Electronic Journal of Mathematical Analysis and Applications
Vol. 10(1) Jan. 2022, pp. 154-166.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

EXISTENCE OF WEAK SOLUTIONS FOR SECOND-ORDER BVPS ON THE HALF-LINE VIA CRITICAL POINT THEORY

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Abstract. By means of the minimization principle and the mountain pass theorem, we prove in this article existence results for weak solutions to the boundary value problem

$$\begin{cases} -(pu')' + qu = f(t,u), \ t > 0 \\ u(0) = u(+\infty) = 0, \end{cases}$$

where $p,\ q : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are measurable functions with $p, q > 0$ a.e. in $\mathbb{R}^+$ and $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function.

1. Introduction

In this article we will prove by means of critical point theory existence results for a class of second order boundary value problems (BVPs for short) posed on the half-line. Existence results for such a kind of problems have been the object of a large amount of reseach articles during the last three decades. Regarding the recent mathematical results for BVPs set on noncompact intervals, we refer the reader to, e.g., [1, 3, 4, 6, 8, 9, 10] and references therein.

We are concerned in this work by existence of weak solutions to the problem

$$\begin{cases} -(pu')' + qu = f(t,u), \ \text{a.e. } t > 0 \\ u(0) = u(+\infty) = 0, \end{cases} \tag{1.1}$$

where $p,\ q : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are measurable functions such that $p, q > 0$ a.e. in $\mathbb{R}^+$ and $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, that is:

(i) $f(\cdot, u)$ is measurable, for each $u \in \mathbb{R}$,

(ii) $f(t, \cdot)$ is continuous, for a.e. $t \geq 0$.

In all this work, we assume that there exists a function $\gamma \in C^1(\mathbb{R}^+)$ such that

$$\gamma > 0 \text{ a.e. in } \mathbb{R}^+ \text{ and } \frac{\gamma}{\sqrt{p}}, \frac{\gamma'}{\sqrt{q}} \in L^2(\mathbb{R}^+) \tag{1.2}$$

and

$$\begin{cases} \text{for all } R > 0, \text{ there exist } \varphi_R \in L^1(0, +\infty) \text{ such that} \\ \left| f(t, \frac{x}{\gamma(t)}) \right| \leq \gamma(t)\, \varphi_R(t) \text{ for a.e. } t > 0 \text{ and all } x \in [-R, R] \end{cases} \tag{1.3}$$

2010 *Mathematics Subject Classification.* 34B15, 34B40, 35B38.

Key words and phrases. BVPs on the half; Minimization principle; Mountain pass theorem.
Submitted Jan. 7, 2021. Revised Feb. 5, 2021.

and we set

$$\Delta = \max\left(\left|\frac{\gamma}{\sqrt{p}}\right|_2, \left|\frac{\gamma'}{\sqrt{q}}\right|_2\right), \tag{1.4}$$

where for $p \in [1, \infty]$, $|\cdot|_p$ denotes the standard norm of $L^p(\mathbb{R}^+)$.

Here for a measurable function $\omega > 0$ a.e. in $\mathbb{R}^+$ and $p \in [1, \infty)$, we denote by $L^p_\omega(\mathbb{R}^+)$ the Banach space

$$L^p_\omega\left(\mathbb{R}^+\right) = \left\{u : \mathbb{R}^+ \to \mathbb{R} : \int_0^{+\infty} \omega(s) |u(s)|^p \, ds < \infty\right\}$$

endowed with its norm $|\cdot|_{p,\omega}$ where for $u \in L^p_\omega(\mathbb{R}^+)$,

$$|u|_{p,\omega} = \left(\int_0^{+\infty} \omega(s) |u(s)|^p \, ds\right)^{1/p}.$$

This work is mainly motivated by those in [3] and [9]. The authors of [3] have considered the particular case of problem (1.1) where $p = 1$, $q \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\inf_{t>T} q(t) > 0$ for some $T > 0$ and $f(t, u) = ug(t, u)$ with g: $\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Under suitable conditions on the function g, they obtain by means the global bifurcation theory an existence result for nodal solutions to problem (1.1). Notice that under such an hypothesis, the weight q is unintegrable and may be unbounded.

In [9] is considered the case of problem (1.1) where $p = q = 1$ and $f(t, u) = \lambda m(t)h(t, u)$, with λ a real parameter, $m : \mathbb{R}^+ \to (0, +\infty)$ is integrable and h: $\mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Employing a critical point theory approach, authors of [9] obtain under suitable conditions on the function h existence results for weak solutions to the BVP (1.1).

By a weak solution to problem (1.1), we mean a function $u \in H$ such that for all $v \in H$

$$\int_0^{+\infty} p(t)u'(t)\, v'(t)dt + \int_0^{+\infty} q(t)u(t)\, v(t)dt = \int_0^{+\infty} f(t, u(t))v(t)dt,$$

where H is a Hilbert space which will be defined in Section 3. In other words, u is a weak solution to problem (1.1) if and only if u is a critical point to the Euler action functional $J : H \to \mathbb{R}$ where for $u \in H$

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t, u(t))dt,$$

and $F(t, u) = \int_0^u f(t, s)ds$.

