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ASSOCIATION SCHEME WITH PBIB DESIGNS FOR MINIMUM CO-INDEPENDENT DOMINATING SETS OF CIRCULANT GRAPHS

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Abstract. A dominating set $D \subseteq V$ is a co-independent dominating (CID)-set a graph $G = (V, E)$ if $\Delta(\langle V - D\rangle) = 0$. The co-independent domination number $\gamma_{ci}(G)$ is the minimum cardinality of a co-independent dominating set and γ_{ci}-set is a minimum co-independent dominating set of G. In this paper, we obtain the total number of γ_{ci}-sets in certain class of circulant graphs apart from strongly regular graphs which are the blocks of Partially Balanced Incomplete Block (PBIB) designs with m-association schemes for $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

1. Introduction

A graph $G = (V, E)$ be a finite, undirected, without loops or multiple edges and $p = |V|$ and $q = |E|$ are the number of vertices and edges of G respectively. An open and closed neighborhood of a vertex u of G means that $N(u) = \{v \in V(G) \colon uv \in E(G)\}$ and $N[u] = N(u) \cup \{u\}$. Any undefined graph theoretical terms and notations are not presented here can be found in ([9]).

In graph theory, circulants are very well known for a long time and circulant graphs are belong to families of cayley graphs, whose adjacency matrix is a circulant. For a given positive integer p, let s_1, s_2, ..., s_t be a sequence of integers where $0 < s_1 < s_2 < \ldots < s_t < \frac{p+1}{2}$. Then the *circulant graph* $C_p(s_1, s_2, \ldots, s_t)$ for $1 \leq h \leq t$ is the graph on p vertices v_1, v_2, ..., v_p with vertex v_h adjacent to each vertex $v_{h \pm s_l (mod\ p)}$. The values of s_t are called its *jump sizes*. For more details, we refer to ([20]).

Circulant graphs having varieties of names such as star polygon graphs, cyclic graphs, distributed loop networks, chrodal rings, multi fixed step graphs, point-symmetric graphs and diophantine structures in Russian.

In the field of computer science and discrete mathematics, application of circulant graphs are very vast, some of them are computer network designs, telecommunication networking, distributed computation, few real massively parallel processing

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systems, large area communication networks and projects and for more details one can be seen in ([16]).

Partially Balanced Incomplete Block (PBIB) designs are an important class of block designs and also, a very important class of incomplete block designs, which are improved by Bose and Nair ([1]) in 1939, these are having m associate classes and it include the class of binary, equireplicate and proper designs.

For ν vertices (objects, elements), the following conditions satisfy m classes of association scheme are given below :
(i) Two vertices are said to be m^{th} associates where $1 \leq k \leq m$, if the relation being symmetric.
(ii) Every vertex α has n_k k^{th} associates.
(iii) Two vertices are said to be k^{th} associates of x and y, if the number of vertices which are a^{th} associates of x and b^{th} associates of y is equal to p^k_{ab}. Further, k^{th} associates are independent of x and y. Thus $p^k_{ab} = p^k_{ba}$.

On ν vertices, a PBIB design with its association scheme is defined as follows.

A PBIB design contains of ν vertices and ρ sets (called blocks) of size g, $g < \nu$ such that
(i) ν vertices are contained exactly in r blocks and g distinct vertices contains in every block.
(ii) If x and y are of k^{th} associates which are exactly in λ_k blocks, where $1 \leq k \leq m$.

Applications of PBIB designs are very wide and which deserve to be mentioned. These are used to investigate the genetic properties and potentials of inbred lines or particulars in plants and animal breeding trials and one can refer ([12] and [18]).

The first kind parameters are the numbers ν, ρ, r, g, λ_1, λ_2, $\ldots$, λ_m, whereas the second kind parameters are the numbers n_1, n_2, $\ldots$, n_m, p^k_{ab} $(1 \leq a, b, k \leq m)$. For more details, we refer to ([6]).

A simple graph of order p is said to be strongly regular graph with parameters (p, l, σ, μ) if the graph is either complete or edgeless and
(i) all the vertices of the graph G are adjacent to l vertices,
(ii) if x and y are two adjacent vertices, then they are adjacent to σ vertices,
(iii) if x and y are two non adjacent vertices, then they are adjacent to μ vertices.

