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CONVERGENCE OF ANALYTICAL SOLUTION OF THE INITIAL-BOUNDARY VALUE MOVING MASS PROBLEM OF BEAMS RESTING ON WINKLER FOUNDATION

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Abstract. In this paper [1], a versatile analytical technique was developed to solve the moving load problem of beams resting on elastic foundation. The technique involves the use of generalized finite integral transform, the expansion of heaviside function in series form and a modification of Struble's asymptotic technique. In this paper, the convergence of the series solution obtained is established. Thereby bring into focus the elegant feature integrating both theory and applications of this robust technique.

1. Introduction

This paper is sequel to an earlier paper [1], where an analytical solution was obtained for the problem of a uniform cantilever beam Bernoulli-Euler beam resting on a winkler elastic foundation and transversed by distributed masses. The governing equation is the fourth order partial differential equation [2,3,4,5,6] with variable and singular coefficients. The solution technique which is analytical involves using the method of generalized finite integral transform which is used to remove the singularity in the governing partial differential equation. The expression of Heaviside function as a Fourier series and the use of the modified struble's asymptotic technique [7] to solve the problem of the flexural vibrations of a Bernoulli-Euler beam under fixed and free end conditions. The purpose of the paper is to establish the convergence of the series solution of the initial-value problem. In particular, it is established that the solution so obtained is not only a formal solution, but it is the actual solution of the problem [8].

2. Methodology

The equation of motion of the beam in [1] is symbolically written in the form

$$L[\psi(x,t) - Q(x,t)] = 0, \tag{1}$$

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where L is the differential operator with variable coefficients, $\psi(x,t)$ is the beam response displacement, $Q(x,t)$ is the load acting on the beam, and x and t are spatial and time coordinates respectively.

The equation (1) is solved firstly using generalized finite integral transform defined by

$$\psi(m,t) = \int_0^L \psi_m(x,t)U(x)dx, \tag{2}$$

The inverse

$$\psi_m(m,t) = \sum_{m=1}^{\infty} \frac{\mu}{\sigma_m}\psi_m(x,t)U(x), \tag{3}$$

where
$U(x)$ is the general kernel chosen so that the clamped free end boundary conditions are satisfied and σ_m is defined as

$$\sigma_m = \int \mu U^2(x)dx, \tag{4}$$

In addition, the property of the heaviside unit step function is expressed in series representation as

$$H[x-ut] = \frac{1}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{sin(2n+1)\pi(x-ut)}{2n+1};\ 0 < x < L \tag{5}$$

where u is the velocity of the moving load. In order to simplify and solve the resulting ordinary differential equations, the modified struble's technique is resorted to. The technique requires that the asymptotic solution of the homogeneous part of (1) be of the form

$$\psi(x,t) = g(m,t)cos[\omega_{mf} - \beta(m,t)] + \eta\psi_1 + 0(\eta^2) \tag{6}$$

where $g(m,t)$ and $\beta(m,t)$ are slowly varying functions or equivalently

$$\begin{aligned} \frac{dg(m,t)}{dt} &\to 0(\eta)\ \frac{d^2g}{dt^2}(m,t) \to 0(\eta^2) \\ \frac{d\beta(m,t)}{dt} &\to 0(\eta)\ \frac{d^2\beta}{dt^2}(m,t) \to 0(\eta^2) \end{aligned} \tag{7}$$

where $\to$ implies "is of ". Expression for $\psi(x,t)$ is obtained through the method of integral transformation in conjuction with the initial condition as:

$$\begin{aligned} \bar{\psi}(x,t) = \sum_{m=1}^{\infty} &\frac{\mu pg}{y_{bj}(\alpha u_i^4 - y_{bj}^4)}\{(\alpha u_i^2 + y_{bj}^2)\{y_{bj}(cos\alpha u_i t - cos y_{bj}t) + \\ &A_m(y_{bj}sin\alpha u_i t - \alpha u_i sin y_{bj}t)\} + B_m\{y_{bj}(sinh\alpha u_i t - sin y_{bj}t)\} - \\ &c_m y_{bj}(cosh\alpha u_i t - cos y_b t) \end{aligned} \tag{8}$$

where,

$$\alpha = \frac{A_m}{L} \tag{9}$$

and p, g and A_m are the load, acceleration due to gravity and mode frequency respectively. Equation (8) represents the transverse-displacement response to a moving mass of the beam on a constant winkler's foundation for all variants of boundary conditions.