By a classical solution to problem (1.1), we mean a function $u : \mathbb{R}^+ \to \mathbb{R}$ such that u and pu' are locally absolutely continuous on $\mathbb{R}^+$ satisfying both the differential equation and the boundary conditions in (1.1).

The first difficulty encountred when studying such BVPs consists in the lack of compacteness due to the unboundeness of the domain. Assumption (1.2) allows us to overcome this difficulty, indeed under this hypothesis, we will prove that the Hilbert space H is compactely embeded in some subspace of $C(\mathbb{R}^+)$. Notice that Hypothesis (1.2) is not in contradiction with the condition imposed on the weight q in [3]. Indeed, all the weights q satisfying $q(t) > 0$ for all $t \in \mathbb{R}^+$ and the condition imposed in [3], satisfy Assumption (1.2) with, for instance, $\gamma(t) = e^{-t}$. Hypothesis (1.3) is so important as Hypothesis (1.2), it intervenes in the proof that the functional J is well defined and is of class C^1 on the space H.

The paper is organized as follows. In Section 2, we present the theoritical background related to the critical point theory and used in the proofs of the main results of this work. Section 3 is devoted for preliminaries and in the last section we present the main results and their proofs.

2. Abstract background

First let us recall some basic concepts related to the critical point theory. Let $(X, \|\cdot\|)$ be a Banach space, X^* be its topological dual and $J : X \longrightarrow \mathbb{R}$ be a functional.

The functional J is said

(1) to be Gâteaux differentiable at $u \in X$ if there exists an operator $J'(u) \in X^*$, called the Gâteaux derivative of J at u, such that for all $v \in X$
$$\lim_{t\to 0} \frac{J(u+tv) - J(u) - \langle J'(u), v\rangle}{t} = 0,$$
(2) to be of class C^1 on X if J is Gâteaux differentiable at any $u \in X$ and $u \longrightarrow J'(u)$ is continuous. In this case, if $J'(u) = 0$ for some $u \in X$, then u is said to be a critical point of J and if J achieves a local minimum at some $v \in X$, then v is a critical point of J.

(3) to be sequentially weakly lower semi-continuous (swlsc for short) if for every sequence $(u_n) \subset X$ converging weakly to u, we have $\liminf_{n\to+\infty} J(u_n) \geq J(u)$,

(4) to satisfy Palais-Smale condition (PS for short) if $J \in C^1(X, \mathbb{R})$ and any sequence (u_n) for which $(J(u_n))$ is bounded and $J'_G(u_n) \longrightarrow 0$ as $n \to +\infty$ has a convergent subsequence.

The following two results are respectively commonly known by the minimization principle and the mountain pass theorem. They will be used to prove the main existence results of solutions to problem (1.1).

Theorem 2.1 ([5]). *Assume that X is reflexive and the functional J is*

i): *swlsc and*
ii): *coercive, that is $\forall (u_n) \subset X$, $\lim_{n\to+\infty} \|u_n\| = \infty$ implies $\lim_{n\to+\infty} J(u_n) = +\infty$.*

Then J is lower bounded on X and achieves its lower bound at some point u_0.

Theorem 2.2 ([2], [11]). *Assume that $J \in C^1(X, \mathbb{R})$,and J satisfies PS. If*

1): $J(0) = 0$,
2): *there exist $\rho,\alpha > 0$ such that $J(u) \geq \alpha$ for $\|u\| = \rho$ and*
3: *there exists $u_0 \in X$ such that $\|u_0\| > \rho$ and $J(u_0) < \alpha$,*

then J admits a critical point u_* characterized by
$$J(u_*) = \inf_{\gamma\in\Gamma} \left(\max_{t\in[0,1]} J(\gamma(t)) \right),$$
where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = u_0\}$.

3. Preliminaries

In this work, we use the following spaces

$$H = \left\{u : \mathbb{R}^+ \longrightarrow \mathbb{R} : u(0) = u(+\infty) = 0,\ \sqrt{q}u \in L^2\left(\mathbb{R}^+\right) \text{ and } \sqrt{p}u' \in L^2\left(\mathbb{R}^+\right)\right\},$$

and

$$E_\gamma = \left\{u : \mathbb{R}^+ \to \mathbb{R} : \gamma u \in C\left(\mathbb{R}^+, \mathbb{R}\right) \text{ and } \lim_{t\to+\infty} \gamma\left(t\right) u(t) = l \in \mathbb{R}\right\}.$$

Endowed with the inner product

$$(u, v) = \int_0^{+\infty} pu'v' + \int_0^{+\infty} quv,$$

and the induced norm $\|\cdot\|$, H becomes a Hilbert space.

Since $\gamma > 0$ a.e. in $\mathbb{R}^+$, $\|\cdot\|_\gamma$ where for $u \in E_\gamma$, $\|u\|_\gamma = \sup_{t\geq 0}\left(\gamma\left(t\right)|u(t)|\right)$ define a norm on E_γ and $\left(E_\gamma, \|\cdot\|_\gamma\right)$ is a Banach space.

The following lemma is an adapted version of Corduneanu's compactness criterion ([7], p. 62) for the case of the space E_γ. It will be used to show that H is compactly embedded in E_γ.