A subset $D \subseteq V$ is a dominating set of a graph G, if every vertex of $V - D$ is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the cardinality of minimum dominating set of G. The concept of domination theory was started since a long time, the theory of domination and its related concepts, (see [7], [10] and [11]). An independent dominating set (ID-set) D is a subset of vertices V of a graph G if $\Delta(\langle D \rangle) = 0$. An independent domination number $\gamma_i(G)$ is the minimum ID-set of G. Analogously, we initiate the domination parameter as co-independent dominating set as follows : a co-independent dominating set D is a subset of vertices V of a graph G, if $\Delta(\langle V - D \rangle) = 0$. Further, co-independent domination number $\gamma_{ci}(G)$ is the minimum CID-set of G. A γ_{ci}-set is a minimum CID-set D of G with $|D| = \gamma_{ci}(G)$.

In ([19]), Slater introduced the concept of number of dominating sets of G, which he denoted by $HED(G)$ in honor of Steve Hedetniemi. In this paper, we will use $\tau_{ci}(G)$ to denote the total number of γ_{ci}-set of a graph G. From domination and its related parameters to some certain kinds of graphs, we initiate the PBIB designs with its association schemes by us in ([2], [3], [4], [5], [14] and [15]). For more details, we refer to ([8] and [21]).

2. Preliminary results

In 2006, a domination parameter was introduced by Kulli and Janakiram ([13]), called a strong split domination. A dominating set D of V is a strong split dominating set of a graph G, if the induced subgraph $\langle V-D\rangle$ is totally disconnected with at least two vertices. Further, the cardinality of minimum strong split dominating set of vertices, called strong split domination number $\gamma_{ss}(G)$ and there is a relation between strong split dominating set and co-independent dominating set.

Theorem 2.1. [2] *Let a dominating set D be a γ_{ss}-set of G, then every γ_{ss}-set is a γ_{ci}-set if and only if $|V-D| \geq 2$.*

Also, in 2015, Nader Jafari Rad and Marcin Krzywkowski ([17]) introduced 2-outer independent domination number. A subset D of V is a 2-outer independent dominating set of a graph G, each vertex of $\langle V-D\rangle$ has minimum two neighbors in D and is independent. The cardinality of minimum set of vertices is called 2-outer independent domination number of G, $\gamma^2_{oi}(G)$ and the relation between them are given below,

Theorem 2.2. *Every 2-outer independent dominating set of a graph G is a co-independent dominating set, but the converse need not be true.*

3. $\gamma_{ci}(G)$ and $\tau_{ci}(G)$ for circulant graph

3.1. **Circulant graph $C_p(1)$.** The jump size of circulant graph is one, known as cycle C_p with $p \geq 3$ vertices. That is, $C_p(1) \cong C_p$; $p \geq 3$, except when $p = 2$. The circulants $C_4(1)$ and $C_5(1)$ are strongly regular graphs.

Theorem 3.1. *For any circulant graph $G_1 = C_p(1)$ with $p \geq 2$ vertices,*
(i) $\gamma_{ci}(G_1) = \left\lceil \dfrac{p}{2} \right\rceil$,
(ii) $\tau_{ci}(G_1) = \begin{cases} 2 & \text{if } p \text{ is even,} \\ p & \text{if } p \text{ is odd.} \end{cases}$

Proof. Let $G_1 = C_p(1)$ be a circulant graph with $p \geq 3$ vertices.
(i) The proof is due to ([13]).
(ii) Here, the following cases arise,
Case 1. If $p = 2n$; $n \geq 1$ and $\gamma_{ci}(G_1) = n$, then the γ_{ci}-set of G_1 is bounded between n times of K_1's and is fixed by the choice of the first K_1. Since there exists exactly one γ_{ci}-set G_1 containing one neglecting vertex v_1, where as in other γ_i-set of G_1 containing the vertices v_2 and v_p. Hence $\tau_{ci}(G_1) = 2$.
Case 2. If $p = 2n+1$; $n \geq 1$ and $\gamma_{ci}(G_1) = n+1$, then γ_{ci}-set of G_1 comprises of only K_1's and is fixed by the placement of the only vertex which is adjacent to $\lfloor \frac{p}{2} \rfloor$ distinct K_1's in γ_{ci}-set. Hence $\tau_{ci}(G_1) = p$. □

3.2. **Circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$.** The Circulant graph with jump size $\lfloor \frac{p}{2} \rfloor$ with $p \geq 4$ vertices, is $C_p(\lfloor \frac{p}{2} \rfloor)$. The circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$; $p = 2n$, $n \geq 1$ vertices contain n times of K_2's and they are disconnected, which are not strongly regular and connected graph is not considered in this section.