THEOREM: The series in equation (8) is convergent.

PROOF: It is noted that equation (8) can be written in the form

$$\bar{\psi}(x,t) = \sum_{m=1}^{n} \frac{\mu pg C_m^2}{y_{bj}(\alpha u_i^4 - y_{bj}^4)}\{y_{bj}\ddot{\psi}(ut) - y_{bj}^3\psi(ut) + 2\alpha u_i^2(C_m \alpha u_i sin y_{bj} t - y_{bj} cos y_{bj})\}\psi(x) \tag{10}$$

where

$$\psi(x) = sin\frac{\lambda_m x}{L} + A_m cos\frac{\lambda_m x}{L} + B_m sinh\frac{\lambda_m x}{L} + C_m cosh\frac{\lambda_m x}{L} \tag{11}$$

and

$$\psi(ut) = sin\frac{\lambda_m ut}{L} + A_m cos\frac{\lambda_m ut}{L} + B_m sinh\frac{\lambda_m ut}{L} + C_m cosh\frac{\lambda_m ut}{L} \le \sum_{m=1}^{n} |\frac{\mu pg C_m^2}{y_{bj}(\alpha u_i^4 - y_{bj}^4)}\{y_{bj}\ddot{\psi}(ut) - y_{bj}^3\psi(ut) + 2\alpha u_i^2(C_m \alpha u_i sin y_{bj} t) - y_{bj}(cos y_{bj})\}\psi(x)| \tag{12}$$

where, A_m, B_m and C_m are constants to be determined using the boundary conditions.
Since $\psi(x)$ are the normal functions for transversed vibration of the uniform beam, they are bounded [9], that is

$$|\psi(x)| \le \delta_0 < \infty \tag{13}$$

also,

$$|\psi'(x)| \le \delta_0^a < \infty \tag{14}$$

$$|\psi''(x)| \le \delta_0^b < \infty \tag{15}$$

$$|\psi'''(x)| \le \delta_0^c < \infty \tag{16}$$

Similarly, [9], $\psi(ut), \dot{\psi}(ut), \ddot{\psi}(ut), \dddot{\psi}(ut)$ are bounded and we have

$$|\psi(ut)| \le \delta_1 < \infty \tag{17}$$

$$|\dot{\psi}(ut)| \le \delta_2 < \infty \tag{18}$$

$$|\ddot{\psi}(ut)| \le \delta_3 < \infty \tag{19}$$

Consequently, equation (9) can be written as

$$|\bar{\psi}(ut)| = |\sum_{m=1}^{\infty} \frac{\mu pg C_m^2}{y_{bj}(\alpha u_i^4 - y_{bj}^4)}\{y_{bj}\ddot{\psi}(ut) - y_{bj}^3\psi(ut) + 2\alpha u_i^2(C_m \alpha u_i sin y_{bj} t - y_{bj} cos y_{bj})\}\psi(x)| \tag{20}$$

$$|\bar{\psi}(ut)| \le |\sum_{m=1}^{\infty} \frac{\mu pg C_m^2}{y_{bj}(\alpha u_i^4 - y_{bj}^4)}\{y_{bj}\ddot{\psi}(ut) - y_{bj}^3\psi(ut) + 2\alpha u_i^2(C_m \alpha u_i sin y_{bj} t - y_{bj} cos y_{bj})\}\psi(x)| \tag{21}$$

$$|\bar{\psi}(ut)| \le |\sum_{m=1}^{\infty} \frac{\mu pg C_m^2}{y_{bj}(\alpha u_i^4 - y_{bj}^4)}\{y_{bj}\delta_3 - y_{bj}^3\delta_1 + 2C_m\alpha_k^3 - 2y_{bj}\alpha u_i^2\}|\delta_0 \tag{22}$$

Considering the first series on the righthand side of the above inequality, that is

$$\lambda_0 = \sum_{m=1}^{n} \frac{\delta_1}{y_{bj}^4 - \alpha u_i^4} \tag{23}$$

To investigate the above series, we recall the following definitions

$$\alpha u_i = \frac{\lambda_i u_t}{L} \tag{24}$$

$$\eta_{mm} = \frac{2N_{aj}^2 - \lambda(F_1 H_A(r,k,m))N_{aj}^2 - \Gamma_1 H_B(r,k,m)}{2N_{aj}^2} \tag{25}$$

