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CONJECTURE OF LU, LI AND YANG CONCERNING DIFFERENTIAL MONOMIALS

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Abstract. This paper aims to prove the uniqueness result for differential monomial of an entire function and its higher order derivative sharing polynomials under suitable conditions. In this regard, the concepts of normal families are employed to obtain the result. Examples are provided to reinforce the sharpness of the conditions considered.

1. Introduction

Throughout this article, the phrase "entire function" means that the function is analytic everywhere in $\mathbb{C}$. The fundamentals of Nevanlinna theory can be read in [2, 4, 13]. The notation $E = \{x : x \in \mathbb{R}^+\}$ represents the finite linear measure. Let $\mathcal{F} = \{f : f\ is\ non-constant\ entire\ function\ in\ \mathbb{C}\}$. For $f, g \in \mathcal{F}$ and $b \in \mathbb{C} \cup \{\infty\}$, if $f - b$ and $g - b$ have the identical zeros including multiplicities then f and g share b CM (Counting Multiplicities), if the multiplicities are ignored, then f and g share b IM (Ignoring Multiplicities) and if $1/f$ and $1/g$ share 0 CM then, f and g share ∞ CM [16].

Definition 1.1. [3] *Let $E_k(b; f)$ denote the set of all b points of f. The multiplicity m of b is counted m times if $m \leq k$ and is counted $k + 1$ times if $m > k$. If $E_k(b; f) = E_k(b; g)$ then f, g share b with weight k.*

Throughout this article, the notation (b, k) denotes that f, g shares b with weight k. f, g shares $(b, 0)[(b, \infty)] \Longleftrightarrow f, g$ shares b IM[(CM)]. Let $q \in \mathbb{Z}^+$ then $N_q(r, \frac{1}{f-b})$ denotes the counting function of f whose b-points are counted with the multiplicity q, the counting function $N_{(q}(r, \frac{1}{f-b})$ of f means those b-points counted with proper multiplicity whose multiplicities are greater than or equal to q and $N_{q)}(r, \frac{1}{f-b})$ denotes the counting function of f whose b-points counted with proper multiplicity where the multiplicities are less than or equal to q. Correspondingly the reduced counting functions are given by $\overline{N}_q(r, \frac{1}{f-b}), \overline{N}_{(q}(r, \frac{1}{f-b})$ and $\overline{N}_{q)}(r, \frac{1}{f-b})$ where the multiplicities are ignored [12, 11]. For $\phi(z) \in \mathcal{F}$, if $T(r, \phi) = S(r, f)$ then ϕ is called the "small function" of f.

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Definition 1.2. [13] *Let $f \in \mathcal{F}$. The order $\rho(f)$ is defined as*

$$\rho(f) = \limsup_{r\to\infty} \frac{log^{+}T(r,f)}{log\, r}.$$

In 1920, R. Nevanlinna stated that if two entire functions f and g share five distinct values IM then the functions are identical or unique and the condition of sharing five values was inevitable. In this perspective, an entire function and its derivative sharing two values was first studied by Rubel and Yang in 1977. In fact the result obtained was as follows:

Theorem 1.1. [9] *For $f, g \in \mathcal{F}$ and $a \neq b \in \mathbb{C}$, if f and f' share a, b CM then $f \equiv g$.*

The IM counterpart to the above theorem was given by Mues and Steinmetz in 1979 [8]. Uniqueness result considering the power of an entire function was first obtained jointly by Yang and Zhang [14]. In 2009 the second author provided the improvised version by extending the sharing condition to small function as follows:

Theorem 1.2. [15] *Let $f \in \mathcal{F}$ and $n, k \in Z^{+}$. If $a(z)$ is a small function of f and suppose f^n and $(f^n)^{(k)}$ share $a(z)$ CM and $n \geq k+1$ then $f^n \equiv (f^n)^{(k)}$ and f is of the form $f = ce^{\frac{\lambda}{n}z}$ where c is a non zero complex constant and $\lambda^k = 1$.*

In 2011, Lu and Yi [5] proved that the conclusion of Theorem 1.2 was valid when the sharing condition was a polynomial as well but the function necessarily had to be transcendental.

