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ON ASYMPTOTICALLY INVARIANT λ-STATISTICAL $\widetilde{\phi}$-EQUIVALENT TRIPLE SEQUENCES

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Abstract. In this manuscript, our concern is to introduce the concepts of asymptotically invariant λ-statistical $\widetilde{\phi}$-equivalent for triple sequences. Some interesting and basic properties concerning them will be studied.

1. Introduction

Fast [4] presented a generalization of the usual concept of sequential limit which their called statistical convergence. Statistical convergent sequences have been studied by Fridy [5], Gürdal et al. [9], Mursaleen and Edely [15], Nabiev et al. [16], Savaş [21], and others ([2, 3, 5, 6, 7, 8, 9, 10, 20, 22, 23]). Several authors including Mursaleen and Edely [14], Savaş and Nuray [26], Schaefer [27], and others have studied invariant convergent sequences. Also, the readers should refer to the monographs [1] and [13] for the background on the sequence spaces and related topics.

Marouf [11] gave the definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson [17] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Recently, the concept of asymptotically equivalent was generalized by Patterson and Savaş [18], Savaş [24], Savaş and Başarır [25] and Yamancı and Gürdal [28].

In this paper, we introduce the asymptotically invariant λ-statistical $\widetilde{\phi}$-equivalent which are some combinations of the definitions for asymptotically equivalent, λ-statistical limit, Orlicz function, σ-convergence and triple sequences. In addition to these definitions, natural inclusion theorems will also be presented.

2. Definitions and Notations

We now recall the following basic concepts which will be needed throughout the paper.

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The idea of the statistical convergence of sequence of real numbers is based on the notion of natural density of subsets of $\mathbb{N}$, the set of all positive integers which is defined as follows: A real number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n n^{-1} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. Here the set of all statistically convergent sequences will be abbreviated by S.

The idea of λ-statistical convergence of sequences $x = (x_k)$ of real numbers has been studied by Mursaleen [12]. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$. The generalized de la Vallèe-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$ If $\lambda_n = n$, then (V, λ)-summability reduces to $(C, 1)$-summability. We write

$$[C, 1] = \left\{ x : \exists L \in \mathbb{R},\ \lim_n n^{-1} \sum_{k=1}^{n} |x_k - L| = 0 \right\}$$

and

$$[V, 1] = \left\{ x : \exists L \in \mathbb{R},\ \lim_n \lambda_n^{-1} \sum_{k \in I_n} |x_k - L| = 0 \right\}$$

for the sets of sequences $x = (x_k)$ which are strongly Cesáro summable and strongly (V, λ)-summable to a number L.

A real number sequence $x = (x_k)$ is said to be λ-statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, S_λ-$\lim x = L$ or $x_k \to L\,(S_\lambda)$.

Remark 1 If $\lambda_n = n$, then S_λ is the same as S.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional φ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

(i) $\varphi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, ...)$ and
(iii) $\varphi\left(x_{\sigma(n)}\right) = \varphi(x_n)$ for all $x \in \ell_\infty$.

The mapping σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $n, m \in \mathbb{Z}^+$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, φ extends the limit functional on c, the space of convergent sequences, in the sense that $\varphi(x_n) = \lim x_n$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit. The space V_σ, the set of bounded sequences whose invariant means are equal, can be shown that

$$V_\sigma = \left\{ x \in \ell_\infty : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L,\ \text{uniformly in } n \right\}.$$

In [27], Schaefer proved that a bounded sequence $x = (x_k)$ of real numbers is σ-convergent to L if and only if

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} x_{\sigma^i(m)} = L,$$

uniformly in m. A sequence $x = (x_k)$ is said to be strongly σ-convergent to L if there exists a number L such that

$$\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \left|x_{\sigma^i(m)} - L\right| = 0,$$

as $k \to \infty$, uniformly in m. We write $[V_\sigma]$ as the set of all strong σ-convergent sequences. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent of all of its Banach limits coincide. The spaces of almost convergent sequences and strongly almost convergent sequences are defined respectively by

$$\widehat{c} = \left\{x \in \ell_\infty : \lim_m t_{mn}(x) \text{ exists uniformly in } n\right\}$$

and

$$[\widehat{c}] = \left\{x \in \ell_\infty : \lim_m t_{mn}(|x - le|) \text{ exists uniformly in } n \text{ for some } l \in \mathbb{C}\right\}$$

where $t_{mn}(x) = \frac{x_n + x_{n+1} + \ldots + x_{n+m}}{m+1}$ and $e = (1, 1, \ldots)$. Taking $\sigma(m) = m + 1$, we obtain $[V_\sigma] = [\widehat{c}]$.

