dx + 3k\int_0^L (S-Z-U_x)\,(\overline{s}-\overline{z}-\overline{u}_x)\;dx + 3b\int_0^L Z_x\,\overline{z}_x\;dx.
\end{aligned}
$$

Using (2.8) and straightforward calculations we have

$$
\begin{aligned}
&\langle \mathcal{A}W, W\rangle_{\mathcal{H}} \\
&\quad = 3k\int_0^L \left[(S-Z-U_x)\,(\overline{s}-\overline{z}-\overline{u}_x) - \left(\overline{S-Z-U_x)\,(\overline{s}-\overline{z}-\overline{u}_x}\right)\right]\;dx \\
&\qquad + \int_0^L \left[Z_x\,\overline{z}_x - \overline{Z_x\,\overline{z}_x}\right]\;dx + b\int_0^L \left[S_x\,\overline{s}_x - \overline{S_x\,\overline{s}_x}\right]\;dx\;dx \\
&\qquad - \beta\int_0^L |Z|^2\;dx - 4\delta\int_0^L |S|^2\;dx - 3k\int_0^L d(x)\,|S_x-U|^2\;dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
\langle \mathcal{A}W,\,W\rangle_{\mathcal{H}} &= 6\,i\,k\,Im\int_0^L (S-Z-U_x)\,(\overline{s}-\overline{z}-\overline{u}_x)\;dx \\
&\quad + 2\,i\,Im\int_0^L Z_x\,\overline{z}_x\;dx + 2\,i\,b\,Im\int_0^L S_x\,\overline{s}_x\;dx \\
&\quad - \beta\int_0^L |Z|^2\;dx - 4\,\delta\int_0^L |S|^2\;dx - 3\,k\int_0^L d(x)\,|S_x-U|^2\;dx.
\end{aligned}
$$

Taking the real part we obtain

$$
\begin{aligned}
Re\langle \mathcal{A}W,\,W\rangle_{\mathcal{H}} &= -\beta\int_0^L |Z|^2\;dx - 4\delta\int_0^L |S|^2\;dx - 3k\int_0^L d(x)\,|S_x-U|^2\;dx \\
&= -\beta\int_0^L |Z|^2\;dx - 4\delta\int_0^L |S|^2\;dx - 3k\,d_0\int_\alpha^\beta |S_x-U|^2\;dx \le 0.
\end{aligned}
\tag{3.6}
$$

It follows that $\mathcal{A}$ is a dissipative operator. □

Lemma 3.2. *The operator $\mathcal{A}$ is Maximal.*

Proof. Let $F = (f_1,\, f_2,\, f_3,\, f_4,\, f_5,\, f_6) \in \mathcal{H}$, we must show that there exists a unique $W = (u,\, u_t,\, z,\, z_t,\, s,\, s_t)^T \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}W = F$, that is,

$$\begin{cases} u_t = f_1 \quad \text{in} \quad H_0^1(0,\, L),\\ -\, k\,(s - z - u_x)_x + k\, d(x)\,(s_{xt} - u_t) = \rho_1\, f_2 \quad \text{in} \quad L^2(0,\, L),\\ z_t = f_3 \text{ in} \quad H_0^1(0,\, L),\\ b\, z_{xx} + k\,(s - z - u_x) - \beta\, z_t = \rho_2\, f_4 \quad \text{in} \quad L^2(0,\, L),\\ s_t = f_5 \quad \text{in} \quad H_0^1(0,\, L),\\ b\, s_{xx} - 3\,k\,(s - z - u_x) + 3\,k\,[d(x)\,(s_{xt} - u_t)]_x - 4\,\delta\, s_t = \rho_2\, f_6 \quad \text{in} \quad L^2(0,\, L), \end{cases} \tag{3.7}$$

with the boundary conditions

$$u(0,\, t) = u(L,\, t) = z(0,\, t) = z(L,\, t) = s(0,\, t) = s(L,\, t) = 0. \tag{3.8}$$

Replacing $(3.7)_{1,\,3,\,5}$ into $(3.7)_{2,\,4,\,6}$ respectively we have

$$\begin{cases} -\, k\,(s - z - u_x)_x = \rho_1\, f_2 - k\, d(x)\,(f_{5x} - f_1) \quad \text{in} \quad L^2(0,\, L),\\ b\, z_{xx} + k\,(s - z - u_x) = \rho_2\, f_4 + \beta\, f_3 \quad \text{in} \quad L^2(0,\, L),\\ b\, s_{xx} - 3\,k\,(s - z - u_x) + 3\,k\,[d(x)\,(f_{5x} - f_1)]_x = \rho_2\, f_6 + 4\,\delta\, f_5 \quad \text{in} \quad L^2(0,\, L). \end{cases} \tag{3.9}$$

Let $(\Phi_1,\, \Phi_2,\, \Phi_3) \in [H_0^1(0,\, L)]^3$. Then multiplying $(3.9)_{1,\,2,\,3}$ by $\overline{\Phi}_1$, $\overline{\Phi}_2$ and $\overline{\Phi}_3$ respectively, integrating over $(0,\, L)$, using (3.8) and performing straightforward calculations we obtain

$$\begin{aligned} &k \int_0^L (s - z - u_x)\,\overline{\Phi}_{1x}\; dx = \rho_1 \int_0^L f_2\,\overline{\Phi}_1\; dx - k \int_0^L d(x)\,(f_{5x} - f_1)\,\overline{\Phi}_1\; dx,\\ &-\; b \int_0^L z_x\,\overline{\Phi}_{2x}\; dx + k \int_0^L (s - z - u_x)\,\overline{\Phi}_2\; dx = \rho_2 \int_0^L f_4\,\overline{\Phi}_2\; dx + \beta \int_0^L f_3\,\overline{\Phi}_2\; dx,\\ &-\; b \int_0^L s_x\,\overline{\Phi}_{3x}\; dx - 3\,k \int_0^L (s - z - u_x)\,\overline{\Phi}_3\; dx\\ &\qquad\qquad = \rho_2 \int_0^L f_6\,\overline{\Phi}_3\; dx - 3\,k \int_0^L d(x)\,(f_{5x} - f_1)\,\overline{\Phi}_{3x}\; dx + 4\,\delta \int_0^L f_5\,\overline{\Phi}_3\; dx. \end{aligned}$$

We define

$$\mathcal{B}\,((u,\, z,\, s),\; (\Phi_1,\, \Phi_2,\, \Phi_3)) = \mathcal{L}((\Phi_1,\, \Phi_2,\, \Phi_3)) \tag{3.10}$$

for all $(\Phi_1,\, \Phi_2,\, \Phi_3) \in [H_0^1(0,\, L)]^3$, where

$$\begin{aligned} \mathcal{B}\,((u,\, z,\, s),\; (\Phi_1,\, \Phi_2,\, \Phi_3)) = {}& k \int_0^L (s - z - u_x)\,\overline{\Phi}_{1x}\; dx\\ &-\; b \int_0^L z_x\,\overline{\Phi}_{2x}\; dx + k \int_0^L (s - z - u_x)\,\overline{\Phi}_2\; dx\\ &-\; b \int_0^L s_x\,\overline{\Phi}_{3x}\; dx - 3\,k \int_0^L (s - z - u_x)\,\overline{\Phi}_3\; dx \end{aligned}$$

and

$$\begin{aligned}\mathcal{L}((\Phi_1,\,\Phi_2,\,\Phi_3)) = \rho_1 \int_0^L f_2\,\overline{\Phi}_1\,dx - k\int_0^L d(x)\,(f_{5x}-f_1)\,\overline{\Phi}_1\,dx \\ +\ \rho_2\int_0^L f_4\,\overline{\Phi}_2\,dx + \beta\int_0^L f_3\,\overline{\Phi}_2\,dx +\ \rho_2\int_0^L f_6\,\overline{\Phi}_3\,dx \\ -\,3\,k\int_0^L d(x)\,(f_{5x}-f_1)\,\overline{\Phi}_{3x}\,dx + 4\,\delta\int_0^L f_5\,\overline{\Phi}_3\,dx.\end{aligned}$$

It is not difficult to see $\mathcal{B}$ is a sesquilinear and continuous form on $[H_0^1(0,\,L)]^3 \times [H_0^1(0,\,L)]^3$ and $\mathcal{L}$ is a linear and continuous form on $[H_0^1(0,\,L)]^3$. Thereby, $\mathcal{B}$ is a coercive form on $[H_0^1(0,\,L)]^3 \times [H_0^1(0,\,L)]^3$. Hence, it follows by the Lax-Milgram Theorem that (3.10) admits a unique solution $(u,\,z,\,s) \in [H_0^1(0,\,L)]^3$. Taking test-functions $(\Phi_1,\,\Phi_2,\,\Phi_3) \in [\mathcal{D}(0,\,L)]^3$ we have that (3.9) holds in the distributional sense, from which we deduce that $(u,\,z,\,s) \in [H^2(0,\,L) \cap H_0^1(0,\,L)]^3$, while $[d(x)\,(f_{5x}-f_1)]_x \in L^2(0,\,L)$. Thereby, $U = (u,\,f_1,\,z,\,f_3,\,s,\,f_5)^T \in \mathcal{D}(\mathcal{A})$ is the unique solution of $\mathcal{A}U = F$. Thus $\mathcal{A}$ is an isomorphism and since $\rho(\mathcal{A})$ is open set of $\mathbb{C}$ (see [14]: Theorem 6.7, Chapter III), we easily get $R(\lambda\,I - \mathcal{A}) = \mathcal{H}$ for a sufficiently small $\lambda > 0$. This and using the dissipativity of $\mathcal{A}$, imply that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$ and that is m-dissipative in $\mathcal{H}$ (see [21]: Theorems 4.5, 4.6), the lemma is proved.