Theorem 3.2. *For any circulant graph $G_2 = C_p(\lfloor \frac{p}{2} \rfloor)$ with $p \geq 4$ vertices,*
(i) $\gamma_{ci}(G_2) = \left\lceil \dfrac{p}{2} \right\rceil$,
(ii) $\tau_{ci}(G_2) = p$.

Proof. Let $G_2 = C_p(\lfloor \frac{p}{2} \rfloor)$ be any circulant graph with $p \geq 4$ vertices.
(i) Clearly, $\gamma_{ci}(C_4(2)) = 2$ and even $p \neq 4$, any γ-set of $C_p(\lfloor \frac{p}{2} \rfloor)$ is the co-independent dominating set, so that $\gamma_{ci}(C_p(\lfloor \frac{p}{2} \rfloor)) \leq \gamma(C_p(\lfloor \frac{p}{2} \rfloor))$. Immediately, the other inequality is $\gamma_{ci}(C_p(\lfloor \frac{p}{2} \rfloor)) = \gamma(C_p(\lfloor \frac{p}{2} \rfloor))$ and hence $\gamma_{ci}(G_2) = \left\lceil \dfrac{p}{2} \right\rceil$.
(ii) For total number of minimum independent dominating sets, we have the following cases,
Case 1. If $p = 2n$; $n \geq 2$ vertices and $\gamma_{ci}(G_2) = n$, then γ_{ci}-set of G_2 is bounded between $(n+1)$ times of K_2's and they are disjoint unions. Fix exactly K_1 from each three times of K_2's and its induced subgraph becomes totally disconnected, repeat the process for remaining K_2's. Hence $\tau_{ci}(G_2) = p$.
Case 2. If $p = 2n+1$; $n \geq 2$ vertices and $\gamma_{ci}(G_2) = n+1$, then γ_{ci}-set of G_2 comprises $(2n+1)$ times of K_1's which is connected. For totally disconnected induced subgraphs, pick three times of K_1's and repeat the process. Hence $\tau_{ci}(G_2) = p$. □

3.3. **Circulant graph $C_p(1, \lfloor \frac{p}{2} \rfloor)$.** Here we consider the circulant graph only with jump sizes 1 and $\lfloor \frac{p}{2} \rfloor$; $p \geq 5$ vertices, that is, $C_p(1, \lfloor \frac{p}{2} \rfloor)$.

Theorem 3.3. *For any circulant graph $G_3 = C_p(1, \lfloor \frac{p}{2} \rfloor)$ with $p = 2n$; $n \geq 3$ vertices,*
(i) $\gamma_{ci}(G_3) = \frac{p}{2}$,
(ii) $\tau_{ci}(G_3) = 2$.

Proof. Let $G_3 = C_p(1, \lfloor \frac{p}{2} \rfloor)$ with $p = 2n$; $n \geq 3$ vertices.
(i) If D is an γ_{ci}-set of G_3, then it dominates all the vertices of $V - D$ and it is also a dominating set, which implies that $|D| = |V - D|$. Hence $\gamma_{ci}(G_3) = \frac{p}{2}$ follows.
(ii) If $\gamma_{ci}(G_3) = 2n$, then γ_{ci}-set of G_3 consists n times of K_1's, which are independent and are fixed by the choice of the first K_1, then there exists exactly one γ_{ci}-set of G_3 containing the vertex v_1 and there is one γ_{ci}-set of G_3, neglecting the vertex v_1 such as γ_{ci}-set of G_3 contains the vertex v_2 and the vertex v_p. Thus $\tau_{ci}(G_3) = 2$. □

3.4. **Circulant graph with odd jump sizes.** The circulant graph with jump size $1, 3, \ldots, \lfloor \frac{p}{2} \rfloor$ with $p \geq 6$ vertices is known as a complete bipartite graph K_{p_1,p_2} with $p_1 = p_2$, that is, $C_p(1, 3, \ldots, \lfloor \frac{p}{2} \rfloor) \cong K_{p_1,p_2}$. If the sequence of an odd jump size from 1 to $\lfloor \frac{p}{2} \rfloor$, then $C_p(1, 3, \ldots, \lfloor \frac{p}{2} \rfloor)$ is strongly regular graph.