Lemma 3.1. *A nonempty subset Ω of E_γ is relatively compact if the following conditions hold:*

(a) *Ω is bounded in E_γ,*
(b) *the sets $\{u : u(t) = \gamma\left(t\right) x(t),\ x \in \Omega\}$ is locally equicontinuous on $[0, +\infty)$, and*
(c) *the sets $\{u : u(t) = \gamma\left(t\right) x(t),\ x \in \Omega\}$ is equiconvergent at $+\infty$; that is for all $\epsilon > 0$ there is $T_\epsilon > 0$ such that $|\gamma\left(t\right) u(t) - \lim_{t\to+\infty}\left(\gamma\left(t\right) u(t)\right)| < \epsilon$ for all $t > T_\epsilon$ and all $u \in \Omega$.*

Lemma 3.2. *Assume that Hypothesis (1.2) holds then the Hilbert H is compactly embedded in the Banach space E_γ.*

Proof. For all $u \in H$ and $t_2 \geq t_1 \geq 0$, we have

$$\begin{aligned}
|\gamma\left(t_2\right) u(t_2) - \gamma\left(t_1\right) u(t_1)| &= \left|\int_{t_1}^{t_2} \left(\gamma\left(s\right) u(s)\right)' ds\right| \\
&\leq \left|\int_{t_1}^{t_2} \gamma\left(s\right) u'(s) ds\right| + \left|\int_{t_1}^{t_2} \gamma'\left(s\right) u(s) ds\right| \\
&= \left|\int_{t_1}^{t_2} \frac{\gamma\left(s\right)}{\sqrt{p(s)}} \left(\sqrt{p(s)}u'(s)\right) ds\right| \\
&\quad + \left|\int_{t_1}^{t_2} \frac{\gamma'\left(s\right)}{\sqrt{q(s)}} \sqrt{q(s)}u(s) ds\right|.
\end{aligned}$$

Since by Assumption (1.2) $\frac{\gamma}{\sqrt{q}}, \frac{\gamma'}{\sqrt{q}} \in L^2(\mathbb{R}^+)$, applying Holder's inequality we obtain

$$\begin{aligned} |\gamma(t_2)\,u(t_2) - \gamma(t_1)\,u(t_1)| &= \left| \int_{t_1}^{t_2} (\gamma(s)\,u(s))'\,ds \right| && (3.1) \\ &\leq \|u\| \left(\left(\int_{t_1}^{t_2} \frac{(\gamma(s))^2}{p(s)} ds \right)^{1/2} \right. \\ &\quad \left. + \left(\int_{t_1}^{t_2} \frac{(\gamma'(s))^2}{q(s)} ds \right)^{1/2} \right). && (3.2) \end{aligned}$$

Clearly, (3.1) shows that γu is continuous on $\mathbb{R}^+$.

Now taking in (3.1) $t_1 = 0$, we obtain by letting $t_2 \to +\infty$

$$\left| \int_0^{+\infty} (\gamma(s)\,u(s))'\,ds \right| \leq \Delta \|u\| < \infty,$$

where Δ is that given in 1.4.

Hence, for all $t > 0$ we have

$$\begin{aligned} \left| \gamma(t)\,u(t) - \int_0^{+\infty} (\gamma(s)\,u(s))'\,ds \right| &= \left| \int_t^{+\infty} (\gamma(s)\,u(s))'\,ds \right| \\ &\leq \|u\| \left(\left(\int_t^{+\infty} \frac{(\gamma(s))^2}{p(s)} ds \right)^{1/2} \right. \\ &\quad \left. + \left(\int_t^{+\infty} \frac{(\gamma'(s))^2}{q(s)} ds \right)^{1/2} \right) \end{aligned}$$

leading to

$$\lim_{t \to +\infty} \left| \gamma(t)\,u(t) - \int_0^{+\infty} (\gamma(s)\,u(s))'\,ds \right| = 0,$$

that is

$$\lim_{t \to +\infty} \gamma(t)\,u(t) = \int_0^{+\infty} (\gamma(s)\,u(s))'\,ds.$$

Again, (3.1) with $t_1 = 0$ and $t_2 = t$ gives

$$|\gamma(t)\,u(t)| \leq \Delta \|u\|$$

then

$$\|u\|_\gamma = \sup_{t>0} \gamma(t)\,|u(t)| \leq \Delta \|u\|. \tag{3.3}$$

All the above show that $H \subset E_\gamma$ and the embeding of H in E_γ is continuous.

Now, we prove by means of Lemma 3.1 that the embeding of H in E_γ is compact. To this aim, let Ω be a subset bounded in H by $r > 0$. We have from (3.3) that Ω is bounded in E_γ by $r\Delta > 0$.

Let $[\xi, \eta] \subset \mathbb{R}^+$ and let $t_1, t_2 \in [\xi, \eta]$ with $t_1 < t_2$. For all $u \in \Omega$ we obtain from (3.1)

$$|\gamma(t_2)\,u(t_2) - \gamma(t_1)\,u(t_1)| \leq r \left(\left(\int_{t_1}^{t_2} \frac{(\gamma(s))^2}{p(s)} ds \right)^{1/2} + \left(\int_{t_1}^{t_2} \frac{(\gamma'(s))^2}{q(s)} ds \right)^{1/2} \right).$$

This with the uniform continuity of the mappings $t \longrightarrow \int_0^t \frac{(\gamma(s))^2}{p(s)}ds$ and $t \longrightarrow \int_0^t \frac{\left(\gamma'(s)\right)^2}{q(s)}ds$ on $[\xi, \eta]$ leads to the equicontinuity of Ω on $[\xi, \eta]$.