□

Proposition 3.3. *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0–semigroup of contractions $\{\mathcal{S}_{\mathcal{A}}(t)\}_{t\geq 0}$.*

Proof. By the lemma 3.1, the operator $\mathcal{A}$ is a dissipative and from lemma 3.2 we have that $\mathcal{A}$ is maximal. Then, according the well known Lumer-Phillips Theorem [21] we have that the operator $\mathcal{A}$ generates a C_0–semigroup of contractions $e^{t\,\mathcal{A}}$ in $\mathcal{H}$.

□

Thereby we have the following result, which gives the well-posedness of (3.3)-(3.4).

Theorem 3.4. *For all $W_0 \in \mathcal{H}$, the system* (3.3)-(3.4) *admits a unique weak solution*

$$W(t) = e^{t\,\mathcal{A}}\,W_0 \in C^0(\mathbb{R}^+,\ \mathcal{H}). \tag{3.11}$$

Moreover, if $W_0 \in \mathcal{D}(\mathcal{A})$, then $W(t)$ is the unique strong solution with the following regularity

$$W(t) \in C^0(\mathbb{R}^+,\ \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}^+,\ \mathcal{H}). \tag{3.12}$$

4. Strong stability

In this section, we prove the strong stability (Theorem 4.3) of the system (2.7). The idea is to use the Theorem 2.3 due to Arendt-Batty [3]. According to Theorem 2.3, we will prove that the operator $\mathcal{A}$ has no pure imaginary eigenvalues and $\sigma(\mathcal{A}) \cap i\,\mathbb{R}$ is countable.

Lemma 4.1. *For all $\lambda \in \mathbb{R}$, $(i\,\lambda\,I - \mathcal{A})$ is injective, that is,*

$$\mathrm{Ker}(i\,\lambda\,I - \mathcal{A}) = \{0\}, \quad \forall\ \lambda \in \mathbb{R}.$$

Proof. By the Lemma 3.2, $0 \in \varrho(\mathcal{A})$. Then we will show the result for $\lambda \in \mathbb{R}^*$. For this aim, suppose that there exists a real number $\lambda \neq 0$ and $W = (u,\, U,\, z,\, Z,\, s,\, S)^T \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}W = i\,\lambda\,W. \tag{4.1}$$

In fact, we have

$$\begin{cases} U = i\,\lambda\,u, \\ -\,k\,(s-z-u_x)_x + k\,d(x)\,(S_x-U) = i\,\lambda\,\rho_1\,U, \\ Z = i\,\lambda\,z, \\ b\,z_{xx} + k\,(s-z-u_x) - \beta\,Z = i\,\lambda\,\rho_2\,Z, \\ S = i\,\lambda\,s, \\ b\,s_{xx} - 3\,k\,(s-z-u_x) + 3\,k\,[d(x)\,(S_x-U)]_x - 4\,\delta\,S = i\,\lambda\,\rho_2\,S. \end{cases} \tag{4.2}$$

From (4.1) we have

$$\langle i\,\lambda\,W,\; W\rangle_{\mathcal{H}} = \langle \mathcal{A}W,\; W\rangle_{\mathcal{H}} \iff i\,\lambda\,\|W\|_{\mathcal{H}} = \langle \mathcal{A}W,\; W\rangle_{\mathcal{H}}. \tag{4.3}$$

Taking the real part and using (3.6) we obtain

$$\begin{aligned} 0 &= Re\,[\,i\,\lambda\,\|W\|_{\mathcal{H}}\,] \\ &= -\;\beta\int_0^L |Z|^2\;dx - 4\,\delta\int_0^L |S|^2\;dx - 3\,k\,d_0\int_\alpha^\beta |S_x-U|^2\;dx \le 0. \end{aligned} \tag{4.4}$$

Thereby we have

$$\begin{cases} Z = 0 \quad \text{in} \quad (0,\, L), \\ S = 0 \quad \text{in} \quad (0,\, L), \\ S_x - U = 0 \quad \text{in} \quad (\alpha,\, \beta). \end{cases} \tag{4.5}$$

Using $(4.2)_{3,\,5}$ into $(4.5)_{1,\,2}$ and then fact that $\lambda \neq 0$ we have

$$\begin{cases} z = 0 \quad \text{in} \quad (0,\, L), \\ s = 0 \quad \text{in} \quad (0,\, L). \end{cases} \tag{4.6}$$

Replacing $(4.6)_2$ into $(4.5)_3$ we have

$$U = 0 \quad \text{in} \quad (\alpha,\, \beta). \tag{4.7}$$

Now, replacing $(4.2)_{1,\,5}$ into $(4.5)_3$ and using that $\lambda \neq 0$ we obtain

$$s_x - u = 0 \iff s_x = u \quad \text{in} \quad (\alpha,\, \beta). \tag{4.8}$$

Therefore from (4.7) into (4.8) we obtain

$$u = 0 \quad \text{in} \quad (\alpha,\, \beta). \tag{4.9}$$

Thereby $W = (0,\, 0,\, 0,\, 0,\, 0,\, 0)$ in $(\alpha,\, \beta)$.

On the other hand, (4.2) can be written in $(0,\, \alpha) \cup (\beta,\, L)$, that is,

$$\begin{cases} \lambda^2\,\rho_1\,u - k\,(s-z-u_x)_x = 0 \quad \text{in} \quad (0,\, \alpha) \cup (\beta,\, L), \\ \lambda^2\,\rho_2\,z + b\,z_{xx} + k\,(s-z-u_x) = 0 \quad \text{in} \quad (0,\, \alpha) \cup (\beta,\, L), \\ \lambda^2\,\rho_2\,s + b\,s_{xx} - 3\,k\,(s-z-u_x) = 0 \quad \text{in} \quad (0,\, \alpha) \cup (\beta,\, L). \end{cases} \tag{4.10}$$

Let $\mathcal{W} = (u,\, u_x,\, z,\, z_x,\, s,\, s_x)^T$ and we define

$$\mathcal{W}_x = \mathcal{B}_\lambda \, \mathcal{W} \tag{4.11}$$

where differentiating in x-variable, $x \in (0,\, \alpha)$ we have

$$\underbrace{\begin{pmatrix} u_x \\ u_{xx} \\ z_x \\ z_{xx} \\ s_x \\ s_{xx} \end{pmatrix}}_{\mathcal{W}_x} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\lambda^2 \rho_1}{k} & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{k}{b} & -\frac{(\lambda^2\,\rho_2 - k)}{b} & 0 & -\frac{k}{b} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{3\,k}{b} & -\frac{3\,k}{b} & 0 & -\frac{(\lambda^2\,\rho_2 - 3\,k)}{b} & 0 \end{pmatrix}}_{\mathcal{B}_\lambda} \underbrace{\begin{pmatrix} u \\ u_x \\ z \\ z_x \\ s \\ s_x \end{pmatrix}}_{\mathcal{W}}. \tag{4.12}$$

From (4.6), (4.7) and (4.9) and the regularity (Theorem 3.4), we have $\mathcal{W}(\alpha) = 0$. From (4.11) we have that the solution is given by

$$\mathcal{W}(x) = e^{\mathcal{B}_\lambda\,(x-\alpha)}\, \mathcal{W}(\alpha). \tag{4.13}$$

Thereby, from (4.13) and using that $\mathcal{W}(\alpha) = 0$, we get

$$\mathcal{W} = 0. \tag{4.14}$$

This way from (4.14) and using that $u(0) = z(0) = s(0) = 0$, it follows that

$$u = 0, \qquad z = 0, \quad \text{and} \quad s = 0 \quad \text{in} \quad (0,\, \alpha). \tag{4.15}$$

Therefore from (4.15), $(4.2)_{1,\,3,\,5}$ and the fact that $\lambda \neq 0$, we obtain

$$U = 0 \quad \text{in} \quad (0,\, \alpha). \tag{4.16}$$

From (4.15) and the regularity of z and s, we obtain

$$z(\alpha) = 0 \quad \text{and} \quad s(\alpha) = 0, \tag{4.17}$$

where using (4.9) and the same idea given above we obtain

$$u = 0, \quad z = 0 \quad \text{and} \quad s = 0 \quad \text{in} \quad (\alpha,\, \beta). \tag{4.18}$$

This way, from $(4.2)_{1,\,3,\,5}$ and the fact that $\lambda \neq 0$, we obtain

$$U = 0 \quad \text{in} \quad (\alpha,\, \beta). \tag{4.19}$$

Now, let $\widetilde{\mathcal{W}} = (u,\, u_x,\, z,\, z_x,\, s,\, s_x)^T$. From (4.19) and the regularity of u, z and s we have $\widetilde{\mathcal{W}}(\beta) = 0$ and the system (4.10) in $(\beta,\, L)$ implies

$$\widetilde{\mathcal{W}}_x = \mathcal{B}_\lambda \, \widetilde{\mathcal{W}} \quad \text{in} \quad (\beta,\, L), \tag{4.20}$$

where $\mathcal{B}_\lambda$ is defined in (4.12). This way, we have

$$\widetilde{\mathcal{W}}(x) = e^{\mathcal{B}_\lambda\,(x-\beta)}\, \widetilde{\mathcal{W}}(\beta) = 0. \tag{4.21}$$

Therefore, $(4.2)_{1,\,3,\,5}$ we conclude

$$U = 0 \quad \text{in} \quad (\beta,\, L). \tag{4.22}$$

Finally, from (4.16), (4.19) and (4.22) we obtain

$$U = 0 \quad \text{in} \quad (0,\, L). \tag{4.23}$$

The Lemma 4.1 follows.