Theorem 3.4. *For any circulant graph $G_4 = C_p(1, 3, \ldots, \lfloor \frac{p}{2} \rfloor)$ with $p = 4n - 2$; $n \geq 2$ vertices,*
(i) $\gamma_{ci}(G_4) = \frac{p}{2}$,
(ii) $\tau_{ci}(G_4) = 2$.

Proof. Let $G_4 = C_p(1, 3, \ldots, \lfloor \frac{p}{2} \rfloor)$ be any circulant graph with $p = 4n - 2$; $n \geq 2$ vertices.
(i) The proof is due to ([13]).
(ii) If $\gamma_{ci}(G_4) = 2n - 1$, then γ_{ci}-set of G_4 consists $(2n-1)$ times of K_1's, which are independent and are fixed by the choice of the first K_1, then there exists exactly one γ_{ci}-set of G_4 containing the vertex v_1 and there is one γ_{ci}-set of G_4, neglecting the vertex v_1 such as γ_{ci}-set of G_4 contains the vertex v_2 and the vertex v_p. Thus $\tau_{ci}(G_4) = 2$. □

Theorem 3.5. *For any circulant graph $G_5 = C_p(1, 3, \ldots, \lfloor \frac{p}{2} \rfloor - 1)$ with $p = 4n$; $n \geq 2$ vertices,*

(i) $\gamma_{ci}(G_5) = \frac{p}{2}$,
(ii) $\tau_{ci}(G_5) = 2$.

Proof. This proof is same as to Theorem 3.4. □

3.5. **Circulant graph with even jump sizes.** The jump size of circulant graph is $2, 4, \ldots, \lfloor \frac{p}{2} \rfloor$ is a $C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)$ with $p \geq 4$ vertices. The circulant graphs $C_5(2)$, $C_6(2)$, $C_8(2, 4)$, $C_{10}(2, 4)$, $C_{12}(2, 4, 6)$ are some of the examples of strongly regular graphs.

Theorem 3.6. *For any circulant graph* $G_6 = C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)$ *with* $p \geq 4$ *vertices,*
(i) $\gamma_{ci}(G_6) = p - 2$,
(ii) $\tau_{ci}(G_6) = p$.

Proof. Let $G_6 = C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor)$ be a circulant graph with $p = 4n$ and $4n + 1$ vertices; $n \geq 1$.
(i) Since the Circulant graph G_6 is a $(2n - 1)$-regular for $p = 4n$ or $2n$-regular for $p = 4n + 1$ vertices, $n \geq 1$. Hence the result follows.
(ii) If $\gamma_{ci}(G_6) = p - 2$, then there exist a γ_{ci}-set and are co- independent. Hence $\tau_{ci}(G_6) = p$. □

Theorem 3.7. *For any circulant graph* $G_7 = C_p(2, 4, \ldots, \lfloor \frac{p}{2} \rfloor - 1)$ *with* $p = 4n + 1$*;* $n \geq 1$*vertices,*
(i) $\gamma_{ci}(G_7) = p - 2$,
(ii) $\tau_{ci}(G_7) = p$.

Proof. This proof is same as to Theorem 3.6. □

3.6. **Circulant graph without jump size 1.** The circulant graph with jump size of $2, 3, \ldots, \lfloor \frac{p}{2} \rfloor$ is a $C_p(2, 3, \ldots, \lfloor \frac{p}{2} \rfloor)$ with $p \geq 4$ vertices. The circulant graphs of $C_4(2)$ and $C_5(2)$ are the only strongly regular graphs without jump size 1.

Theorem 3.8. *For any circulant graph* $G_8 = C_p(2, 3, \ldots, \lfloor \frac{p}{2} \rfloor)$ *with* $p \geq 4$ *vertices,*
(i) $\gamma_{ci}(G_8) = p - 2$,
(ii) $\tau_{ci}(G_8) = p$.