$$\eta_{mm} = \sigma_m[1 - \frac{\lambda}{2}(F_1 H_A(r,k,m) - \frac{\Gamma_1 H_B(r,k,m)}{N_{aj}^2})] \tag{26}$$

where

$$H_A(r,k,m) = \frac{u}{2}R_E(k,m) + \frac{2u}{\pi}\sum_{n=0}^{\infty}\frac{cos(2n+1)\pi ut}{2n+1}(R_F(r,k,m) - R_G(r,k,m)) \tag{27}$$

$$H_B(r,k,m) = \frac{u^2}{2}R_A(k,m) + \frac{u^2}{\pi}\sum_{n=0}^{\infty}\frac{cos(2n+1)\pi ut}{2n+1}(R_H(r,k,m) - R_I(r,k,m)) \tag{28}$$

But

$$\sigma_m = \omega_m S_{mm} \tag{29}$$

$$S_{mm} = \frac{1}{\sqrt{EI}N_{aj}L^2u_i^2}(EIH_A(t) - NH_B(t) + H_C(t) + H_0(t) + H_E(t)) \tag{30}$$

At this juncture, we recall that

$$\omega_m^2 = \frac{\eta_m^4 EI}{L_{\bar{m}}^4} \tag{31}$$

Where ω_m is the natural circular frequency of the free vibration of the beam and;

$$H_A(t) = \int_0^L \psi(x,t)\psi_m^{iv}(x)dx \tag{32}$$

$$H_B(t) = \int_0^L \psi_m^{iv}(x,t)\psi(x)dx \tag{33}$$

$$H_C(t) = \int_0^L H[x-ut]\ddot{\psi}(x,t)\psi_m(x)dx \tag{34}$$

$$H_D(t) = 2Mu^2\int_0^L H[x-ut]\dot{\psi}'(x,t)\psi_m(x)dx \tag{35}$$

$$H_E(t) = Mu^2\int_0^L H[x-ut]\dot{\psi}''(x,t)\psi_m(x)dx \tag{36}$$

From equation (22), it is straightforward to show that

$$\eta_{mm} = \omega_m S_{mm}\{1 - \frac{\lambda}{2}(F_1 H_A(r,k,m) - \frac{\Gamma_1 H_B(r,k,m)}{\omega_m^2 S_{mm}^2 N_{aj}^2})\} \tag{37}$$

Similarly, from (23) and (30), one obtains

$$\alpha_k^4 = \omega_k^2 S_2 k \tag{38}$$

$$S_2 k = \frac{Mu^4}{EIL^4} \tag{39}$$

Using (36) and (37), one obtains

$$\lambda_a = \sum_{m=1}^{n} \frac{\delta_3}{S_{mp}\omega_m^4 S_{2k}\omega_k^2} \tag{40}$$

where,

$$S_{mp} = S_{rp}^A\{1 - \frac{\lambda}{2}(F_1 H_A(r,k,m) - \frac{H_B(r,k,m)}{\omega_m^2 S_{rp}^2 N_{aj}^2})\} \tag{41}$$

$$|H_A(t)| \leq \int_0^L |\psi(x,t)||\psi_m^{iv}(x)|dx \leq \delta_0\delta_1 L < \infty \tag{42}$$

$$|H_B(t)| \leq \int_0^L |\psi^{iv}(x,t)||\psi_m(x)|dx \leq \delta_0\delta_1 L < \infty \tag{43}$$

$$|H_C(t)| \leq \int_0^L |H[x-ut]\ddot{\psi}(x,t)||\psi_m(x)|dx < \delta_3\delta_0 L^2 < \infty \tag{44}$$

$$|H_D(t)| \leq \int_0^L |2MUH[x-ut]\dot{\psi}'(x,t)||\psi_m(x)|dx < \delta_4\delta_0 L^3 < \infty \tag{45}$$

$$|H_E(t)| \leq \int_0^L |MU^2 H[x-ut]\psi''(x,t)||\psi_m(x)|dx < \delta_5\delta_0 L^4 < \infty \tag{46}$$

In view of the above

$$\begin{aligned} |S_{mp}| \leq \frac{u^2}{\sqrt{EI}(\delta_0 L^3 \alpha u_i^2)}&(EI\delta_2\delta_1 L - N\delta_2\delta_0 L + \delta_3\delta_0 L^2 + \delta_4\delta_0 L^3 + \\ \delta_0^2 L)^{\frac{1}{2}} < \infty& \end{aligned} \tag{47}$$