□

Lemma 4.2. *For all $\lambda \in \mathbb{R}$, we have*

$$R\,(i\,\lambda\, I - \mathcal{A}) = \mathcal{H}. \tag{4.24}$$

Proof. By the Lemma 3.2, $0 \in \varrho(\mathcal{A})$. We will show the result for $\lambda \in \mathbb{R}^*$. For this aim, let $\mathcal{F} = (f_1,\, f_2,\, f_3,\, f_4,\, f_5,\, f_6)^T \in \mathcal{H}$, we want to find $W = (u,\, U,\, z,\, Z,\, s,\, S)^T \in \mathcal{D}(\mathcal{A})$ solution of

$$(i\,\lambda\, I - \mathcal{A})\, W = \mathcal{F}, \tag{4.25}$$

that is

$$\begin{cases} i\,\lambda\, u - U = f_1, \\ i\,\lambda\, \rho_1\, U + k\,(s - z - u_x)_x - k\, d(x)\,(S_x - U) = \rho_1\, f_2, \\ i\,\lambda\, z - Z = f_3, \\ i\,\lambda\, \rho_2\, Z - b\, z_{xx} - k\,(s - z - u_x) + \beta\, Z = \rho_2\, f_4, \\ i\,\lambda\, s - S = f_5, \\ i\,\lambda\, \rho_2\, S - b\, s_{xx} + 3\,k\,(s - z - u_x) - 3\,k\,[d(x)\,(S_x - U)]_x + 4\,\delta\, \rho_2\, S = \rho_2\, f_6, \end{cases} \tag{4.26}$$

with the boundary conditions

$$u(0) = u(L) = z(0) = z(L) = s(0) = s(L) = 0. \tag{4.27}$$

From $(4.26)_{1,\,3,\,5}$ we have

$$\begin{cases} U = i\,\lambda\, u - f_1, \\ Z = i\,\lambda\, z - f_3, \\ S = i\,\lambda\, s - f_5. \end{cases} \tag{4.28}$$

Replacing $(4.28)_{1,\,2,\,3}$ into $(4.26)_{2,\,4,\,6}$ we obtain

$$-\,\lambda^2\, u + \frac{k}{\rho_1}\,[(s - z - u_x)_x - i\,\lambda\, d(x)\,(s_x - u)] = g_1(x), \tag{4.29}$$

$$-\,\lambda^2\, z - \frac{1}{\rho_2}\,[b\, z_{xx} + k\,(s - z - u_x)] + i\,\lambda\,\beta\, z = g_1(x), \tag{4.30}$$

$$-\,\lambda^2\, s - \frac{1}{\rho_2}\,[b\, s_{xx} - 3\,k\,(s - z - u_x) + 3\,i\,\lambda\, k\,[d(x)\,(s_x - u)]_x] + 4\,i\,\lambda\,\delta\, s = g_3(x), \tag{4.31}$$

where

$$\begin{cases} g_1(x) = f_2 + i\,\lambda\, f_1 - \frac{k}{\rho_1}\, d(x)\,[f_{5x} - f_1] \in H^{-1}(0,\, L), \\ g_2(x) = f_4 + (i\,\lambda + b)\, f_3 \in H^{-1}(0,\, L), \\ g_3 = f_6 + i\,\lambda\, f_5 + 4\,\delta\, f_5 - \frac{3\,k}{\rho_2}\,[d(x)\,(f_{5\,x} - f_1)]_x \in H^{-1}(0,\, L). \end{cases} \tag{4.32}$$

On the other hand, for all $\widetilde{W} = (u,\, z,\, s)^T \in \widetilde{\mathcal{H}} = [H_0^1(0,\, L)]^3$, we define the linear operator

$$\widetilde{\mathcal{L}} : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}' := [H^{-1}(0,\, L)]^3 \tag{4.33}$$

by

$$\widetilde{\mathcal{L}}\left(\widetilde{W}\right)=\begin{pmatrix}\frac{k}{\rho_1}\left[(s-z-u_x)_x-i\,\lambda\,d(x)\,(s_x-u)\right]\\ -\frac{1}{\rho_2}\left[b\,z_{xx}+k\,(s-z-u_x)\right]+i\,\lambda\,b\,z\\ -\frac{1}{\rho_2}\left[b\,s_{xx}-3\,k\,(s-z-u_x)+3\,i\,\lambda\,k\,[d(x)\,(s_x-u)]_x\right]+4\,i\,\lambda\,\delta\,s\end{pmatrix}. \tag{4.34}$$

Claim: $\widetilde{\mathcal{L}}$ is an isomorphism. In fact, let $\Psi=(\rho_1\,\Psi_1,\,\rho_2\,\Psi_2,\,\rho_2\,\Psi_3)^T\in\widetilde{\mathcal{H}}$, then

$$\begin{aligned}\langle\widetilde{\mathcal{L}}\left(\widetilde{W}\right),\,\Psi\rangle_{\widetilde{\mathcal{H}}'\times\widetilde{\mathcal{H}}}&=\langle k\,(s-z-u_x)_x-i\,k\,\lambda\,d(x)\,(s_x-u),\,\Psi_1\rangle_{H^{-1}(0,\,L)\times H_0^1(0,\,L)}\\&+\ \langle -\,b\,z_{xx}-k\,(s-z-u_x)+i\,\rho_2\,\lambda\,b\,z,\ \Psi_2\rangle_{H^{-1}(0,\,L)\times H_0^1(0,\,L)}\\&+\ \langle -\,b\,s_{xx}+3\,k\,(s-z-u_x)-3\,i\,\lambda\,k\,[d(x)\,(s_x-u)]_x\\&+4\,i\,\rho_2\,\lambda\,\delta\,s,\ \Psi_3\rangle_{H^{-1}(0,\,L)\times H_0^1(0,\,L)}.\end{aligned}$$

Then is not difficult to show that

$$\begin{aligned}\langle\widetilde{\mathcal{L}}\left(\widetilde{W}\right),\,\Psi\rangle_{\widetilde{\mathcal{H}}'\times\widetilde{\mathcal{H}}}&=-k\int_0^L(s-z-u_x)\,\overline{\Psi}_{1x}\,dx-ik\lambda\int_0^L d(x)\,(s_x-u)\,\overline{\Psi}_1\,dx\\&+b\int_0^L z_x\,\overline{\Psi}_{2x}\,dx-k\int_0^L(s-z-u_x)\,\overline{\Psi}_2\,dx+i\rho_2\lambda\,b\int_0^L z\,\overline{\Psi}_2\,dx\\&+b\int_0^L s_x\,\overline{\Psi}_{3x}\,dx+3k\int_0^L(s-z-u_x)\,\overline{\Psi}_3\,dx\\&+3\,i\,\lambda\,k\int_0^L d(x)\,(s_x-u)\,\overline{\Psi}_{3x}\,dx+4i\lambda\,\rho_2\,\delta\int_0^L s\,\overline{\Psi}_3\,dx\end{aligned}$$

defines a continuous sesquilinear form which is coercive on $\mathcal{H}$. Therefore, using the Lax-Milgram theorem, we deduce that $\widetilde{\mathcal{L}}$ is an isomorphism from $\widetilde{\mathcal{H}}$ onto $\widetilde{\mathcal{H}}'$. On the other hand, let $\widetilde{W}=(u,\,z,\,s)^T$ and $\mathcal{G}=(g_1,\,g_2,\,g_3)^T$, then (4.29), (4.30) and (4.31) can be transformed into the following form

$$\left(I-\lambda^2\,\widetilde{\mathcal{L}}^{-1}\right)\widetilde{W}=\widetilde{\mathcal{L}}^{-1}\mathcal{G}. \tag{4.35}$$

Using $I:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}'$ is a compact operator and $\widetilde{\mathcal{L}}^{-1}:\widetilde{\mathcal{H}}'\to\widetilde{\mathcal{H}}$ is an isomorphism, then the operator $I-\lambda^2\,\widetilde{\mathcal{L}}^{-1}$ is Fredholm of index zero. Then, by the Fredholm alternative, (4.35) admits a unique solution $\widetilde{W}\in\widetilde{\mathcal{H}}$ if and only if $I-\lambda^2\,\widetilde{\mathcal{L}}^{-1}$ is injective. Let $\boldsymbol{W}=(\boldsymbol{u},\,\boldsymbol{z},\,\boldsymbol{s})^T\in\widetilde{\mathcal{H}}$ such that

$$\boldsymbol{W}-\lambda^2\,\widetilde{\mathcal{L}}^{-1}\boldsymbol{W}=0\iff\lambda^2\,\boldsymbol{W}-\widetilde{\mathcal{L}}\boldsymbol{W}=0. \tag{4.36}$$

Equivalently, we have

$$-\,\lambda^2\,\boldsymbol{u}+\frac{k}{\rho_1}\left[(\boldsymbol{s}-\boldsymbol{z}-\boldsymbol{u}_x)_x-i\,\lambda\,d(x)\,(\boldsymbol{s}_x-\boldsymbol{u})\right]=0, \tag{4.37}$$

$$-\,\lambda^2\,\boldsymbol{z}-\frac{1}{\rho_2}\left[b\,\boldsymbol{z}_{xx}+k\,(\boldsymbol{s}-\boldsymbol{z}-\boldsymbol{u}_x)\right]+i\,\lambda\,b\,\boldsymbol{z}=0, \tag{4.38}$$

$$-\,\lambda^2\,\boldsymbol{s}-\frac{1}{\rho_2}\left[b\,\boldsymbol{s}_{xx}-3\,k\,(\boldsymbol{s}-\boldsymbol{z}-\boldsymbol{u}_x)+3\,i\,\lambda\,k\,[d(x)\,(\boldsymbol{s}_x-\boldsymbol{u})]_x\right]+4\,i\,\lambda\,\delta\,\boldsymbol{s}=0. \tag{4.39}$$

It is not difficult to show that if $\boldsymbol{W} = (\boldsymbol{u},\, \boldsymbol{z},\, \boldsymbol{s})^T$ if a solution of (4.37)-(4.39), then $\widetilde{\boldsymbol{W}} = (\boldsymbol{u},\, i\,\lambda\,\boldsymbol{u},\, \boldsymbol{z},\, i\,\lambda\,\boldsymbol{z},\, \boldsymbol{s},\, i\,\lambda\,\boldsymbol{s})^T \in \mathcal{D}(\mathcal{A})$ and satisfies: $i\,\lambda\,\boldsymbol{W} - \mathcal{A}\boldsymbol{W} = 0$. Thereby, using Lemma 4.1 we obtain $\widetilde{\boldsymbol{W}} = 0$ and we conclude that $I - \lambda^2\,\widetilde{\mathcal{L}}^{-1}$ is injective. Since to Fredholm's alternative, (4.35) admits a unique solution $\widetilde{W} \in \mathcal{H}$ and

$$u,\, z,\, s \in H^2(0,\, L), \qquad 3\,k\,d(x)\,[i\,\lambda\,s_x - f_{5\,x} - (i\,\lambda\,u - f_1)]_x \in L^2(0,\, L).$$

Finally, setting $U = i\,\lambda\,u - f_1$, $Z = i\,\lambda\,z - f_3$, and $S = i\,\lambda\,s - f_5$ we have that $W \in \mathcal{D}(\mathcal{A})$ is a unique solution of (4.25). The Lemma 4.2 follows.