Proof. Let $G = G_8$ be a circulant graph with $p \geq 4$ vertices.
(i) Since the Circulant graph G_8 is a $(n + 1)$-regular with $p = 2n + 3$ or $2n + 4$ vertices, $n \geq 1$. Hence the result follows.
(ii) If $\gamma_{ci}(G_8) = n + 1$, then there exist γ_{ci}-set of G_8 comprises of only K_1's and is fixed by assigning the only vertex which is adjacent to two distinct K_1's in γ_{ci}-set of G_8. Hence $\tau_{ci}(G_8) = p$. □

3.7. **Circulant graph** $C_p(1, 2, \ldots, \lfloor \frac{p}{2} \rfloor)$**.** The jump size of circulant graph is $1, 2, \ldots, \lfloor \frac{p}{2} \rfloor$, known as complete graph K_p with $p \geq 3$ vertices, that is, $C_p\left(1, 2, \ldots, \lfloor \frac{p}{2} \rfloor\right) \cong K_p$. The complete graph K_p is strongly regular for all $p \geq 3$. The status of the trivial singleton graph K_1 is unclear. Opinions differ on K_2 is a strongly regular graph, since it has no well-defined μ parameter, it is preferable to consider as not to be a strongly regular.

Theorem 3.9. *For any circulant graph* $G_9 = C_p(1, 2, 3, \ldots, \lfloor \frac{p}{2} \rfloor)$ *with* $p \geq 3$ *vertices,*
(i) $\gamma_{ci}(G_9) = p - 1$,
(ii) $\tau_{ci}(G_9) = p$.

Proof. Let $G_9 = C_p(1, 2, 3, \dots, \lfloor \frac{p}{2} \rfloor)$ with $p \geq 3$ vertices. We have
(i) Since the Circulant graph G_9 is a $(p-1)$-regular with $p \geq 3$ vertices. Hence the result follows.
(ii) By (i), we have $\gamma_{ci}(G_9) = p - 1$ and there exist a γ_{ci}-set of G_9 comprises of $(p-1)$ times of K_1's which is adjacent to $(p-1)$-regular vertices in γ_{ci}-set of G_9. Thus $\tau_{ci}(G_9) = p$. □

4. Matrix representation of circulant graphs via Association schemes

Using the definition of association scheme with m classes and by above theorems, we construct association schemes for the graphs G_1 and G_2 as Type 2 and 3; G_3, G_4 and G_5 as Type 2; G_6, G_7 and G_8 as Type 3; G_9 as Type 1 respectively.

The following three cases of tables can be constructed and they are,

Type 1. Matrix representation of circulant graph $C_p(s_1, s_2, \dots, s_t)$ with the association scheme are as follows,

	Association scheme						
Elements	*First*	*Second*	$\cdots$	k	$\cdots$	$\frac{p-1}{2}$	$\frac{p}{2}$
v_1	v_p, v_2	v_{p-1}, v_3	$\cdots$	$v_{(p-(k-1))(mod\ p)}$, $v_{(1+k)(mod\ p)}$	$\cdots$	$v_{1+\frac{p-1}{2}}$, $v_{1+\frac{p-1}{2}+1}$	$v_{1+\frac{p}{2}}$
v_2	v_1, v_3	v_p, v_4	$\cdots$	$v_{(p-(k-2))(mod\ p)}$, $v_{(2+k)(mod\ p)}$	$\cdots$	$v_{2+\frac{p-1}{2}}$, $v_{2+\frac{p-1}{2}+1}$	$v_{2+\frac{p}{2}}$
v_3	v_2, v_4	v_1, v_5	$\cdots$	$v_{(p-(k-3))(mod\ p)}$, $v_{(3+k)(mod\ p)}$	$\cdots$	$v_{3+\frac{p-1}{2}}$, $v_{3+\frac{p-1}{2}+1}$	$v_{3+\frac{p}{2}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_a	$v_{(a-1)(mod\ p)}$, $v_{(a+1)(mod\ p)}$	$v_{(a-2)(mod\ p)}$, $v_{(a+2)(mod\ p)}$	$\cdots$	$v_{(p-(k-a))(mod\ p)}$, $v_{(a+k)(mod\ p)}$	$\cdots$	$v_{(a+\frac{p-1}{2})(mod\ p)}$, $v_{(a+\frac{p-1}{2}+1)(mod\ p)}$	$v_{(a+\frac{p}{2})(mod\ p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	v_{p-1}, v_1	v_{p-2}, v_2	$\cdots$	v_{p-k}, v_k	$\cdots$	$v_{\frac{p-1}{2}}$, $v_{\frac{p-1}{2}+1}$	$v_{\frac{p}{2}}$

Table 1. Association schemes of $C_p(s_1, s_2, \dots, s_t)$.