Moreover, for all $u \in \Omega$ we have

$$\begin{aligned} \left|\gamma(t)\,u(t) - \lim_{t\to+\infty}(\gamma(t)\,u(t))\right| &= \left|\int_t^{+\infty}(\gamma u)'(s)ds\right| \\ &\leq r\left(\left(\int_t^{+\infty}\frac{(\gamma(s))^2}{p(s)}ds\right)^{1/2}\right. \\ &\quad \left. + \left(\int_t^{+\infty}\frac{(\gamma'(s))^2}{q(s)}ds\right)^{1/2}\right) \\ &\longrightarrow 0 \text{ as } t \longrightarrow +\infty. \end{aligned}$$

Proving the equiconvergence of Ω. This ends the proof. □

Now, we will prove that the functional J introduced in the section Introduction is well defined on the Hilbert space H.

Lemma 3.3. *Assume that Hypotheses (1.2) and (1.3) hold, then the functional J is well defined on H and is continuously differentiable. Moreover, for all $u \in H$ we have*

$$\begin{aligned} (J'(u), v) &= \int_0^{+\infty} p(t)u'(t)\,v'(t)dt + \int_0^{+\infty} q(t)u(t)\,v(t)dt \\ &\quad - \int_0^{+\infty} f(t, u(t))v(t)dt. \end{aligned}$$

Proof. We have to prove that the functional

$$J_0(u) = \int_0^{+\infty} F(t, u(t))dt$$

is well defined and continuously differentiable on H.

For all $u \in H$ with $\|u\|_\gamma \leq R$, we have

$$\begin{aligned} \left|\int_0^{+\infty} F(t, u(t))dt\right| &= \left|\int_0^{+\infty}\left(\int_0^{u(t)} f(t,s)ds\right)dt\right| \\ &= \left|\int_0^{+\infty}\left(\int_0^{u(t)} f(t, \frac{\gamma(t)\,s}{\gamma(t)})ds\right)dt\right| \\ &= \left|\int_0^{+\infty}\left(\int_0^{\gamma(t)u(t)} \frac{1}{\gamma(t)} f(t, \frac{\xi}{\gamma(t)})d\xi\right)dt\right| \\ &\leq \int_0^{+\infty}\left(\int_0^{|\gamma(t)u(t)|}\left|\frac{1}{\gamma(t)} f(t, \frac{\xi}{\gamma(t)})\right|d\xi\right)dt \\ &\leq R\int_0^{+\infty}\varphi_R(t)dt < \infty. \end{aligned}$$

In order to prove the Gâteaux-differentiability, let $u, v \in H$ with $\|v\|_\gamma + \|v\|_\gamma \leq R$ for some $R > 0$ and s small. By the mean value theorem for all $t \in \mathbb{R}^+$, there is $\eta_{t,s} \in (0,1)$ such that

$$\begin{aligned} |F(t,u(t)+sv(t)) - F(t,u(t))| &= |sf(t,u(t)+\eta_{t,s}sv(t))v(t)| \\ &\leq \left| f\left(t, \frac{\gamma(t)\,(u(t)+\eta_{t,s}v(t))}{\gamma(t)}\right) v(t)\right| \\ &\leq \gamma(t)\,|v(t)|\,\varphi_R(t) \\ &\leq \|v\|_\gamma\,\varphi_R(t). \end{aligned}$$

Since $\varphi_R \in L^1(\mathbb{R}^+)$, we obtain from Lebesgue dominated convergence theorem that

$$\lim_{s\to 0}\int_0^{+\infty} (F(t,u(t)+sv(t)) - F(t,u(t)))\,dt = \int_0^{+\infty} f(t,u(t))v(t)dt,$$

that is

$$(J_0'(u), v) = \int_0^{+\infty} f(t,u(t))v(t)dt.$$

It remains to prove that J_0' is continuous. Let $(u_n) \subset H$ with $u_n \longrightarrow u$ in H. We have

$$\begin{aligned} \|J_0'(u_n) - J_0'(u)\| &= \sup_{\|v\|=1}\left|\int_0^{+\infty} (f(t,u_n(t)) - f(t,u_n(t)))\,v(t)dt\right| \\ &\leq \sup_{\|v\|=1}\|v\|_\gamma\int_0^{+\infty}\frac{1}{\gamma(t)}\left(|f(t,u_n(t)) - f(t,u_n(t))|\right)dt \\ &\leq \Delta\int_0^{+\infty}\frac{1}{\gamma(t)}\left(|f(t,u_n(t)) - f(t,u_n(t))|\right)dt. \end{aligned}$$

Since Lemma 3.2 guarantees that $u_n(t) \longrightarrow u(t)$ for all $t \in \mathbb{R}^+$, applying Lebesgue dominated convergence theorem we obtain from Hypothesis (1.3) that $J_0'(u_n) \longrightarrow J_0'(u)$ as $n \to \infty$. This ends the proof. □

The following proposition provides sufficient conditions for a weak solution to be a classical solution.