□

Now we will prove the main theorem of this section:

Theorem 4.3. *The C_0-semigroup of contraction $\left(e^{t\,\mathcal{A}}\right)_{t\geq 0}$ is strongly stable in $\mathcal{H}$, that is, for all $W_0 \in \mathcal{H}$, the solution of (3.3) satisfies*

$$\lim_{t\to+\infty} \|e^{t\,\mathcal{A}}W_0\|_{\mathcal{H}} = 0.$$

Proof. We have that the operator $\mathcal{A}$ has no pure imaginary, that is, $\sigma_p(\mathcal{A}) \cap i\,\mathbb{R} = \emptyset$ (Lemma 4.1).On the pther hand, using the closed graph theorem of Banach and Lemma 4.2, we have $\sigma(\mathcal{A}) \cap i\,\mathbb{R} = \emptyset$. Now using Theorem 2.3 due to Arendt-Batty [3], we get that the C$_0$-semigroup $\left(e^{t\,\mathcal{A}}\right)_{t\geq 0}$ is strongly stable. The main theorem follows.

□

5. Polynomial Stability

In this section, we will prove the polynomial stability of the system (1.1). Since $i\,\mathbb{R} \subset \varrho(\mathcal{A})$ (Section three), from Theorem 2.4 to prove the main theorem, we still need to prove the following condition

$$\sup_{\lambda\in\mathbb{R}} \left\|(i\,\lambda\,I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}\left(|\lambda|^{\ell}\right). \tag{5.1}$$

We will prove condition (5.1) by a contradiction argument. For this purpose, we suppose that (5.1) is false, then there exists

$$\left[\left(\lambda^{(n)},\; W^{(n)} = \left(u^{(n)},\, U^{(n)},\, z^{(n)},\, Z^{(n)},\, s^{(n)},\, S^{(n)}\right)^T\right)\right]_{n\geq 1} \subseteq \mathbb{R}^* \times \mathcal{D}(\mathcal{A})$$

with

$$|\lambda^{(n)}| \to \infty \quad \text{and} \quad \left\|W^{(n)}\right\|_{\mathcal{H}} = \left\|\left(u^{(n)},\, U^{(n)},\, z^{(n)},\, Z^{(n)},\, s^{(n)},\, S^{(n)}\right)^T\right\|_{\mathcal{H}} = 1, \tag{5.2}$$

such that

$$\left(\lambda^{(n)}\right)^{\ell}\left(i\,\lambda^{(n)}\,I - \mathcal{A}\right)W^{(n)} = F^{(n)} = (f_1^{(n)},\, f_2^{(n)},\, f_3^{(n)},\, f_4^{(n)},\, f_5^{(n)},\, f_6^{(n)}) \to 0. \tag{5.3}$$

Here for simplicity, we omit the index n. Thereby from (5.3) we have

$$\begin{cases} i\,\lambda\,u - U = \lambda^{-\ell}\,f_1 \iff U = i\,\lambda\,u - \lambda^{-\ell}\,f_1, \\ i\,\lambda\,\rho_1\,U + k\,(s - z - u_x)_x - k\,d(x)\,(S_x - U) = \rho_1\,\lambda^{-\ell}\,f_2, \\ i\,\lambda\,z - Z = \lambda^{-\ell}\,f_3 \iff Z = i\,\lambda\,z - \lambda^{-\ell}\,f_3, \\ i\,\lambda\,\rho_2\,Z - b\,z_{xx} - k\,(s - z - u_x) + \beta\,Z = \rho_2\,\lambda^{-\ell}\,f_4, \\ i\,\lambda\,s - S = \lambda^{-\ell}\,f_5 \iff S = i\,\lambda\,s - \lambda^{-\ell}\,f_5, \\ i\,\lambda\,\rho_2\,S - b\,s_{xx} + 3\,k\,(s - z - u_x) - 3\,k\,[d(x)\,(S_x - U)]_x + 4\,\delta\,\rho_2\,S = \rho_2\,\lambda^{-\ell}\,f_6. \end{cases} \tag{5.4}$$

Replacing $(5.4)_{1,\,3,\,5}$ into $(5.4)_{2,\,4,\,6}$ respectively and performing straightforward calculations we obtain

$$\lambda^2\,\rho_1\,u - k\,(s - z - u_x)_x + k\,d(x)\,(S_x - U) = -\,\rho_1\,\lambda^{-\,\ell}\,f_2 - i\,\rho_1\,\lambda^{-\,\ell+1}\,f_1, \tag{5.5}$$

$$\lambda^2\,\rho_2\,z + b\,z_{xx} + k\,(s - z - u_x) - \beta\,Z = -\,\rho_2\,\lambda^{-\,\ell}\,f_4 - i\,\rho_2\,\lambda^{-\,\ell+1}\,f_3, \tag{5.6}$$

$$\begin{aligned} \lambda^2\,\rho_2\,s + b\,s_{xx} - 3\,k\,(s - z - u_x) + 3\,k\,[d(x)\,(S_x - U)]_x + 4\,\delta\,S \\ = -\,\rho_2\,\lambda^{-\,\ell}\,f_6 - i\,\rho_2\,\lambda^{-\,\ell+1}\,f_5. \end{aligned} \tag{5.7}$$

Here we will check the condition (5.1) by finding a contradiction with (5.2) by proving that $\|W\|_{\mathcal{H}} = \mathcal{O}(1)$. From (5.4), and the fact that $\ell = 2$,

$$\|W\|_{\mathcal{H}} = 1 \quad \text{and} \quad \|F\|_{\mathcal{H}} \approx \mathcal{O}(1), \tag{5.8}$$

we have that

$$\begin{cases} \|u\| \approx \mathcal{O}\left(|\lambda|^{-1}\right), \quad \|z\| \approx \mathcal{O}\left(|\lambda|^{-1}\right) \quad \text{and} \quad \|s\| \approx \mathcal{O}\left(|\lambda|^{-1}\right), \\ \|u_{xx}\| \approx \mathcal{O}\left(|\lambda|\right), \quad \|z_{xx}\| \approx \mathcal{O}\left(|\lambda|\right) \quad \text{and} \quad \|s_{xx}\| \approx \mathcal{O}\left(|\lambda|\right), \\ [d(x)\,(S_x - U)]_x \approx \mathcal{O}\left(|\lambda|\right). \end{cases} \tag{5.9}$$

Moreover, from Poincaré inequality and the fact $\|F\| = \mathcal{O}(1)$ it follows that

$$\|f_1\| \lesssim \|f_{1x}\| \approx \mathcal{O}(1), \quad \|f_3\| \lesssim \|f_{3x}\| \approx \mathcal{O}(1) \quad \text{and} \quad \|f_5\| \lesssim \|f_{5x}\| \approx \mathcal{O}(1). \tag{5.10}$$

Claim 5.1. *Let* $\ell = 2$. *The solution* $W = (u,\,U,\,z,\,Z,\,s,\,S)^T \in \mathcal{D}(\mathcal{A})$ *of* (5.4) *satisfies*

$$\begin{cases} \int_\alpha^\beta |S_x - U|^2\,dx \lesssim \frac{\mathcal{O}(1)}{\lambda^\ell}, \quad \|S_x - U\|^2 \lesssim \frac{\mathcal{O}(1)}{\lambda^\ell}, \quad \|S_x - U\| \lesssim \frac{\mathcal{O}(1)}{\lambda^{\ell/2}}, \\ \int_\alpha^\beta |s_x - u|^2\,dx \lesssim \frac{\mathcal{O}(1)}{\lambda^{\ell+2}}, \\ \int_\alpha^\beta |S_x|^2\,dx \lesssim \mathcal{O}(1), \\ \int_\alpha^\beta |s_x|^2\,dx \lesssim \frac{\mathcal{O}(1)}{\lambda^2}. \end{cases} \tag{5.11}$$

Proof. Taking the inner product of (5.3) with W we have

$$-\,Re\langle \mathcal{A}W,\,W\rangle_{\mathcal{H}} = \lambda^{-\ell}\,Re\langle F,\,W\rangle_{\mathcal{H}} \le \lambda^{-\ell}\,\|F\|_{\mathcal{H}}\,\|W\|_{\mathcal{H}}.$$

Then taking the real part and using (3.6) we obtain

$$\beta \int_0^L |Z|^2 \, dx + 4\,\delta \int_0^L |S|^2 \, dx + 3\,k\,d_0 \int_\alpha^\beta |S_x - U|^2 \, dx \leq \lambda^{-\ell} \underbrace{\|F\|_{\mathcal{H}}}_{\lesssim \mathcal{O}(1)} \underbrace{\|W\|_{\mathcal{H}}}_{=1}. \tag{5.12}$$

From (5.8) we obtain

$$\beta \int_0^L |Z|^2 \, dx + 4\,\delta \int_0^L |S|^2 \, dx + 3\,k\,d_0 \int_\alpha^\beta |S_x - U|^2 \, dx \lesssim \frac{\mathcal{O}(1)}{\lambda^\ell}. \tag{5.13}$$