With the association scheme for the Table 1, the second kind parameters are given by $n_a = 2$ for $1 \leq a \leq \frac{p-1}{2}$ or $1 \leq a \leq \frac{p}{2} - 1$ and $n_{\frac{p}{2}} = 1$.
The matrices and the possible values of k are given below,

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \cdots & p_{1\ \frac{p-1}{2}}^k \\ p_{21}^k & p_{22}^k & \cdots & p_{2\ \frac{p-1}{2}}^k \\ \vdots & \vdots & \vdots & \vdots \\ p_{(\frac{p-1}{2})1}^k & p_{(\frac{p-1}{2})2}^k & \cdots & p_{(\frac{p-1}{2})\ (\frac{p-1}{2})}^k \end{pmatrix} \text{ and}$$

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \cdots & p_{1\ \frac{p}{2}}^k \\ p_{21}^k & p_{22}^k & \cdots & p_{2\ \frac{p}{2}}^k \\ \vdots & \vdots & \vdots & \vdots \\ p_{\frac{p}{2}1}^k & p_{\frac{p}{2}2}^k & \cdots & p_{\frac{p}{2}\ \frac{p}{2}}^k \end{pmatrix}$$

The possible values of k are:
For $k = 1$,
$p_{ab}^1 = 1$ for $1 \leq a \leq \frac{p-1}{2} - 1$ and $1 \leq a \leq \frac{p}{2} - 1$, $b = a + 1$

$p^1_{ab} = 1$ for $1 \leq b \leq \frac{p-1}{2} - 1$ and $1 \leq b \leq \frac{p}{2} - 1$, $a = 1 + b$
$p^1_{ab} = 1$ for $a = \frac{p-1}{2}$, $b = \frac{p-1}{2}$
For $2 \leq k \leq \frac{p-3}{2}$ and $2 \leq k \leq \frac{p}{2} - 1$,
$p^k_{ab} = 1$ for $1 \leq a \leq \frac{p-3}{2}$ and $1 \leq a \leq \frac{p}{2} - 1$, $a + b = k$, $b = k + a$ and $a + b = p - k$
$p^k_{ab} = 1$ for $1 \leq b \leq \frac{p-3}{2}$ and $1 \leq b \leq \frac{p}{2} - 1$, $a - b = k$ and $a + b = p - k$
For $k = \frac{p-1}{2}$ and $k = \frac{p}{2}$,
$p^k_{ab} = 1$, for $1 \leq a \leq \frac{p-3}{2}$, $a + b = \frac{p-1}{2}$
$p^k_{ab} = 1$, for $1 \leq a \leq \frac{p-1}{2}$, $a + b = \frac{p+1}{2}$ and
$p^k_{ab} = 2$ for $1 \leq a \leq \frac{p}{2} - 1$, $a + b = k$ and the other entries are all zero.

Type 2. Matrix representation of circulant graph $C_p(s_1, s_2, \ldots, s_t)$; $p\ (\geq 4)$ is even, with the association scheme are as follows,

Elements	Association scheme					
	First	Second	$\cdots$	k	$\cdots$	$\frac{p}{2}$
v_1	v_p, v_2	v_{p-1}, v_3	$\cdots$	$v_{(p-(k-1))(mod\ p)}$, $v_{(1+k)(mod\ p)}$	$\cdots$	$v_{1+\frac{p}{2}}$
v_2	v_1, v_3	v_p, v_4	$\cdots$	$v_{(p-(k-2))(mod\ p)}$, $v_{(2+k)(mod\ p)}$	$\cdots$	$v_{2+\frac{p}{2}}$
v_3	v_2, v_4	v_1, v_5	$\cdots$	$v_{(p-(k-3))(mod\ p)}$, $v_{(3+k)(mod\ p)}$	$\cdots$	$v_{3+\frac{p}{2}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_a	$v_{(a-1)(mod\ p)}$, $v_{(a+1)(mod\ p)}$	$v_{(a-2)(mod\ p)}$, $v_{(a+2)(mod\ p)}$	$\cdots$	$v_{(p-(k-a))(mod\ p)}$, $v_{(a+k)(mod\ p)}$	$\cdots$	$v_{(a+\frac{p}{2})(mod\ p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	v_{p-1}, v_1	v_{p-2}, v_2	$\cdots$	v_{p-k}, v_k	$\cdots$	$v_{\frac{p}{2}}$

Table 2. Association schemes of $C_p(s_1, s_2, \ldots, s_t)$; p is even.