Question 1. *What can be said about the result of Theorem 1.2 when the function and its derivative share two different polynomials instead of one?*

In view of the above question, Majumder in 2015 gave a possible answer as follows:

Theorem 1.3. [6] *Let $f \in \mathcal{F}$ be transcendental, $n, k \in Z^{+}$ and Q_1, Q_2 be non vanishing polynomials. Suppose $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM and $n \geq k+1$ then $(f^n)^{(k)}\frac{Q_2}{Q_1} \equiv f^n$ and if $Q_1 \equiv Q_2$ then f is of the form $f = ce^{\frac{\lambda}{n}z}$ where c is a non zero complex constant and $\lambda^k = 1$.*

Question 2. *It is natural to ask what happens to conclusion of Theorem 1.3 when the product of two functions and their derivatives is considered?*

In this direction, Sahoo and Biswas [10] in 2018 considered $f^nP(f) - Q_1$ and $(f^nP(f))^{(k)} - Q_2$ sharing 0 CM to obtain the uniqueness result where $P(f) = \sum a_i f^i, i = 0, 1, \ldots, m$ and $f \in \mathcal{F}$. Extending this result, Majumder [7] in 2019, introduced weighted sharing and used the concepts of normal families to prove uniqueness result. The author considered $P(f) - a_1$ and $(P(f))^{(k)} - a_2$ sharing $(0,1)$ where $f \in \mathcal{F}$ is transcendental, $a_1 = P_1e^{Q}$, $a_2 = P_2e^{Q}$ such that P_1, P_2, Q are polynomials. The main intention of this paper is to check whether the result of [7] holds good when differential monomial is considered in place of $P(f)$. As an affirmative answer, the following result is obtained.

Theorem 1.4. *Let $f(z)$ be a transcendental entire function. Let $\phi_i(z) = A_i(z)e^{B(z)}$ such that $A_i \neq 0$ for $i = 1, 2$ and $B(z)$ are polynomials. Define*
$M[f] = f^{n_0}(f')^{n_1}(f'')^{n_2}\cdots(f^{(k)})^{n_k}$ where $n_0, n_1, n_2, \cdots n_k$ are positive integers. Suppose B is non-constant and if $\rho(f) > 2\, max\{deg(B), 1 + deg(A_2) - deg(A_1)\}$, $M[f] - \phi_1$ and $(M[f])^{(k)} - \phi_2$ share (0,1) and the multiplicities of zeros of $M[f]$

are not less than $k+1$ then $(M[f])^{(k)} \equiv \dfrac{\phi_2}{\phi_1} M[f]$. In addition if $\phi_1 \equiv \phi_2$, then $f = a + Ce^{(\frac{\mu z}{\gamma})}$ where C is a non-zero complex constant, $\mu^k = 1$ and $\gamma = n_0 + n_1 + n_2 + \cdots + n_k$.

Remark 1.5. Suppose if $B(z)$ is a constant the result of Theorem 1.4 holds and the condition of $\rho(f) > 2\, max\{deg(B),\ 1 + deg(A_2) - deg(A_1)\}$ is not necessary.

We now give some examples to show that the conditions assumed in the theorem are necessary.

Example 1.6. *Let $f = z^2$ and $M[f] = f^2 f'$. Clearly $M[f]$ does not have simple zeros. For $A_1 = 11z^4 + z^5$ and $A_2 = 3z^5 + z^4$, $M' - A_2$ and $M - A_1$ share 0 CM but $M' \not\equiv \dfrac{A_2}{A_1} M$ as f is not transcendental function.*

Example 1.7. *Let $M[f] = f(f')^2$ where $f(z) = e^z + 1$ and $M[f]$ does not have simple zeros. $M' - A_2$ and $M - A_1$ shares 0 CM where $A_1 = e^{2z}$ and $A_2 = 2e^{2z}$. In this case $B(z) = e^{2z}$ and $deg(B) = 1$ which does not satisfy the condition $\rho(f) > 2\, max\{deg(B), 1 + deg(A_2) - deg(A_1)\}$ and hence $M' \not\equiv \dfrac{A_2}{A_1} M$*

Example 1.8. *Define $M[f] = f'$ where $f(z) = e^z + z$. It is clear that the zeros of $M[f]$ are simple. $M' - A_2 = e^z - 1$ and $M - A_1 = e^z - 1$ share 0 CM where $A_1 = 2$, $A_2 = 1$ and $B(z)$ is a constant. $M' = e^z \not\equiv \dfrac{A_2}{A_1} M$ as the zeros of $M[f]$ cannot be less than $k+1$*

Example 1.9. *Let $f(z) = ze^{z^2} + 2z$ and $M[f] = f'$. $M' - (4z^3 + 6z)$ and $M - (2z^2 + 3)$ share (0,1) CM where $A_1 = 2z^2 + 3$, $A_2 = 4z^3 + 6z$ and $B(z)$ is a constant. $M' \not\equiv \dfrac{A_2}{A_1} M$ which shows that the condition zeros of $M[f] \geq k+1$ is sharp.*