From (5.13) we have that $(5.11)_1$ follows. Differentiating $(5.4)_5$ in x-variable we have

$$i\,\lambda\, s_x - S_x = \lambda^{-\ell}\, f_{5x}. \tag{5.14}$$

Subtracting $(5.4)_1$ with (5.14) we obtain

$$i\,\lambda\,(s_x - u) = (S_x - U) + \lambda^{-\ell}\,(f_{5x} - f_1).$$

Then using $(a+b)^2 \leq 2\,a^2 + 2\,b^2$ we have

$$\begin{aligned}\int_\alpha^\beta |s_x - u|^2\, dx &\leq \frac{2}{\lambda^2}\int_\alpha^\beta |S_x - U|^2\, dx + \frac{2}{\lambda^{2\,\ell+2}}\int_\alpha^\beta |f_{5x} - f_1|^2\, dx \\ &\leq \frac{2}{\lambda^2}\int_\alpha^\beta |S_x - U|^2\, dx + \frac{4}{\lambda^{2\,\ell+2}}\int_\alpha^\beta |f_{5x}|^2\, dx + \frac{4}{\lambda^{2\,\ell+2}}\int_\alpha^\beta |f_1|^2\, dx \\ &\leq \frac{2}{\lambda^2}\int_\alpha^\beta |S_x - U|^2\, dx + \frac{4}{\lambda^{2\,\ell+2}}\,\|f_{5x}\|^2 + \frac{4}{\lambda^{2\,\ell+2}}\|f_1\|^2.\end{aligned}$$

Now using (5.10) together with we obtain

$$\int_\alpha^\beta |s_x - u|^2\, dx \approx \frac{2}{\lambda^2}\,\frac{\mathcal{O}(1)}{\lambda^{2\,\ell}} + \frac{4}{\lambda^{2\,\ell+2}}\,\mathcal{O}(1) + \frac{4}{\lambda^{2\,\ell+2}}\,\mathcal{O}(1) \quad \lesssim \frac{1}{\lambda^{2\,\ell+2}}\,\mathcal{O}(1), \tag{5.15}$$

then $(5.11)_2$ follows. On the other hand, using (5.13)

$$\begin{aligned}\int_\alpha^\beta |S_x|^2\, dx &= \int_\alpha^\beta |S_x - U + U|^2\, dx \\ &\leq 2\int_\alpha^\beta |S_x - U|^2\, dx + 2\int_\alpha^\beta |U|^2\, dx \\ &\lesssim \frac{2}{\lambda^\ell}\,\mathcal{O}(1) + 2\int_\alpha^\beta |U|^2\, dx.\end{aligned} \tag{5.16}$$

Using that U is uniformly bounded in $L^2(0,\, L)$, we get $(5.11)_3$. Differentiating $(5.4)_5$ we have

$$i\,\lambda\, s_x = S_x + \lambda^{-\ell}\, f_{5x}.$$

Elevating square the above expression, integrating over $(\alpha,\,\beta)$ and using $(a+b)^2 \le 2\,a^2 + 2\,b^2$ together with $(5.11)_3$ and (5.10) we obtain

$$\begin{aligned}\int_\alpha^\beta |s_x|^2\,dx &\le \frac{2}{\lambda^2}\int_\alpha^\beta |S_x|^2\,dx + \frac{2}{\lambda^{2\,\ell+2}}\int_\alpha^\beta |f_{5x}|^2\,dx \\ &\lesssim \frac{2}{\lambda^2}\,\mathcal{O}(1) + \frac{2}{\lambda^{2\,\ell+2}}\,\mathcal{O}(1) \\ &\lesssim \left(\frac{2}{\lambda^2}+\frac{2}{\lambda^2}\right)\mathcal{O}(1) \lesssim \frac{4}{\lambda^2}\,\mathcal{O}(1) \approx \frac{C}{\lambda^2}\,\mathcal{O}(1),\end{aligned} \tag{5.17}$$

then we have $(5.11)_4$. The Claim 5.1 follows.

□

For all $0 < \varepsilon < \frac{(\beta-\alpha)}{10}$, we fix the following cut-off functions:

$\chi_j \in C^\infty([0,\,L])$, $j \in \{1,\,\ldots,\,4\}$ such that $0 \le \chi_j \le 1$, for all $x \in [0,\,L]$ and

$$\chi_j(x) = \begin{cases} 1 & \text{if} \quad x \in [\alpha + j\,\varepsilon,\, \beta - j\,\varepsilon], \\ 0 & \text{if} \quad x \in [0,\, \alpha + (j-1)\,\varepsilon] \cup [\beta + (1-j)\,\varepsilon,\, L]. \end{cases} \tag{5.18}$$

Claim 5.2. *Let $\ell = 2$. The solution $W = (u,\,U,\,z,\,Z,\,s,\,S)^T \in \mathcal{D}(\mathcal{A})$ of (5.4) satisfies the following estimates*

$$\begin{cases} \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |S|^2\,dx \approx \mathcal{O}(1), \\ \\ \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |\lambda\,s|^2\,dx \approx \mathcal{O}(1). \end{cases} \tag{5.19}$$

Proof. Multiplying $(5.4)_6$ by $-\,i\,\lambda^{-1}\,\chi_1(x)\,\overline{S}$ and integrating over $(\alpha,\,\beta)$ we have

$$\begin{aligned}\rho_2\int_\alpha^\beta \chi_1(x)\,|S|^2\,dx &= -\frac{i\,b}{\lambda}\int_\alpha^\beta \chi_1(x)\,s_{xx}\,\overline{S}\,dx + \frac{3\,i\,k}{\lambda}\int_\alpha^\beta \chi_1(x)\,(s-z-u_x)\,\overline{S}\,dx \\ &- \frac{3\,i\,k}{\lambda}\int_\alpha^\beta \chi_1(x)\,[d(x)\,(S_x-U)]_x\,\overline{S}\,dx \\ &+ \frac{4\,i\,\delta\,\rho_2}{\lambda}\int_\alpha^\beta \chi_1(x)\,|S|^2\,dx - \frac{i\,\rho_2}{\lambda^{\ell+1}}\int_\alpha^\beta \chi_1(x)\,f_6\,\overline{S}\,dx.\end{aligned} \tag{5.20}$$

Using the Cauchy-Schwarz inequality and the fact that S is uniformly bounded in $L^2(0,\,L)$ together with (5.10) and $\chi_1(\alpha) = \chi_1(\beta) = 0$ we have

$$\begin{aligned}\rho_2\int_\alpha^\beta \chi_1(x)\,|S|^2\,dx &\approx \frac{i\,b}{\lambda}\int_\alpha^\beta \chi_1(x)\,s_x\,\overline{S}_x\,dx + \frac{3\,i\,k}{\lambda}\int_\alpha^\beta \chi_1(x)\,(s-z-u_x)\,\overline{S}\,dx \\ &- \frac{3\,i\,k}{\lambda}\int_\alpha^\beta \chi_1(x)\,[d(x)\,(S_x-U)]_x\,\overline{S}\,dx + \frac{\mathcal{O}(1)}{\lambda^{\ell+1}}.\end{aligned} \tag{5.21}$$

Moreover, using the fact that $(s-z-u_x)$ and S are uniformly bounded in $L^2(0,\,L)$, together with Claim 5.1 we get

$$\rho_2\int_\alpha^\beta \chi_1(x)\,|S|^2\,dx \approx \underbrace{-\frac{3\,i\,k}{\lambda}\int_\alpha^\beta \chi_1(x)\,[d(x)\,(S_x-U)]_x\,\overline{S}\,dx}_{I_1} + \mathcal{O}(1). \tag{5.22}$$

Integrating by parts, using the definition of $d(x)$ and the fact that $\chi_1(\alpha) = \chi_1(\beta) = 0$ it follows that

$$\begin{aligned} I_1 &= \frac{3\,i\,k}{\lambda}\int_\alpha^\beta \chi_{1x}(x)\,d(x)\,(S_x - U)\,\overline{S}\,dx + \frac{3\,i\,k}{\lambda}\int_\alpha^\beta \chi_1(x)\,d(x)\,(S_x - U)\,\overline{S}_x\,dx \\ &= \frac{3\,i\,k\,d_0}{\lambda}\int_\alpha^\beta \chi_{1x}(x)\,(S_x - U)\,\overline{S}\,dx + \frac{3\,i\,k\,d_0}{\lambda}\int_\alpha^\beta \chi_1(x)\,(S_x - U)\,\overline{S}_x\,dx. \quad (5.23) \end{aligned}$$

From Claim 5.1 and the fact that S is uniformly bounded in $L^2(0,\,L)$ we obtain

$$I_1 \approx \frac{\mathcal{O}(1)}{\lambda^{\frac{\ell}{2}+1}}. \quad (5.24)$$

Replacing (5.24) into (5.22) we obtain

$$\rho_2\int_\alpha^\beta \chi_1(x)\,|S|^2\,dx \approx \mathcal{O}(1). \quad (5.25)$$

From (5.25) and the definition (5.19), we have $(5.18)_1$. On the other hand, from $(5.4)_5$ and using that $(P+Q)^2 \le 2\,P^2 + 2\,Q^2$ we have

$$|\lambda\,s|^2 = |i\,\lambda\,s|^2 = |S + \lambda^{-\ell}\,f_5|^2 \le 2\,|S|^2 + 2\,|\lambda^{-\ell}\,f_5|^2 = 2\,|S|^2 + 2\,\lambda^{-2\,\ell}\,|f_5|^2.$$

Then

$$\begin{aligned} \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |\lambda\,s|^2\,dx &\le 2\int_{\alpha+\varepsilon}^{\beta-\varepsilon} |S|^2\,dx + 2\,\lambda^{-2\,\ell}\int_{\alpha+\varepsilon}^{\beta-\varepsilon} |f_5|^2\,dx \\ &\approx 2\;\mathcal{O}(1) + 2\,\lambda^{-2\,\ell}\;\mathcal{O}(1) \\ &\lesssim 2\;\mathcal{O}(1). \end{aligned} \quad (5.26)$$

From $(5.19)_1$ we have the first estimate in the right-hand side and for the second term we use the fact that $\|f_5\| \approx \mathcal{O}(1)$. Then we obtain $(5.19)_2$. The Claim 5.2 follows.