With the association scheme for the Table 2, the second kind parameters are given by $n_a = 2$ for $1 \leq a \leq \frac{p}{2} - 1$ and $n_{\frac{p}{2}} = 1$.
The matrix and the possible values of k are given below,

$$P^k = \begin{pmatrix} p^k_{11} & p^k_{12} & \cdots & p^k_{1\ \frac{p}{2}} \\ p^k_{21} & p^k_{22} & \cdots & p^k_{2\ \frac{p}{2}} \\ \vdots & \vdots & \vdots & \vdots \\ p^k_{\frac{p}{2}1} & p^k_{\frac{p}{2}2} & \cdots & p^k_{\frac{p}{2}\ \frac{p}{2}} \end{pmatrix}$$

The possible values of k are:
For $k = 1$,
$p^1_{ab} = 1$ for $1 \leq a \leq \frac{p}{2} - 1$, $b = a + 1$
$p^1_{ab} = 1$ for $a = 1 + b, 1 \leq b \leq \frac{p}{2} - 1$
For $2 \leq k \leq \frac{p}{2} - 1$,
$p^k_{ab} = 1$ for $1 \leq b \leq \frac{p}{2} - 1$, $a + b = k$, $b - a = k$ and $a + b = p - k$
$p^k_{ab} = 1$ for $1 \leq b \leq \frac{p}{2} - 1$, $a - b = k$ and $a + b = p - k$
For $k = \frac{p}{2}$,
$p^k_{ab} = 2$ for $1 \leq a \leq \frac{p}{2} - 1$ and $a + b = k$ and the other entries are all zero.

Type 3. Matrix representation of circulant graph $C_p(s_1, s_2, \ldots, s_t)$; $p\ (\geq 3)$ is odd, with the association scheme are as follows,

Elements	Association scheme					
	First	Second	$\cdots$	k	$\cdots$	$\frac{p-1}{2}$
v_1	$v_p,\ v_2$	$v_{p-1},\ v_3$	$\cdots$	$v_{(p-(k-1))(mod\ p)}$, $v_{(1+k)(mod\ p)}$	$\cdots$	$v_{1+\frac{p-1}{2}},\ v_{1+\frac{p-1}{2}+1}$
v_2	$v_1,\ v_3$	$v_p,\ v_4$	$\cdots$	$v_{(p-(k-2))(mod\ p)}$, $v_{(2+k)(mod\ p)}$	$\cdots$	$v_{2+\frac{p-1}{2}},\ v_{2+\frac{p-1}{2}+1}$
v_3	$v_2,\ v_4$	$v_1,\ v_5$	$\cdots$	$v_{(p-(k-3))(mod\ p)}$, $v_{(3+k)(mod\ p)}$	$\cdots$	$v_{3+\frac{p-1}{2}},\ v_{3+\frac{p-1}{2}+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_a	$v_{(a-1)(mod\ p)}$, $v_{(a+1)(mod\,p)}$	$v_{(a-2)(mod\ p)}$, $v_{(a+2)(mod\,p)}$	$\cdots$	$v_{(p-(k-a))(mod\,p)}$, $v_{(a+k)(mod\,p)}$	$\cdots$	$v_{(a+\frac{p-1}{2})(mod\ p)}$, $v_{(a+\frac{p-1}{2}+1)(mod\ p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	$v_{p-1},\ v_1$	$v_{p-2},\ v_2$	$\cdots$	$v_{p-k},\ v_k$	$\cdots$	$v_{\frac{p-1}{2}},\ v_{\frac{p-1}{2}+1}$

Table 3. Association schemes of $C_p(s_1, s_2, \ldots, s_t)$; p is odd.