Proposition 3.4. *Assume that Hypotheses (1.2) and (1.3) hold and* $1/p,\ q \in L^1_{loc}(\mathbb{R}^+)$. *If* u *is a weak solution to problem (1.1), then* u *is a classical solution to problem (1.1).*

Proof. Let $u \in H$ be a critical point of J, then u is a weak solution to problem (1.1). Since $C_0^\infty(\mathbb{R}^+) \subset H$, for all $v \in C_0^\infty(\mathbb{R}^+)$ we have

$$\int_0^{+\infty} p(t)u'(t)\,v'(t)dt = -\int_0^{+\infty} (q(t)u(t) - f(t,u(t)))\,v(t)dt.$$

Set

$$\begin{aligned} U(t) &= q(t)u(t) - f(t,u(t)) \text{ and} \\ V(t) &= \int_0^t U(s)ds. \end{aligned}$$

Since $q \in L^1_{loc}(\mathbb{R}^+)$ and $u \in C_\gamma(\mathbb{R}^+) \subset C_\gamma(\mathbb{R}^+)$, Hypothesis (1.3) implies that $U \in L^1_{loc}(\mathbb{R}^+)$ and V is locally absolutely continuous on $\mathbb{R}^+$.

Thus, for all $v \in C_0^{\infty}(\mathbb{R}^+)$ we have

$$\begin{aligned}\int_0^{+\infty} V(t)v'(t)dt &= \int_0^{+\infty}\left(\int_0^t U(s)ds\right)v'(t)dt \\ &= \int_0^{+\infty} U(s)\left(\int_s^{+\infty} v'(t)dt\right)ds \\ &= -\int_0^{+\infty} U(s)v(s)ds \\ &= \int_0^{+\infty} p(t)u'(t)\,v'(t)dt,\end{aligned}$$

leading to

$$\int_0^{+\infty}(p(t)u'(t)-V(t))\,v'(t)dt = 0 \text{ for all } v \in C_0^{\infty}\left(\mathbb{R}^+\right)$$

then to

$$p(t)u'(t) = V(t)+\alpha = \alpha + \int_0^t (q(s)u(s) - f(s,u(s)))\,ds \text{ a.e. } t \geq 0.$$

This implies that

$$u(x) = \int_0^x \frac{\alpha}{p(t)}dt + \int_0^x \left(\frac{1}{p(t)}\int_0^t (q(s)u(s)-f(s,u(s)))\,ds\right)dt \tag{3.4}$$

and pu' is locally absolutely continuous on $\mathbb{R}^+$ with

$$(pu')'(t) = q(t)u(t) - f(t,u(t)) \text{ a.e. } t \geq 0. \tag{3.5}$$

Hence, from (3.4) and (3.5) we conclude that u is a classical solution of problem (1.1), ending the proof. □

4. Main results

The begening of this section concerns the existence of positive eigenvalues to the the linear eigenvalue problem

$$\begin{cases} -(pu')'(t) + q(t)u(t) = \mu m(t)u(t), \text{ a.e. } t > 0 \\ u(0) = u(+\infty) = 0, \end{cases} \tag{4.1}$$

where $m : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is measurable such that $m > 0$ a.e. in $\mathbb{R}^+$ and μ is a positive real parameter.

By a positive eigenvalue to the bvp (4.1), we mean a positive real value μ_1 such that there is $\phi \in H$ with $\phi > 0$ a.e. in $\mathbb{R}^+$ and the pair (μ_1, ϕ) satisfies the differential equation in (4.1).

Theorem 4.1. *Assume that Hypothesis (1.2) holds then for all measurable function $m : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $m > 0$ a.e. in $\mathbb{R}^+$ $\dfrac{m}{q} \in L^{\infty}(\mathbb{R}^+)$ and $\dfrac{\sqrt{m}}{\gamma} \in L^2(\mathbb{R}^+)$, the Problem (4.1) admits a unique positive eigenvalue $\mu_1(m)$ satisfying*

$$\mu_1(m) = \inf_{u \in H\setminus\{0\}} \frac{\|u\|^2}{|u|_{2,m}^2}.$$

Proof. Let $\Phi : H \to \mathbb{R}$ be the functional defined for $u \in H\setminus\{0\}$ by $\Phi(u) = \frac{\|u\|^2}{|u|^2_{2,m}}$. Since $\dfrac{m}{q} \in L^\infty(\mathbb{R}^+)$, for all $u \in H$ the following estimte holds:

$$|u|^2_{2,m} = \int_0^{+\infty} m(t)\,(u(t))^2\,dt \leq \left|\frac{m}{q}\right|_\infty |u|^2_{2,q} \leq \left|\frac{m}{q}\right|_\infty \|u\|^2 .$$

This shows that for all $u \in H\setminus\{0\}$,

$$\Phi(u) = \frac{\|u\|^2}{|u|^2_{2,m}} \geq \left|\frac{m}{q}\right|_\infty^{-1} > 0$$

and

$$\mu_1(m) = \inf_{u\in H\setminus\{0\}} \Phi(u) \geq \left|\frac{m}{q}\right|_\infty^{-1} > 0.$$