□

Claim 5.3. *Let $\ell = 2$. The solution $W = (u,\,U,\,z,\,Z,\,s,\,S)^T \in \mathcal{D}(\mathcal{A})$ of (5.4) satisfies the following estimates*

$$\begin{cases} \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |u_x|^2\,dx \approx \mathcal{O}(1), \\ \\ \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |\lambda\,u|^2\,dx \approx \mathcal{O}(1). \end{cases} \quad (5.27)$$

Proof. Multiplying $(5.4)_6$ by $\chi_2(x)\,\overline{u}_x$ integrating over $(\alpha+\varepsilon,\,\beta-\varepsilon)$ and and using the fact that u_x is uniformly bounded in $L^2(0,\,L)$ and $\|f_6\|\approx\mathcal{O}(1)$ we have

$$\begin{aligned}
&\underbrace{i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,S\,\overline{u}_x\;dx}_{J_1}-\underbrace{3\,k\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,[d(x)\,(S_x-U)]_x\,\overline{u}_x\;dx}_{J_2}\\
&-\;3\,k\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,|u_x|^2\;dx+3\,k\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,(s-z)\,\overline{u}_x\;dx\\
&-\;b\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,s_{xx}\,\overline{u}_x\;dx+4\,\delta\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,S\,\overline{u}_x\;dx\\
&=\rho_2\,\lambda^{-\ell}\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,f_6\,\overline{u}_x\;dx\lesssim C\,\lambda^{-\ell}\;\mathcal{O}(1)\approx\frac{\mathcal{O}(1)}{\lambda^{\ell}}.
\end{aligned}\tag{5.28}$$

Using the fact that u_x is uniformly bounded in $L^2(0,\,L)$, and from (5.9), $\|z\|\approx\mathcal{O}(|\lambda|^{-1})$ and $\|s\|\approx\mathcal{O}(|\lambda|^{-1})$ we get

$$\begin{cases}
\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,(s-z)\,\overline{u}_x\;dx\approx\mathcal{O}(1),\\
\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,s_{xx}\,\overline{u}_x\;dx\approx\mathcal{O}(1),\\
\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,S\,\overline{u}_x\;dx\approx\mathcal{O}(1).
\end{cases}\tag{5.29}$$

Then

$$\begin{aligned}
&\underbrace{-\,i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,S\,\overline{u}_x\;dx}_{J_1}+\underbrace{3\,k\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,[d(x)\,(S_x-U)]_x\,\overline{u}_x\;dx}_{J_2}\\
&\qquad+\;3\,k\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,|u_x|^2\;dx\approx\mathcal{O}(1).
\end{aligned}\tag{5.30}$$

On the other hand, using integration by parts, the definition of χ_2, the Claim 5.2 ($(5.19)_1$), and the fact that $\|u\|\approx\mathcal{O}(|\lambda|^{-\ell})$ we obtain

$$\begin{aligned}
J_1&=\;-\,i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,S_x\,\overline{u}\;dx-i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_{2x}(x)\,S\,\overline{u}\;dx\\
&\approx\;-\,i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,S_x\,\overline{u}\;dx+C\,\mathcal{O}(|\lambda|^{-\ell})\;\mathcal{O}(1)\\
&\lesssim\;-\,i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,S_x\,\overline{u}\;dx+\mathcal{O}(1).
\end{aligned}\tag{5.31}$$

Moreover

$$\begin{aligned}
-\;i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,S_x\,\overline{u}\;dx&=-\,i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,(S_x-U+U)\,\overline{u}\,dx\\
&=-\,i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,(S_x-U)\,\overline{u}\,dx-i\,\lambda\,\rho_2\int_{\alpha+\varepsilon}^{\beta-\varepsilon}\chi_2(x)\,U\,\overline{u}\,dx.
\end{aligned}\tag{5.32}$$

From Claim 5.1 (see $(5.11)_1$) and the fact that $\|u\| \approx \mathcal{O}(|\lambda|^{-1})$ we have

$$-\,i\,\lambda\,\rho_2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,S_x\,\overline{u}\,dx = -\,i\,\lambda\,\rho_2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,U\,\overline{u}\,dx + \mathcal{O}(|\lambda|^{-\ell/2}). \qquad (5.33)$$

Replacing $(5.4)_1$, that is, $U = i\,\lambda\,u - \lambda^{-\ell}\,f_1$ into (5.33) we have

$$\begin{aligned} -\,i\,\lambda\,\rho_2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,S_x\,\overline{u}\,dx &= -\,i\,\lambda\,\rho_2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,(i\,\lambda\,u - \lambda^{-\ell}\,f_1)\,\overline{u}\,dx + \mathcal{O}(|\lambda|^{-\ell/2}) \\ &= \lambda\,\rho_2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,|\lambda\,u|^2\,dx + i\,\lambda^{-\ell+1}\,\rho_2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,f_1\,\overline{u}\,dx + \mathcal{O}(|\lambda|^{-\ell/2}). \end{aligned}$$

Now, using $\|u\| \approx \mathcal{O}(|\lambda|^{-1})$ and $\|f_1\| \approx \mathcal{O}(1)$ we obtain

$$-\,i\,\lambda\,\rho_2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,S_x\,\overline{u}\,dx \approx \lambda\,\rho_2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,|\lambda\,u|^2\,dx + \mathcal{O}(|\lambda|^{-\ell/2}). \qquad (5.34)$$

Replacing (5.34) into (5.31) we obtain

$$J_1 \approx \lambda\,\rho_2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,|\lambda\,u|^2\,dx + \mathcal{O}(1). \qquad (5.35)$$

On the other hand, using integration by parts and the definition of $\chi_2(x)$ we have

$$\begin{aligned} J_2 &= -\,3\,k \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,[d(x)\,(S_x - U)]_x\,\overline{u}_x\ dx \\ &= 3\,d_0\,k \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_{2x}(x)\,d_0\,(S_x - U)\,\overline{u}_x\ dx + 3\,d_0\,k \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,(S_x - U)\,\overline{u}_{xx}\ dx. \end{aligned}$$

From Claim 5.1 and the fact that u_x is uniformly bounded in $L^2(0,\,L)$, $\|u_{xx}\| \approx \mathcal{O}(|\lambda|)$ (see (5.9)) we obtain

$$J_2 \approx \mathcal{O}(|\lambda|^{-\ell/2}). \qquad (5.36)$$

Replacing (5.35) and (5.36) into (5.30)

$$\int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,|u_x|^2\,dx \approx \mathcal{O}(1) \qquad \text{and} \qquad \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \chi_2(x)\,|\lambda\,u|^2\,dx \approx \mathcal{O}(1). \qquad (5.37)$$

Finally, from the above estimation and the definition of $\chi_2(x)$, we obtain (5.30). The Claim 5.3 follows.

□

Claim 5.4. *Let* $\ell = 2$. *The solution* $W = (u,\,U,\,z,\,Z,\,s,\,S)^T \in \mathcal{D}(\mathcal{A})$ *of* (5.4) *satisfies the following estimates*

$$\begin{cases} \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |z_x|^2\,dx \approx \mathcal{O}(1), \\ \\ \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |\lambda\,z|^2\,dx \approx \mathcal{O}(1). \end{cases} \qquad (5.38)$$

Proof. Multiplying (5.5) by $\rho_1^{-1}\,\chi_3(x)\,\overline{z}_x$, integrating over $(\alpha + 2\,\varepsilon,\,\beta - 2\,\varepsilon)$ and using the definition of $d(x)$ together with the fact that zx_x is uniformly bounded in $L^2(0,\,L)$, $\|f_1\| \approx \mathcal{O}(1)$ and $\|f_2\| \approx \mathcal{O}(1)$ we have

$$\lambda^2\,\rho_1 \int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,u\,\overline{z}_x\,dx - \frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,(s-z-u_x)_x\,\overline{z}_x\,dx + \frac{k\,d_0}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,(S_x-U)\,\overline{z}_x\,dx \approx \mathcal{O}(|\lambda|^{-1})$$

or

$$\lambda^2\,\rho_1 \int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,u\,\overline{z}_x\,dx - \frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,s_x\,\overline{z}_x\,dx + \frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,|z_x|^2\,dx + \frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,u_{xx}\,\overline{z}_x\,dx + \frac{k\,d_0}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,(S_x-U)\,\overline{z}_x\,dx \approx \mathcal{O}(|\lambda|^{-1}).$$

Then

$$\frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,|z_x|^2\,dx \lesssim -\,\lambda^2 \int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,u\,\overline{z}_x\,dx + \frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,s_x\,\overline{z}_x\,dx - \frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,u_{xx}\,\overline{z}_x\,dx - \frac{k\,d_0}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,(S_x-U)\,\overline{z}_x\,dx + \mathcal{O}(|\lambda|^{-1}). \tag{5.39}$$

From Claim 5.1, (5.18) ($j=3$) and the fact that z_x is uniformly bounded in $L^2(0,\,L)$, it follows that ($\ell=2$)

$$\begin{cases} \frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,s_x\,\overline{z}_x\,dx \approx \mathcal{O}(1) \\ -\frac{k\,d_0}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,(S_x-U)\,\overline{z}_x\,dx \approx \mathcal{O}(|\lambda|^{-\,\ell+1}) \approx \mathcal{O}(|\lambda|^{-\,1}). \end{cases} \tag{5.40}$$