Consequently,

$$S_{mp} \leq \delta_{51}^4\{1 - \frac{\lambda}{2}(F_1\delta_0\delta_1 + \frac{\Gamma_1\delta_2\delta_0}{\omega_m^2\delta_{51}^4\delta_0^2 L})\} \leq \delta_5 < \infty. \tag{48}$$

Thus

$$|\lambda_a| \leq \sum_{m=1}^{\infty} |\frac{\delta_3}{S_{mp}\omega_m^4 - S_{2k}\omega_k^2}| \leq \sum_{m=1}^{n} |\frac{\delta_3}{\delta_5\omega_m^4}| \tag{49}$$

$$= \frac{\delta_3}{\delta_5}\sum_{m=1}^{n} |\frac{1}{\omega_m^4}| \tag{50}$$

It is noted that the natural circular frequency ω_m, the eigen frequencies of the free vibration of beam, one known to be real and hence form a countable set except possibly for a finite number of $|\omega_m|$, generally

$$|\omega_{m+1}| > |\omega_m|. \tag{51}$$

Thus

$$|\frac{1}{\omega_m}| > |\omega_{m+1}| \tag{52}$$

Firstly, the convergence of the series $\sum_m |\frac{1}{\omega_m}|$ is sought. Using the ratio test,

$$m \overset{lim}{\rightarrow} \infty \{\frac{\frac{1}{\omega_{m+1}}}{\frac{1}{\omega_m}}\} = C < 1. \tag{53}$$

Hence, $\sum_{m=1}^{\infty} |\frac{1}{\omega_m}|$ is convergent. Similarly, $\sum_{m=1}^{\infty} \frac{1}{\omega_m}$ converges absolutely. In view of this, $\sum(\frac{1}{\omega})^r, r > 1$ is also absolutely convergent since the sum, difference

and product of two absolutely convergent series is absolutely convergent [10].
For convenience, we set

$$\sum_{m=1}^{r} \frac{1}{\omega_m} = \eta_1 < \infty. \tag{54}$$

It follows that

$$|\lambda_a| \leq \frac{\delta_3}{\delta_6}\eta_1^4 \tag{55}$$

showing that λ_a is uniformly convergent.
The same argument can be applied to other series in equation (19) to obtain

$$\lambda_b = \sum_{m=1}^{n} \frac{\eta_3^2\delta_1}{y_{bj}^4 - \alpha u_i^4} \tag{56}$$

$$= \sum_{m=1}^{n} \frac{\sqrt{S}_{mp}\delta_1}{S_{mp}\omega_m^2 - S_{2m}}. \tag{57}$$

Consequently

$$\begin{aligned}
|\lambda_b| &\leq \sum_{m=1}^{n} |\frac{\sqrt{S}_{mp}\delta_1}{S_{mp}\omega_m^2 - S_{2m}}| \\
&\leq \sum_{m=1}^{n} |\frac{\delta_6\delta_1}{\delta_5\omega_m^2 - S_{2m}}|; \text{ where } \delta_6 = \sqrt{\delta_5} \\
&\leq \sum_{m=1}^{n} |\frac{\delta_6\delta_1}{\delta_5\omega_m^2 - S_{2m}}| \\
&\frac{\delta_6\delta_1}{\delta_5} \sum_{m=1}^{n} |\frac{1}{\omega_m^2}| \leq |\frac{\delta_6\delta_1}{\delta_5} n_1^2.
\end{aligned} \tag{58}$$

Similarly

$$\lambda_c = \sum_{m=1}^{n} \frac{C_m S_3}{(S_{mp})^{\frac{1}{4}}\omega_m^{\frac{1}{2}}} \tag{59}$$

where

$$S_3 = \frac{U^3}{L^3}(\frac{\bar{m}}{EI})^{\frac{3}{4}}$$

Since the values of the constant C_m in the beam functions are bounded, we set

$$|C_m| \leq \delta_7 < \infty \tag{60}$$

$$\begin{aligned}
|\lambda_c| &\leq \sum_{m=1}^{n} |\frac{C_m S_3}{(S_{mp})^{\frac{1}{4}}\omega_m^{\frac{1}{2}}(S_{1m}\omega_m^2 - S_{2m})}| \\
&\leq \sum_{m=1}^{n} |\frac{\delta_7 S_3}{\delta_5^{\frac{1}{4}}\omega_m^{\frac{1}{2}}(\delta_5\omega_m^2 - S_{2m})}| \\
&\leq \sum_{m=1}^{n} |\frac{\delta_7 S_3}{\delta_5^{\frac{5}{4}}\omega_m^{\frac{3}{2}}}| \\
&= \frac{\delta_7 S_3}{\delta_5^{\frac{5}{4}}} \sum_{m=1}^{n} \frac{1}{w^{\frac{3}{2}}}
\end{aligned}$$

$$\leq \frac{\delta_7 S_3 \eta_1^{\frac{3}{2}}}{\delta_5^{\frac{5}{4}}} \tag{61}$$