Let (u_n) be a minimizing sequence of Φ and $v_n = \dfrac{|u_n|}{|u_n|_{2,m}}$. We have then $\lim_{n\to\infty}\Phi(v_n) = \lim_{n\to\infty}\|v_n\|^2 = \mu_1(m)$. Since (v_n) is bounded in H reflexive, (v_n) converges weakly in H to some v then by Lemma 3.2, (v_n) converges, up to a subsequence, to v in E_γ. So, we have

$$\begin{aligned} |v_n - v|^2_{2,m} &= \int_0^{+\infty} m(t)\,(v_n(t) - v(t))^2\,dt \\ &= \int_0^{+\infty} \frac{m(t)}{\gamma^2(t)}\,(\gamma(t)\,(v_n(t) - v(t)))^2\,dt \\ &\leq \left|\frac{\sqrt{m}}{\gamma}\right|_2^2 \|v_n - v\|^2_\gamma \longrightarrow 0 \text{ as } n \to +\infty \end{aligned}$$

and

$$|v|_{2,m} = \lim_{n\to+\infty} |v_n|_{2,m} = 1.$$

At the end, the swlsc of the norm leads to

$$\mu_1(m) = \lim\inf_{n\to\infty} \Phi(v_n) = \lim\inf_{n\to\infty} \|v_n\|^2 \geq \|v\|^2 = \Phi(v) \geq \mu_1(m).$$

The lemma is proved. □

Example 4.2. *Consider the bvp (4.1) with* $p(t) = q(t) = 1+t$ *and* $m(t) = (1+t)^{-6}$. *Taking* $\gamma(t) = (1+t)^{-1}$ *we have*

$$\frac{\gamma(t)}{\sqrt{p(t)}} = \frac{1}{(1+t)^{3/2}},\ \frac{\gamma'(t)}{\sqrt{q(t)}} = \frac{-1}{(1+t)^3},$$
$$\frac{m(t)}{q(t)} = \frac{1}{(1+t)^7} \leq 1 \text{ and } \frac{\sqrt{m(t)}}{\gamma(t)} = \frac{1}{(1+t)^2}.$$

Hence, for such a case of p, q, m *the bvp (4.1) admits a positive eigenvalue.*

Theorem 4.3. *Assume that Hypothesis (1.2) holds and there exists two measurable functions* $a, b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *such that*

$$\begin{cases} a(t) > 0 \ a.e.\ t \geq 0,\ \dfrac{a}{q} \in L^\infty(\mathbb{R}^+),\ \dfrac{\sqrt{a}}{\gamma} \in L^2(\mathbb{R}^+) \\ \mu_1(a) > 1,\ \dfrac{b}{\sqrt{q}} \in L^2(\mathbb{R}^+),\ \dfrac{b}{\gamma} \in L^1(\mathbb{R}^+) \text{ and} \\ |f(t,u)| \leq a(t)\,|u| + b(t) \text{ for a.e. } t \geq 0 \text{ and all } u \in \mathbb{R}. \end{cases} \tag{4.2}$$

Then Problem (1.1) admits a weak solution.

Proof. Notice that Hypothesis (4.2) implies that the nonlinearity f satisfies Assumption (1.3) with $\varphi_R = \frac{aR}{\gamma^2} + \frac{b}{\gamma}$ for all $R > 0$. Therefore, we have by Lemma 3.3 that the functional J is well defined on H and is continuously differentiable. Furthermore, for all $u \in H$ we have

$$\begin{aligned} J(u) &\geq \frac{\|u\|^2}{2} - \frac{1}{2}\int_0^{+\infty} a(t)\,|u(t)|^2\,dt - \int_0^{+\infty} b(t)\,|u\,(t)|\,dt \\ &\geq \frac{\|u\|^2}{2}\left(1 - \frac{1}{\mu_1(a)}\right) - \left|\frac{b}{\sqrt{q}}\right|_2 \|u\| \end{aligned}$$

This together with the condition $\mu_1(a) > 1$ in Hypothesis (4.2) lead to $\lim_{n\to+\infty} J\,(u_n) = +\infty$ for all sequence (u_n) satisfying $\lim_{n\to+\infty} \|u_n\| = +\infty$; that is J is coercive.

Let us prove that J is swlsc. Let (u_n) be a sequence in H such that $u_n \rightharpoonup u$ in H, we obtain from Lemma 3.2, that up to a subsequence, (u_n) converges to u in E_γ and in particular, $u_n(t) \to u(t)$ as $n \to +\infty$ for all $t \geq 0$. Consequently, $F(t, u_n(t)) \longrightarrow F(t, u(t))$ as $n \to +\infty$ for all $t \geq 0$.

Let $r > 0$ be such that for all $n \geq 1$, $\|u_n\|_\gamma \leq r$. Since

$$\begin{aligned} |F(t,u_n(t)) - F(t,u(t))| &\leq \frac{a(t)}{2}\left(|u_n(t)|^2 + |u(t)|^2\right) + b(t)\,(|u_n(t)| + |u(t)|) \\ &\leq \frac{a(t)}{(\gamma\,(t))^2} r^2 + \frac{2b(t)}{(\gamma\,(t))} \end{aligned}$$

and

$$\int_0^{+\infty}\left(\frac{a(t)}{(\gamma\,(t))^2} r^2 + \frac{2b(t)}{(\gamma\,(t))}\right) dt < \infty,$$

Lebesgue dominated convergence theorem gives

$$\int_0^{+\infty} |F(t,u_n(t))|\,dt \longrightarrow \int_0^{+\infty} |F(t,u(t))|\,dt \text{ as } n \to +\infty.$$

This with the fact that the norm $\|\cdot\|$ is swlsc, we see that J is swlsc.