Replacing (5.40) into (5.39)

$$\frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,|z_x|^2\,dx \lesssim -\,\lambda^2 \int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,u\,\overline{z}_x\,dx \tag{5.41}$$

$$- \frac{k}{\rho_1}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,u_{xx}\,\overline{z}_x\,dx + \mathcal{O}(1). \tag{5.42}$$

On the other hand, from (5.6) and applying conjugate we have ($\ell=2$)

$$\lambda^2\,\rho_2\,\overline{z} + b\,\overline{z}_{xx} + k\,(\overline{s}-\overline{z}-\overline{u}_x) - \beta\,\overline{Z} = -\,\rho_2\,\lambda^{-\,2}\,\overline{f}_4 + i\,\rho_2\,\lambda^{-\,1}\,\overline{f}_3. \tag{5.43}$$

Multiplying (5.43) by $\rho_2^{-1}\,\chi_3\,u_x$, integrating over $(\alpha+2\,\varepsilon,\,\beta-2\,\varepsilon)$ we obtain

$$\lambda^2 \int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,\overline{z}\,u_x\,dx + \frac{b}{\rho_2}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,\overline{z}_{xx}\,u_x\,dx + \frac{k}{\rho_2}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,(\overline{s}-\overline{z}-\overline{u}_x)\,u_x\,dx - \frac{\beta}{\rho_2}\int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} \chi_3(x)\,\overline{Z}\,u_x\,dx \approx \mathcal{O}(|\lambda|^{-\,1}). \tag{5.44}$$

Integrating the first two terms and the fourth term in (5.44) we obtain

$$\begin{aligned}
&- \lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_3(x)\, u\, \overline{z}_x\, dx - \frac{b}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_3(x)\, u_{xx}\, \overline{z}_x\, dx \\
&+ \frac{\beta}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_3(x)\, u\, \overline{Z}_x\, dx \\
&\lesssim \lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_{3x}(x)\, u\, \overline{z}\, dx + \frac{b}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_{3x}(x)\, u_x\, \overline{z}_x\, dx \\
&\quad + \frac{\beta}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_{3x}(x)\, u\, \overline{Z}\, dx + \frac{k}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_3(x)\, u_x\, (\overline{s} - \overline{z} - \overline{u}_x)\, dx + \mathcal{O}(|\lambda|^{-1}).
\end{aligned} \tag{5.45}$$

From Claim 5.3, using the fact that z_x, Z and $(\overline{s}-\overline{z}-\overline{u}_x)$ are uniformly bounded in $L^2(0,\, L)$ together with $\|z\| \approx \mathcal{O}(|\lambda|^{-\ell+1})$, we get

$$\begin{cases}
\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_{3x}(x)\, u\, \overline{z}\, dx \approx \mathcal{O}(1), \\
\frac{b}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_{3x}(x)\, u_x\, \overline{z}_x\, dx \approx \mathcal{O}(1), \\
\frac{k}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_3(x)\, u_x\, (\overline{s} - \overline{z} - \overline{u}_x)\, dx \approx \mathcal{O}(1).
\end{cases} \tag{5.46}$$

Inserting (5.46) into (5.45) we get

$$- \lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_3(x)\, u\, \overline{z}_x\, dx - \frac{b}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \chi_3(x)\, u_{xx}\, \overline{z}_x\, dx \approx \mathcal{O}(1). \tag{5.47}$$

Replacing (5.47) into (5.42) and using the definition of χ_3, we obtain $(5.38)_1$. On the other hand, multiplying (5.43) by $\chi_4(x)\, z$, integrating by parts over $(\alpha+3\,\varepsilon,\, \beta-3\,\varepsilon)$, using the definition of χ_4 and the fact $\|z\| = \mathcal{O}(1)$ together with $\|f_3\| = \mathcal{O}(1)$ and $\|f_4\| = \mathcal{O}(1)$ we get

$$\begin{aligned}
\rho_2 \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \chi_4(x)\, |\lambda\, z|^2\, dx &\lesssim b \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \chi_4(x)\, |\overline{z}_x|^2\, dx + \beta \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \chi_4(x)\, Z\, z\, dx \\
&\quad - k \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \chi_4(x)\, (\overline{s} - \overline{z} - \overline{u}_x)\, z\, dx + \mathcal{O}(|\lambda|^{-2})
\end{aligned}$$

or

$$\begin{aligned}
\rho_2 \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \chi_4(x)\, |\lambda\, z|^2\, dx &\lesssim b \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \chi_4(x)\, |\overline{z}_x|^2\, dx \\
&\quad - k \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \chi_4(x)\, z\, (\overline{s} - \overline{z} - \overline{u}_x)\, dx + \mathcal{O}(|\lambda|^{-2}).
\end{aligned} \tag{5.48}$$

From (5.48), the first estimation in (5.38) and using the fact that $(\overline{s} - \overline{z} - \overline{u}_x)$ is uniformly bounded in $L^2(0,\, L)$ together with $\|z\| \approx \mathcal{O}(|\lambda|^{-1})$ we obtain

$$\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \chi_4(x)\, |\lambda\, z|^2\, dx \approx \mathcal{O}(1). \tag{5.49}$$

Thereby, from (5.49) and the definition of $\chi_4(x)$, we obtain the second estimative. □

Claim 5.5. *Let $\ell = 2$. Let $\digamma \in C^1([0, L])$ such that $\digamma(0) = \digamma(L) = 0$. The solution $W = (u, U, z, Z, s, S)^T \in \mathcal{D}(\mathcal{A})$ of (5.4) satisfies the following estimates*

$$\int_0^L \digamma_x \left[\rho_1 |\lambda u|^2 + k |u_x|^2 + \rho_2 |\lambda z|^2 + b |z_x|^2 + \rho_2 |\lambda s|^2 + 3 k |d(x) (S_x - U)|^2\right] dx \approx \mathcal{O}(1). \tag{5.50}$$

Proof. First, we multiply (5.5) by $2 \digamma \overline{u}_x$ integrating over $(0, L)$. Later we use the Claim 5.1 and then we use the fact that u_x is uniformly bounded in $L^2(0, L)$ together with $\|u\| \approx \mathcal{O}(|\lambda|^{-1})$, $\|f_1\| \approx \mathcal{O}(1)$, $\|f_1\| \approx \mathcal{O}(1)$.

$$\lambda^2 \rho_1 \int_0^L 2 \digamma u \overline{u}_x \, dx - k \int_0^L 2 \digamma (s - z - u_x)_x \overline{u}_x \, dx + k \int_0^L 2 \digamma d(x) (S_x - U) \overline{u}_x \, dx \approx \mathcal{O}(|\lambda|^{-\ell+1}).$$

Now we take the real part, that is,

$$Re\left\{\lambda^2 \rho_1 \int_0^L 2 \digamma u \overline{u}_x \, dx\right\} + Re\left\{k \int_0^L 2 \digamma u_{xx} \overline{u}_x \, dx\right\} + Re\left\{k \int_0^L 2 \digamma z_x \overline{u}_x \, dx\right\} - Re\left\{k \int_0^L 2 \digamma s_x \overline{u}_x \, dx\right\} + Re\left\{k \int_0^L 2 \digamma d(x) (S_x - U) \overline{u}_x \, dx\right\} \lesssim \mathcal{O}(|\lambda|^{-\ell+1}). \tag{5.51}$$

Using that $2\, Re(h \overline{h}_x) = (|h|^2)_x$ we have

$$\int_0^L \digamma \left(\rho_1 |\lambda u|^2 + k |u_x|^2\right)_x dx + Re\left\{2 k \int_0^L \digamma z_x \overline{u}_x \, dx\right\} - Re\left\{2 k \int_0^L \digamma s_x \overline{u}_x \, dx\right\} + \underbrace{Re\left\{2 k d_0 \int_0^L \digamma (S_x - U) \overline{u}_x \, dx\right\}}_{\approx \mathcal{O}(|\lambda|^{-\ell/2})} \lesssim \mathcal{O}(|\lambda|^{-\ell+1}). \tag{5.52}$$

Multiplying (5.6) by $2 \digamma \overline{z}_x$ and we integrate over $(0, L)$. Later we use the fact that z_x is uniformly bounded in $L^2(0, L)$ together with $\|z\| \approx \mathcal{O}(|\lambda|^{-1})$, $\|s\| \approx \mathcal{O}(|\lambda|^{-1})$, $\|f_3\| \approx \mathcal{O}(1)$ and $\|f_4\| \approx \mathcal{O}(1)$, $2\, Re(h \overline{h}_x) = (|h|^2)_x$ we have

$$\int_0^L \digamma \left(\rho_2 |\lambda z|^2 + b |z_x|^2\right)_x dx - Re\left\{2 k \int_0^L \digamma u_x \overline{z}_x \, dx\right\} + \underbrace{Re\left\{2 k \int_0^L \digamma (s - z) \overline{z}_x \, dx\right\}}_{\approx \mathcal{O}(1)} \lesssim \mathcal{O}(|\lambda|^{1-\ell}). \tag{5.53}$$

On the other hand, let $\mathcal{M} = d(x) (S_x - U)$. From Claim 5.1, the definition of $d(x)$ and the fact that s_x is uniformly in $L^2(0, L)$, we get $\mathcal{M}$ is uniformly bounded in $L^2(0, L)$. Multiplying (5.7) by $2 \digamma \overline{\mathcal{M}}$ and we integrate over $(0, L)$. Taking the real part and using the fact that $\|z\| \approx \mathcal{O}(|\lambda|^{-1})$, $\|S\| \approx \mathcal{O}(|\lambda|^{-1})$, $\|f_5\| \approx \mathcal{O}(1)$ and $\|f_6\| \approx \mathcal{O}(1)$ we obtain

$$\begin{aligned} & Re\left\{2\,\lambda^2\,\rho_2\int_0^L F\,s\,\overline{\mathcal{M}}\,dx\right\} + Re\left\{2\,b\int_0^L F\,s_{xx}\,\overline{\mathcal{M}}\,dx\right\} \\ & + Re\left\{-6\,k\int_0^L F\,(s - z - u_x)\,\overline{\mathcal{M}}\,dx\right\} \\ & + Re\left\{-6\,k\int_0^L F\,\mathcal{M}_x\,\overline{\mathcal{M}}\,dx\right\} + Re\left\{8\,\delta\int_0^L F\,S\,\overline{\mathcal{M}}\,dx\right\} \\ & \qquad = \underbrace{Re\left\{-2\int_0^L F\,(\rho_2\,\lambda^{-\ell}\,f_6 - i\,\rho_2\,\lambda^{-\ell+1}\,f_5)\right\}}_{\approx\,\mathcal{O}(|\lambda|^{-\ell+1})}. \end{aligned}$$