2. Lemmas

Lemma 2.1. [13] *For a finite order entire function g and $k \in \mathbb{Z}^+$, when $r \to \infty$,*

$$m\left(r, \frac{g^{(k)}}{g}\right) = O(log\ r).$$

Lemma 2.2. [4] *Let f be a transcendental entire function and $0 < \delta < \frac{1}{4}$. Suppose that at the point z with $|z| = r$ the inequality $|f(z)| > M(r,f)\nu(r,f)^{-\frac{1}{4}+\delta}$ holds, then there exists a set $F \subset \mathbb{R}^+$ of finite logarithmic measure, $\int_F \frac{1}{t} dt < +\infty$, such that*

$$f^{(m)}(z) = \left(\frac{\nu(r,f)}{z}\right)^m (1 + o(1)) f(z),$$

holds for all $m \geq 0$ and $r \notin F$

3. Proof of the Theorem

Let

$$F = M[f],$$

$$F^* = \frac{F}{\phi_1} \quad and \quad G^* = \frac{F^{(k)}}{\phi_2}.$$

The case when f is of infinite order can be dealt in a similar manner as in case 2 of [7]. Suppose $\rho(f) < \infty$. Since ϕ_i is a small function of f, $\rho(\phi_i) < \rho(f)$ but also $\rho(\phi_i) = \rho(B)$ hence

$$\rho(\phi_i) < \rho(f) = \rho(F) \quad for \quad i = 1,2. \tag{3.1}$$

We see that $\rho(\frac{F}{\phi_1}) \le max\{\rho(F), \rho(\phi_1)\} = \rho(F)$. From (3.1), it follows that $\rho(F) = \rho(\frac{F\phi_1}{\phi_1}) \le max\{\rho(\frac{F}{\phi_1}),\ \rho(\phi_1)\} = \rho(\frac{F}{\phi_1}) = \rho(F^*) < \infty$. Arguing in the same lines, since F is a monomial, $\rho(F) = \rho(F^{(k)}) < \infty$ and hence $\rho(G^*) < \infty$. The following two cases follow.

Case 1: Suppose $B(z)$ is a constant, clearly F^* and G^* share (1,1) except for the zeros of $\phi_i(z)$ for $i = 1,2$. Therefore $\overline{N}(r, 1; F^*) = \overline{N}(r, 1; G^*) + O(log\ r)$. Define

$$\psi = \frac{F^{*\prime}(F^* - G^*)}{F^*(F^* - 1)} = \frac{F^{*\prime}}{F^* - 1}\left(1 - \frac{A_1 F^{(k)}}{A_2 F}\right). \tag{3.2}$$

The following two cases arise.

Case 1.1: Suppose $\psi \not\equiv 0$, it is evident from (3.2) that $F^* \ne G^*$. From lemma (2.1), we see that $m(r, \infty, \psi) = O(log\ r)$. Let α be a zero of F^* of multiplicity $\left(s\left(\sum_{j=0}^{k} n_j\right) - \sum_{j=1}^{k} jn_j\right)$, $s \ge k+1$ such that $\phi_i(\alpha) \ne 0$ for $i = 1,2$. Now α is the zero of F with the same multiplicity and the zero of $F^{(k)}$ with the multiplicity $\left(s(\sum_{j=0}^{k} n_j) - \sum_{j=1}^{k} jn_j\right) - k$. From (3.2), we see that $\psi(z) = O\left(z - \alpha\right)^{s(\sum_{j=0}^{k} n_j) - (\sum_{j=1}^{k} jn_j + k + 1)}$ and $\psi(z)$ is analytic at $z = \alpha$. Let $F^* - 1$ and $G^* - 1$ have common zero say α_1 and $\phi_i(\alpha_1) \ne 0$, $i = 1,2$. Now let α_1 be a zero of $F^* - 1$ of multiplicity s_1. Since F^* and G^* share (1,1) except for the zeros of ϕ_i, $i = 1,2$, it is clear that α_1 is a zero of $G^* - 1$ of multiplicity t_1. Using Taylor's series expansion in the neighborhood of α_1 for F^* and G^*, we get

$$\begin{aligned}
F^*(z) - 1 &= a_{s_1}(z - \alpha_1)^{s_1} + a_{s_1+1}(z - \alpha_1)^{s_1+1} + \ldots, a_{s_1} \ne 0,\\
G^*(z) - 1 &= b_{t_1}(z - \alpha_1)^{t_1} + b_{t_1+1}(z - \alpha_1)^{t_1+1} + \ldots, b_{t_1} \ne 0,\\
F^{*\prime}(z) &= s_1 a_{s_1}(z - \alpha_1)^{s_1-1} + (s_1+1)a_{s_1+1}(z - \alpha_1)^{s_1} + \ldots.
\end{aligned}$$

$$F^*(z) - G^*(z) = \begin{cases} a_{s_1}(z - \alpha_1)^{s_1} + \ldots if \quad s_1 < t_1 \\ -b_{t_1}(z - \alpha_1)^{t_1} - \ldots if \quad s_1 > t_1 \\ (a_{s_1} - b_{t_1})(z - \alpha_1)^{s_1} + \ldots if \quad s_1 = t_1. \end{cases} \tag{3.3}$$