At the end, it is easy to see that J is Gâteaux differentiable. Therefore, from Lemma 2.1 we conclude that J achieves its global minimum at some $u_* \in H$ and u_* is a weak solution of problem (1.1). □

For the following particular case of Problem (1.1) where f is linear,

$$\begin{cases} -\left(pu'\right)'(t) + q(t)u(t) = \lambda\alpha(t)u\,(t) + \beta\,(t)\,, \text{ a.e. } t > 0 \\ u(0) = u(+\infty) = 0, \end{cases} \tag{4.3}$$

with λ is a real parameter and $\alpha, \beta : \mathbb{R}^+ \longrightarrow \mathbb{R}$ are measurable functions, we deduce from Theorem 4.3 the following corollary:

Corollary 4.4. *Assume that Hypothesis (1.2) holds and*

$$\begin{cases} |\alpha(t)| > 0 \ a.e. \ t \geq 0, \ \dfrac{|\alpha|}{q} \in L^\infty(\mathbb{R}^+)\,, \ \dfrac{\sqrt{|\alpha|}}{\gamma} \in L^2(\mathbb{R}^+) \\ \dfrac{|\beta|}{\sqrt{q}} \in L^2(\mathbb{R}^+) \ \text{ and } \dfrac{|\beta|}{\gamma} \in L^1(\mathbb{R}^+)\,. \end{cases}$$

Then for all $\lambda \in (-\mu_1(|\alpha|), \mu_1(|\alpha|))$ Problem (4.3) admits a weak solution.

Theorem 4.5. *Assume that Hypotheses (1.2) and (1.3) hold and the following conditions (4.4) and (4.5),*

$$\begin{cases} \text{there exists } a \in L^1(\mathbb{R}^+) \text{ such that} \\ \lim_{x\to 0} \dfrac{F(t, \frac{x}{\gamma(t)})}{a(t)\, x^2} = 0 \text{ uniformly in } t \geq 0, \end{cases} \tag{4.4}$$

$$\begin{cases} F(t,x) \geq \beta_1(t)\,|x|^\sigma - \beta_2(t) \text{ for a.e. } t > 0 \text{ and all } x \in \mathbb{R}, \\ \sigma F(t,x) \leq x f(t,x) \text{ for a.e. } t > 0 \text{ and all } x \in \mathbb{R}, \\ \text{where } \beta_1, \beta_2 \in L^1(\mathbb{R}^+) \text{ and } \sigma > 2, \end{cases} \tag{4.5}$$

are satisfied. Then Problem (1.1) admits a weak solution.

Proof. Let $\epsilon \in \left(0, \left(2\,|a|_2\,\Delta^2\right)^{-1}\right)$, it follows from Hypothesis (4.4) that there exists $\eta > 0$ such that $\left|F(t, \frac{x}{\gamma'(t)})\right| < \epsilon a(t)\,|x|^2$ for all $x \in (0,\eta)$ and all $t \geq 0$. Therefore, for all $u \in H$ with $\|u\| < \eta\Delta^{-1}$, the following estimate holds.

$$\begin{aligned} \int_0^{+\infty} |F(t,u(t))|\,dt &= \int_0^{+\infty} \left|F(t, \frac{\gamma(t)u(t)}{\gamma(t)})\right| dt \\ &\leq \epsilon \int_0^{+\infty} |a(t)|\,(\gamma(t)u(t))^2\,dt \\ &\leq \epsilon\,|a|_1\,\Delta^2\,\|u\|^2 . \end{aligned}$$

Let $\rho \in \left(0, \eta\Delta^{-1}\right)$ and $\alpha = \frac{1}{2}\left(1 - 2\epsilon\,|a|_1\,\Delta^2\right)\rho^2$. For $\|u\| = \rho$ we have

$$\begin{aligned} J(u) &= \frac{\|u\|^2}{2} - \int_0^{+\infty} F(t,u(t))dt \\ &\geq \frac{\|u\|^2}{2} - \int_0^{+\infty} |F(t,u(t))|\,dt \\ &\geq \frac{1}{2}\left(1 - 2\epsilon\,|a|_1\,\Delta^2\right)\rho^2 = \alpha. \end{aligned}$$

Condition 1) in Theorem 2.2 is satisfied.

For $v_0 \in H$ with $v_0 \neq 0$ and $s \in \mathbb{R}$, we have from (4.5)

$$\begin{aligned} J(sv_0) &= \frac{s^2\,\|v_0\|^2}{2} - \int_0^{+\infty} F(t, sv_0(t))dt \\ &\leq \frac{s^2}{2}\,\|v_0\|^2 - |s|^\sigma \int_0^{+\infty} \beta_1(t)\,|v_0(t)|^\sigma\,dt + \int_0^{+\infty} \beta_2(t)\,dt. \end{aligned}$$

Since $\sigma > 2$, $J(sv_0) \leq 0$ for $|s|$ large. Consequently, Condition 2) in Theorem 2.2 is satisfied.