Finally, considering the last term, that is

$$\lambda_d = \sum_{m=1}^{n} \frac{1}{y_{bj}^4 - \alpha u_i^4} = \sum_{m=1}^{n} \frac{1}{S_{1m}\omega_m^4 - S_{2m}\omega_m^2} \tag{62}$$

It follows from the previous arguments that

$$\begin{aligned} |\lambda_d| &\leq \sum_{m=1}^{n} |\frac{1}{S_{1m}\omega_m^4 - S_{2m}\omega_m^2}| \\ &\leq \sum_{m=1}^{n} |\frac{1}{\delta_5 \omega_m^4} \\ &= \frac{1}{\delta_5} \sum_{m=1}^{n} |\frac{1}{\omega^4}| \\ &\leq \frac{\eta_1^4}{\delta_5} \end{aligned} \tag{63}$$

Employing the inequalities (54), (56), (60) and (62), the uniform convergence of the series in (8) is easily established. Accordingly, (8) is not only a formal solution but is the actual solution to the moving mass.

3.0 CONCLUSION

The present paper focusses on the application of the use of generalized finite integral transform method. This investigation has yielded several interesting findings concerning the employed solution method as well as the response of uniform cantilever beams resting on elastic foundation. Convergence of the closed form solution obtained for the initial-boundary value moving problem in [1] has been proved. This clearly brings into focus the elegance of the versatile technique for the class of problems in the paper. Unlike other conventional methods, an important advantage of the employed solution technique is that it is capable of solving moving beam problem [11] for various forms of classical boundary conditions.

References

[1] O.K. Ogunbamike, I.T. Awolere and A.O. Owolanke, Dynamic response of uniform cantilever beams on elastic foundation. African Journal of Mathematics and Statistics Studies, 4:1, (2021), pp. 47-62.

[2] O. Civalek and O. Kiracioglu, Free vibration analysis of Timoshenko beams by Discrete Singular Convolution Method. International Journal for Numerical Methods in Biomedical Engineering, 26:12, (2010), pp. 1890-1898.

[3] O. Civalek and A. Yavas, Large deflection statitc analysis of rectangular plates on two parameter elastic foundations. International Journal of Science and Technology, 1:1, (2006), pp. 43-50.

[4] S.M. Abdhelghany, K.M. Ewis, A.A. Mahmoud and M.M. Nassar, Dynamic response of non-uniform beam subjected to moving load and resting on non-linear viscoelastic foundation. Beni-Suef University Journal of Basic and Applied Sciences, 4:3, (2015), pp. 192-199.

[5] O. Civalek, B. Uzun, M.O. Yayli and B. Akgoz, size-dependent transverse and longitudinal vibrations of embedded carbon and silica carbide nanotubes by nonlocal finite element method. The European Physical Journal plus, 135:4, (2020), 381.

[6] J.A. Gbadeyan and S.T. Oni, Dynamic behaviour of beams and rectangular plates under moving loads. Journal of sound and Vibration. Journal of Sound and Vibration, 182:5, (1995), pp. 677-695.
[7] A.H. Nayfey, Perturbation methods. Wiley and Sons inc, 1973.
[8] S.T. Oni and O.K.Ogunbamike, Convergence of closed form solution of the initial-boundary value moving mass problem of rectangular plates resting on pasternak foundations, Nigerian Journal of Mathematics and Applications, Vol. 20(2010), pp. 93-101
[9] S. Sadiku and H.H.E. Leipholz, On the dynamics of elastic systems with moving concentrated masses. Ingenieur Archieve Vol.57(1987). pp.223-242.
[10] Adegoke Olubumo: Introduction to real analysis. Heinemann Educational Books (Nig) Limited, 1979
[11] O.K. Ogunbamike and S.T. Oni, Flexural vibration to partially distributed masses of non-uniform Rayleigh beam resting on Vlasov foundation with general boundary conditions. Journal of the Nigerian Mathematical Society, Vol.38(2019), pp 55-88.

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