Substituting the values of (3.3) in (3.2), we get

$$\psi(z) = O((z - \alpha_1)^{m-1}), \tag{3.4}$$

where $m \ge min\{s_1, t_1\}$. Clearly (3.4) shows that $\psi(z)$ is analytic at α_1. Also by the assumption of F^* and G^*, we see that ψ has no poles. Therefore $T(r, \psi) = O(log\ r)$ and hence ψ is a rational function. From (3.4), $s_1 \ge 2$ and since F^* and G^* share $(1,1)$ except for the zeros of $\phi_i(z)$, $i = 1,2$, it follows that $t_1 \ge 2$. Therefore

$$\begin{aligned}
\overline{N}_{(2}(r, 1; F^*) &\le N(r, 0; \psi),\\
&\le T(r, \psi) + O(1),\\
&= O(log\ r) \quad as \quad r \to \infty.
\end{aligned}$$

By the hypothesis of sharing condition, we get $\overline{N}_{(2}(r, G^*) = O(log\ r)$ as $r \to \infty$. Hence $F^* - 1$ and $G^* - 1$ have multiple zeros which are finite. It follows that

$\overline{N}_{(2}(r,\phi_1;F)=\overline{N}_{(2}(r,\phi_2;F^{(k)})=O(log\ r)$ as $r\to\infty$. Now the multiple zeros of $F-\phi_1$ and $F^{(k)}-\phi_2$ are finite. In addition, $F-\phi_1$ and $F^{(k)}-\phi_2$ share (0,1),

$$\therefore\ \frac{F^{(k)}-\phi_2}{F-\phi_1}=\xi e^{\eta}, \tag{3.5}$$

where $\xi\neq 0$ is a rational function and η is a polynomial. From (3.5) we see that $\eta=(log\ \xi)\dfrac{\frac{F^{(k)}}{F}-\frac{\phi_2}{F}}{1-\frac{\phi_1}{F}}$. From lemma 2.2, we have

$$\frac{F^{(k)}(z_r)}{F(z_r)}=\left(\frac{\nu(r,F)}{z_r}\right)^k(1+o(1)), \tag{3.6}$$

possibly outside a set of finite logarithmic measure E, where $m(r,F)=|F(z_r)|$. Since $\rho(H)<\infty$, $log\ \nu(r,F)=O(log\ r)$. Also F is transcendental, $\dfrac{\phi_i}{F}\Big|_{z_r}\to 0$ as $r\to\infty, i=1,2$. Now $|\eta(z_r)|=\left|(log\frac{1}{\xi})\dfrac{\frac{F^{(k)}}{F}-\frac{\phi_2}{F}}{1-\frac{\phi_1}{F}}\right|=O(log\ r)$, for $|z_r|=r\in E$. From this we see that η is constant. Without loss of generality, we write that

$$F^{(k)}-\phi_2\equiv\xi(F-\phi_1)\quad or\quad F^{(k)}\equiv\xi F+\phi_2-\phi_1\xi \tag{3.7}$$

Case 1.1.1: Suppose F has infinitely many zeros. Let $\{z_n\}_{n=1}^{\infty}$ be the zeros of F except for the zeros of ϕ_i, $i=1,2$. Substituting in (3.7), we get $\xi(z_n)=\dfrac{\phi_2(z_n)}{\phi_1(z_n)}$ which implies $F^*\equiv G^*$ which is a contradiction.