Using that $2\,Re(h\,\overline{h}_x) = (|h|^2)_x$ we have

$$\begin{aligned} & Re\left\{2\,\lambda^2\,\rho_2\int_0^L F\,s\,\overline{\mathcal{M}}\,dx\right\} + Re\left\{2\,b\int_0^L F\,s_{xx}\,\overline{\mathcal{M}}\,dx\right\} \\ & + Re\left\{6\,k\int_0^L F\,u_x\,\overline{\mathcal{M}}\,dx\right\} - \underbrace{Re\left\{6\,k\int_0^L F\,(s - z)\,\overline{\mathcal{M}}\,dx\right\}}_{\approx\,\mathcal{O}(1)} \\ & -\,3\,k\int_0^L F\,\left(|\mathcal{M}|^2\right)_x\,dx + Re\left\{8\,\delta\int_0^L F\,S\,\overline{\mathcal{M}}\,dx\right\} \\ & \qquad = \underbrace{Re\left\{-2\int_0^L F\,(\rho_2\,\lambda^{-\ell}\,f_6 - i\,\rho_2\,\lambda^{-\ell+1}\,f_5)\,\overline{\mathcal{M}}\,dx\right\}}_{\approx\,\mathcal{O}(|\lambda|^{-\ell+1})}. \end{aligned} \tag{5.54}$$

On the other hand, using de definition of $\mathcal{M} = d(x)\,(S_x - U)$ and $d(x)$, Claim 5.1 and the fact that u_x is uniformly bounded in $L^2(0,\,L)$, $\|s\| \approx \mathcal{O}(|\lambda|^{-1})$ we have

$$Re\left\{2\,\lambda^2\,\rho_2\int_0^L F\,s\,\overline{\mathcal{M}}\,dx\right\} = \underbrace{Re\left\{2\,\lambda^2\,\rho_2\,d_0\int_0^L F\,s\,(\overline{S}_x - \overline{U})\,dx\right\}}_{\approx\,\mathcal{O}(|\lambda|^{-\ell/2+1})} \tag{5.55}$$

and

$$Re\left\{6\,k\int_0^L F\,u_x\,\overline{\mathcal{M}}\,dx\right\} = \underbrace{Re\left\{6\,k\,d_0\int_0^L F\,u_x\,(\overline{S}_x - \overline{U})\,dx\right\}}_{\approx\,\mathcal{O}(|\lambda|^{-\ell/2})}. \tag{5.56}$$

Replacing (5.55) and (5.56) into (5.54) and using the fact $\ell = 2$ we obtain

$$3\,k\int_0^L F\,\left(|\mathcal{M}|^2\right)_x\,dx \lesssim \mathcal{O}(1). \tag{5.57}$$

Adding (5.52), (5.53) and (5.57) together with the fact that $\ell = 2$ we obtain the Claim 5.5.

□

Let $q_1,\ q_2 \in C^1([0,\, L])$ such that $0 \le q_1(x) \le 1$ and $0 \le q_2(x) \le 1$, for all $x \in [0,\, L]$ and

$$q_1(x) = \begin{cases} 1, & \text{if} \quad x \in [0,\, \gamma_1], \\ 0, & \text{if} \quad x \in [\gamma_2,\, L], \end{cases} \tag{5.58}$$

and

$$q_2(x) = \begin{cases} 0, & \text{if} \quad x \in [0,\, \gamma_1], \\ 1, & \text{if} \quad x \in [\gamma_2,\, L], \end{cases} \tag{5.59}$$

where $0 < \alpha < \gamma_1 < \gamma_2 < \beta < L$.

We consider

$$F = x\, q_1(x) + (x - L)\, q_2(x) = \begin{cases} x, & \text{if} \quad x \in [0,\, \gamma_1], \\ (x - L), & \text{if} \quad x \in [\gamma_2,\, L]. \end{cases} \tag{5.60}$$

Claim 5.6. *We take $\ell = 2$. Let $F \in C^1([0,\, L])$ such that $F(0) = F(L) = 0$. The solution $W = (u,\, U,\, z,\, Z,\, s,\, S)^T \in \mathcal{D}(\mathcal{A})$ of (5.4) satisfies the following estimates*

$$\Upsilon(\alpha + 4\,\varepsilon,\, \beta - 4\,\varepsilon) \approx \mathcal{O}(1) \quad \text{if} \quad \ell = 2, \tag{5.61}$$

where

$$\begin{aligned} \Upsilon(\gamma_1,\ \gamma_2) = & \int_0^{\gamma_1} \left(\rho_1\, |\lambda\, u|^2 + k\, |u_x|^2 + \rho_2\, |\lambda\, z|^2 + b\, |z_x|^2 + \rho_2\, |\lambda\, s|^2\right) dx \\ & + \int_{\gamma_2}^{L} \left(\rho_1\, |\lambda\, u|^2 + k\, |u_x|^2 + \rho_2\, |\lambda\, z|^2 + b\, |z_x|^2 + \rho_2\, |\lambda\, s|^2\right) dx \end{aligned} \tag{5.62}$$

for all $0 < \alpha < \gamma_1 < \gamma_2 < \beta < L$.

Proof. Taking $F = x\, q_1(x) + (x - L)\, q_2(x)$ in Claim 5.5 and using the definition of $d(x)$ with the fact that $0 < \alpha < \gamma_1 < \gamma_2 < \beta < L$ we obtain

$$\begin{aligned} & \int_0^{\gamma_1} \left(\rho_1\, |\lambda\, u|^2 + k\, |u_x|^2 + \rho_2\, |\lambda\, z|^2 + b\, |z_x|^2 + \rho_2\, |\lambda\, s|^2\right) dx \\ & + \int_{\gamma_2}^{L} \left(\rho_1\, |\lambda\, u|^2 + k\, |u_x|^2 + \rho_2\, |\lambda\, z|^2 + b\, |z_x|^2 + \rho_2\, |\lambda\, s|^2\right) dx \\ = & - \int_{\gamma_1}^{\gamma_2} (q_1 + x\, q_{1x}) \left(\rho_1\, |\lambda\, u|^2 + k\, |u_x|^2 + \rho_2\, |\lambda\, z|^2 + b\, |z_x|^2 + \rho_2\, |\lambda\, s|^2 + d_0\, |S_x - U|^2\right) dx \\ & - \int_{\gamma_1}^{\gamma_2} (q_2 + (x - L)\, q_{2x})(\rho_1\, |\lambda\, u|^2 + k\, |u_x|^2 + \rho_2\, |\lambda\, z|^2 + b\, |z_x|^2 + \rho_2\, |\lambda\, s|^2 \\ & + d_0\, |S_x - U|^2) dx + d_0 \int_{\alpha}^{\gamma_2} q_1\, |S_x - U|^2\, dx + d_0 \int_{\gamma_1}^{\beta} q_2\, |S_x - U|^2\, dx. \end{aligned} \tag{5.63}$$

Taking $\gamma_1 = \alpha + 4\,\varepsilon$ and $\gamma_2 = \beta - 4\,\varepsilon$ into (5.63), using Claim 5.1-5.4 to $\ell = 2$ we obtain (5.61).

□

Now we are in position to present the result of polynomial stability.

Theorem 5.7. *There is a constant $C > 0$ such that for every $U_0 \in \mathcal{D}(\mathcal{A})$, we have*

$$E(t) \leq \frac{C}{t}\,\|W_0\|_{\mathcal{D}(\mathcal{A})}, \quad t > 0. \tag{5.64}$$

Proof. From Claim 5.1-5.4 for $\ell = 2$ we have

$$\begin{cases} \int_\alpha^\beta |s_x|^2\,dx \approx \mathcal{O}(\lambda^{-2}) \approx \mathcal{O}(1), \quad \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |S|^2\,dx \approx \mathcal{O}(1), \quad \int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} |u_x|^2\,dx \approx \mathcal{O}(1), \\ \\ \int_{\alpha+2\,\varepsilon}^{\beta-2\,\varepsilon} |\lambda\,u|^2\,dx \approx \mathcal{O}(1), \quad \int_{\alpha+3\,\varepsilon}^{\beta-3\,\varepsilon} |z_x|^2\,dx \approx \mathcal{O}(1), \quad \int_{\alpha+4\,\varepsilon}^{\beta-4\,\varepsilon} |\lambda\,z|^2\,dx \approx \mathcal{O}(1). \end{cases}$$

Thus, from (5.61) the above expression and the fact that $0 < \varepsilon < \frac{(\beta-\alpha)}{10}$ we conclude that $\|W\|_{\mathcal{H}} \approx \mathcal{O}(1)$ in $L^2(0,\,L)$, which we have a contradiction with (5.1). This implies that

$$\sup_{\lambda\in\mathbb{R}} \left\|(i\,\lambda\,I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} = \mathcal{O}\left(\lambda^2\right). \tag{5.65}$$

Using Borichev and Tomilov [5] the theorem follows.

□

Conclusion: We have studied the stabilization for a laminated beam with interfacial slip with one discontinuous local internal viscoelastic damping of Kelvin-Voigt type acting on the axial force under certain boundary conditions. We prove the strong stability of the system by using Arendt-Batty criteria. We prove that the total energy of the system decays polynomially.

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