With the association scheme for the Table 3, the second kind parameters are given by $n_a = 2$ for $1 \leq a \leq \frac{p-1}{2}$ and $n_{\frac{p}{2}} = 1$.
The matrix and the possible values of k are given below,

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \cdots & p_{1\ \frac{p-1}{2}}^k \\ p_{21}^k & p_{22}^k & \cdots & p_{2\ \frac{p-1}{2}}^k \\ \vdots & \vdots & \vdots & \vdots \\ p_{(\frac{p-1}{2})1}^k & p_{(\frac{p-1}{2})2}^k & \cdots & p_{(\frac{p-1}{2})\ (\frac{p-1}{2})}^k \end{pmatrix}$$

The possible values of k are:
For $k = 1$,
$p_{ab}^1 = 1$ for $1 \leq a \leq \frac{p-1}{2} - 1$, $b = a + 1$
$p_{ab}^1 = 1$ for $1 \leq b \leq \frac{p-1}{2} - 1$, $a = 1 + b$
$p_{ab}^1 = 1$ for $a = \frac{p-1}{2}$, $b = \frac{p-1}{2}$
For $2 \leq k \leq \frac{p-3}{2}$,
$p_{ab}^k = 1$ for $1 \leq a \leq \frac{p-3}{2}$, $a + b = k$, $b - a = k$ and $a + b = p - k$
$p_{ab}^k = 1$ for $1 \leq b \leq \frac{p-3}{2}$, $a - b = k$ and $a + b = p - k$
For $k = \frac{p-1}{2}$,
$p_{ab}^k = 1$, for $1 \leq a \leq \frac{p-3}{2}$, $a + b = \frac{p-1}{2}$
$p_{ab}^k = 1$, for $1 \leq a \leq \frac{p-1}{2}$, $a + b = \frac{p+1}{2}$ and the other entries are all zero.

5. PBIB designs and its parameters

By considering above Theorems, Tables and the possible values of k, the parameters of PBIB designs in the following Table:

Circulant Graph		Parameters of PBIB designs					
		ν	ρ	r	g	λ_m	
G_1	$p=2n$; $n\geq 1$	p	2	1	2	0; if $m=1$	1; if $2\leq m\leq \lfloor \frac{p}{2}\rfloor$
	$p=2n+1$; $n\geq 1$	p	p	$\lceil \frac{p}{2}\rceil$	$\lceil \frac{p}{2}\rceil$	$\left\lfloor \frac{1+\sqrt{1+8\lfloor \frac{m}{2}\rfloor}}{2}\right\rfloor$; if m is odd	$\lceil \frac{p}{2}\rceil - \frac{m}{2}$; if m is even
G_2	$p=2n$; $n\geq 1$	p	p	$\lfloor \frac{p}{2}\rfloor$	$\lfloor \frac{p}{2}\rfloor$	$\lfloor \frac{p}{2}\rfloor - m$; $1\leq m\leq \lfloor \frac{p}{2}\rfloor$	
	$p=2n+1$; $n\geq 1$	p				$\lfloor \frac{p}{2}\rfloor - m$; $1\leq m\leq \lfloor \frac{p}{2}\rfloor$	
G_3		p	2	$\frac{p}{2}$	1	0; if m is odd	1; if m is even
G_4		p	2	$\frac{p}{2}$	1		
G_5		p	2	$\frac{p}{2}$	1		
G_6		p	p	$p-2$	$p-2$	$4n-3$; if $m=1$ and $n=1$	$4n-4$; $2\leq m\leq \lfloor \frac{p}{2}\rfloor$
G_7		p	p	$p-2$	$p-2$	$4n-2$; if $m=1$ and $n=1$	$4n-3$; $2\leq m\leq \lfloor \frac{p}{2}\rfloor$
G_8	$p=2n+3$; $n\geq 1$	p	p	$p-2$	$p-2$	$2p+2$; if $m=1$ and $n=1$	$2p+1$; $2\leq m\leq \lfloor \frac{p}{2}\rfloor$
	$p=2n+4$; $n\geq 1$	p				$2p+1$; if $m=1$ and $n=1$	$2p$; $2\leq m\leq \lfloor \frac{p}{2}\rfloor$
G_9		p	p	$p-1$	$p-1$	$\lfloor \frac{p}{2}\rfloor$; $1\leq m\leq \lfloor \frac{p}{2}\rfloor$	

Table 4. PBIB-designs and its parameters.

6. Conclusion

In this paper, we determine the total number of γ_{ci}-set, the partially balanced incomplete block (PBIB) designs and association schemes arising from the γ_{ci}-sets in some classes of circulant graph. Surprisingly, we obtain the total number of γ_{ci}-sets in certain class of circulant graphs apart from strongly regular graphs which are the blocks of PBIB design with at most $\lfloor \frac{p}{2}\rfloor$-association schemes.

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