Case 1.1.2: Suppose F has finitely many zeros then F can be written in the form $F=f^n$, $\xi=\frac{\phi_2}{\phi_1}$ and $\rho(F)<\infty$. Therefore $F(z)=A_3(z)e^{A_4(z)}$ where A_3 is a non-zero polynomial and A_4 is a non constant polynomial. Now $F^{(k)}(z)=\left(A_3(z)A_4^{(k)}(z)+A_5(z)\right)e^{A_4(z)}$, where $A_5=A_4^{(k-1)}A_3'+\overline{A}(A_3'',A_4')$ and $\overline{A}(A_3'',A_4')$ is a differential polynomial in A_3'' and A_4'. Substituting these functions in (3.7), we get $A_3(z)e^{A_4(z)}=\xi(z)A_3(z)e^{A_4(z)}+\phi_2(z)-\xi(z)\phi_1(z)$. Comparing the coefficients, we have $A_3A_4^{(k)}+A_5=\xi A_3$ and $\phi_2-\xi\phi_1\equiv 0$ or in other words $\xi=\dfrac{\phi_2}{\phi_1}$ which is a contradiction.

Case 1.2: Suppose $\psi\equiv 0$ then $F^{*\prime}\not\equiv 0$ as $F(z)$ is a transcendental entire function. Hence, $F^*=G^*$ or $(M[f])^{(k)}\equiv\dfrac{\phi_2}{\phi_1}M[f]$. In particular if $A_1\equiv A_2$ then

$$(M[f])^{(k)}\equiv M[f] \tag{3.8}$$

Let $n_{11},n_{12},\ldots,n_{1n_0}$ each with multiplicity $l_{11},l_{12},\ldots,l_{1n_0}$ respectively be the zeros of f such that $l_{11}+l_{12}+\ldots+l_{1n_0}=n_0$. Let $n_{21},n_{22},\ldots,n_{2n_1}$ be the zeros of f coming from f' each with multiplicity $l_{21},l_{22},\ldots,l_{2n_1}$ such that $l_{21}+l_{22}+\ldots+l_{2n_1}=n_1$. Proceeding in the same way, let $n_{k1},n_{k2},\ldots,n_{kn_k}$ be the zeros of f coming from $f^{(k)}$ each with multiplicity $l_{k1},l_{k2},\ldots,l_{kn_k}$ such that $l_{k1}+l_{k2}+\ldots+l_{kn_k}=n_k$.

Substituting these conditions in (3.8) becomes

$$\begin{gathered}[(f-n_{11})(f-n_{12})\dots(f-n_{1n_0})]\,[(f-n_{21})(f-n_{22})\dots(f-n_{2n_1})]\\ \dots[(f-n_{k1})(f-n_{k2})\dots(f-n_{kn_k})]=\\ ([(f-n_{11})(f-n_{12})\dots(f-n_{1n_0})]\,[(f-n_{21})(f-n_{22})\dots(f-n_{2n_1})]\\ \dots[(f-n_{k1})(f-n_{k2})\dots(f-n_{kn_k})])^{(k)}.\end{gathered}\tag{3.9}$$

Since f is an entire function, it has only one Picard exceptional value say $'a'$. Therefore (3.9) can be written as

$$\begin{aligned}(f-a)^{n_0}(f-a)^{n_1}\dots(f-a)^{n_k}&=((f-a)^{n_0}(f-a)^{n_1}\dots(f-a)^{n_k})^{(k)},\\ (f-a)^{n_0+n_1+\dots+n_k}&=\left((f-a)^{n_0+n_1+\dots+n_k}\right)^{(k)},\\ (f-a)^{\gamma}&=((f-a)^{\gamma})^{(k)},\end{aligned}\tag{3.10}$$

where $\gamma=n_0+n_1+\dots+n_k$. Since $(M[f])^{(k)}$ exists, left hand side of (3.10) does not vanish i.e.,$(f-a)^{\gamma}\not\equiv 0$. Therefore

$$f=a+Ce^{\left(\frac{\mu z}{\gamma}\right)},$$

where C is a non-zero complex constant and $\mu^k=1$.
Case 2: Suppose $B(z)$ is a polynomial whose degree ≥ 1. Let $r_1=2\,max\{deg(B),1+deg(A_2)-deg(A_1)\}\geq 2$ and $r_2=\frac{r_1-2}{2}$. Since $deg(B)\leq\rho(f)<\infty$, it can be written as $2\leq r_1<\rho(f)$ hence $0\leq r_2<\frac{\rho(f)-2}{2}$. For a small positive quantity ϵ, it can be said that $0\leq r_2<r=r_2+\epsilon<\frac{\rho(f)-2}{2}$. Replacing F,H by F^*, F respectively in (3.10) of [7] and proceeding likewise, when the multiplicities of zeros of $M[f]$ are not less than $k+1$, there is a contradiction which proves the theorem.

Open question 1. *Can the sharing condition $(0,1)$ considered be relaxed to $(0,0)$?*

Open question 2. *Keeping the sharing condition intact, can the result of Theorem 1.4 be obtained for a differential polynomial using the concepts of normal families?*

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