At this stage, we have to prove that J satisfy PS. To this aim, let (u_n) be a sequence in H such that $(J(u_n))$ is bounded and $\lim_{n\to+\infty} J'(u_n) = 0$. Consequently, there exists $K > 0$ such that

$$\begin{aligned} K &\geq \sigma J(u_n) - (J'(u_n), u_n) \\ &= \left(\frac{\sigma}{2} - 1\right)\|u_n\|^2 - \int_0^{+\infty} \left(\sigma F(t,u_n(t)) - f(t,u_n(t))u_n(t)\right)dt \\ &\geq \left(\frac{\sigma}{2} - 1\right)\|u_n\|^2 . \end{aligned}$$

This with $\sigma > 2$ imply that the sequence (u_n) is bounded in H and so, up to a subsequence, (u_n) converges to some u in E_γ. Thus, using Lebesgue dominated convergence theorem, we obtain from Hypothesis (1.3) that

$$\lim_{n\to+\infty}\int_0^{+\infty}\left(f(t,u_n(t))-f(t,u(t))\right)\left(u_n(t)-u(t)\right)dt=0.$$

Since $\lim_{n\to+\infty} J'(u_n)=0$ and (u_n) converges weakly to some u, we get

$$\lim_{n\to+\infty}\left(J'(u_n)-J'(u),u_n-u\right)=0.$$

Taking in consideration the fact that

$$\begin{aligned}\frac{1}{2}\left\|u_n-u\right\|^2 &= \left(J'(u_n)-J'(u),u_n-u\right)= \\ & -\int_0^{+\infty}\left(f(t,u_n(t))-f(t,u(t))\right)\left(u_n(t)-u(t)\right)dt,\end{aligned}$$

we see that $\lim_{n\to+\infty}\|u_n-u\|=0$ and (u_n) converges strongly to u in H. Proving that J satisfies the PS condition. All conditions of Theorem 2.2 are then fulfilled, as a consequence, J has a critical point which is a nontrivial weak solution to Problem (1.1). □

Example 4.6. *Consider the case of Problem (1.1) where*

$$\begin{aligned}&p(t)=q(t)=1+t \text{ and} \\ &f(t,u)=\frac{|u|^{\theta-1}u}{\sqrt{t}\,(1+t)^{\xi}} \text{ with } \theta>1 \text{ and } \xi>\theta+\frac{3}{2}.\end{aligned}$$

For such a case, we have $F(t,u)=\frac{|u|^{\theta+1}}{\sqrt{t}(1+t)^{\xi}}$ and it is easy to see that Hypotheses (1.2), (1.3), (4.4) and (4.5) are satisfied with

$$\begin{aligned}&\gamma(t)=\frac{1}{1+t}, && \sigma=\theta+1, \\ &\beta_1(t)=\frac{1}{(\theta+1)\sqrt{t}\,(1+t)^{\xi}}, && \beta_2(t)=0, \\ &a(t)=\frac{1}{\sqrt{t}\,(1+t)^{\xi-\theta-1}}, \\ &\varphi_R(t)=\frac{1}{\sqrt{t}\,(1+t)^{\xi-\theta-1}} \text{ and } g_R(u)=|u|^{\theta} \text{ for all } R>0.\end{aligned}$$

Threrefore, for such a case of Problem (1.1), Theorem 4.5 guarantees existence of a weak solution.

Acknowledgement. The author is thankful to the anonymous referee for his careful reading of the manuscript and for all his comments and suggestions, which led to a substantial improvement of the original manuscript.

References

[1] R. P. Agarwal and D. O'Regan, Infinite interval problems for differential difference and integral equations, Kluwer Academic Publishers, Glasgow,2001.

[2] A. Ambrosetti and P.H. Rabinowitz, Dual variational method in critical point theory and applications, J. Funct. Anal. 14 (1973), 349-381.

[3] A. Benmezai and S. Mellal, Nodal solutions for asymptotically linear second-order BVPs on the half-line, Bull. Iran. Math. Soc., 45 (2019), 315–335.

[4] G. Bonanno and D. O'Regan, A boundary value problem on the half-line via critical point methods, Dynam. Systems Appl., 15 (2006), 395–407.

[5] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer, New-York, 2010.

[6] H. Chen, Z. He and J. Li, Multiplicity of solutions for impulsive differential equation on the half-line via variational methods, Bound. Value Probl. 14 (2016), doi:10.1186/s13661-016-0524-8.

[7] C. Corduneanu, Integral Equations and Stability of Feedback Systems, Academic Press, New York, 1973.

[8] S. Djebali, O. Saifi and S. Zehar, Singular boundary value problems with variable coefficients on the positive half-line, Electron. J. Differential Equations, 2013 (2013), No. 73, 1–18.

[9] M. Galewski, T. Moussaoui, and I. Soufi, On the existence of solutions for a boundary value problem on the half-line, Electron. J. Qual. Theory Differ. Equ. 2018 (2018), No. 12, 1–12.

[10] R. Ma and B. Zhu, Existence of positive solutions for a semipositone boundary value problem on the half line, Comput. Math. Appl., **58** (8) (2009), 1672-1686.

[11] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, CBMS Regional Conference Series in Mathematics 65, Amer. Math. Soc. Providence, RI, 1986.

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