Recently, the concept of statistical convergence for triple sequences was presented by Şahiner, Gürdal and Düden [20] as follows: A function $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a real (complex) triple sequence. A triple sequence (x_{jkl}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{jkl} - L| < \varepsilon$ whenever $j, k, l \geq n_0$.

A subset K of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to have natural density $\delta_3(K)$ if

$$\delta_3(K) = P - \lim_{n,k,l\to\infty} \frac{|K_{nkl}|}{nkl}$$

exists, where the vertical bars denote the number of (n, k, l) in K such that $p \leq n$, $q \leq k$, $r \leq l$. Then, a real triple sequence $x = (x_{pqr})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3\left(\{(n, k, l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : p \leq n, q \leq k, r \leq l,\ \ |x_{pqr} - L| \geq \varepsilon\}\right) = 0.$$

In several literary works, statistical convergence of any real sequence is identified relatively to absolute value. While we have known that the absolute value of real numbers is special of an Orlicz function [19], that is, a function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ in such a way that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\widetilde{\phi}(x) = 0 \text{ if and only if } x = 0 \text{ and } \widetilde{\phi}(x) \to \infty \text{ as } x \to \infty.$$

Further, an Orlicz function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $\triangle_2$ condition, if there exists an positive real number M such that $\widetilde{\phi}(2x) \leq M.\widetilde{\phi}(x)$ for every $x \in \mathbb{R}^+$.

Marouf [11] studied the relationships between the asymptotic equivalence of two sequences in summability theory.

Definition 1 Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Patterson [17] presented a natural combination of the concepts of statistical convergence and asymptotically equivalent to introduce the concept of asymptotically statistically equivalent as follows.

Definition 2 Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left\{ \text{the number of } k < n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} = 0$$

(denoted by $x \overset{S_L}{\sim} y$) and simply asymptotically statistical equivalent if $L = 1$.

3. Main Results

Following the above definitions and results, we aim in this section to introduce some new notions of asymptotically invariant λ-statistical $\widetilde{\phi}$-equivalent of multiple L with the use of triple sequences. In addition to these definition, natural inclusion theorems shall also be presented.

First we define the concept of λ_3-density :

Let $\lambda = (\lambda_n)$, $\mu = (\mu_m)$ and $\nu = (\nu_o)$ be a three non-decreasing sequences of positive real numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \ \lambda_1 = 1$$
$$\mu_{m+1} \leq \mu_m + 1, \ \mu_1 = 1$$

and

$$\nu_{o+1} \leq \nu_o + 1, \ \nu_1 = 1.$$

Let $K \subseteq \mathbb{N} \times \mathbb{N} \times \dot{\mathbb{N}}$. The number

$$\delta_{\lambda_3}(K) = P - \lim_{n,m,o \to \infty} \frac{1}{\lambda_{n,m,o}} \left| \{ j \in I_n, k \in J_m, l \in L_o : (j,k,l) \in K \} \right|$$

is said to the λ_3-density of K, provided the limit exists, where $\lambda_{n,m,o} = \lambda_n \mu_m \nu_o$ and $I_n = [n - \lambda_n + 1, n]$, $J_m = [m - \mu_m + 1, m]$ and $L_o = [o - \nu_o + 1, o]$ for $n, m, o = 1, 2, ...$

We now ready to define the triple λ_3-statistical $\widetilde{\phi}$-convergence.

Definition 3 Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A triple sequence $x = (x_{jkl})$ is said to be triple λ_3-statistically $\widetilde{\phi}$-convergent to L if for every $\varepsilon > 0$,

$$\lim_{n,m,o \to \infty} \frac{1}{\lambda_{n,m,o}} \left| \left\{ j \in I_n, k \in J_m, l \in L_o : \widetilde{\phi}\left(x_{jkl} - L\right) \geq \varepsilon \right\} \right| = 0.$$

In this case we write $S_{\lambda_3}\left(\widetilde{\phi}\right)$-$\lim x = L$ or $x_{jkl} \to L\left(S_{\lambda_3}\left(\widetilde{\phi}\right)\right)$. If $\lambda_{n,m,o} = nmo$, for all n, m, o, and $\widetilde{\phi}(x) = |x|$, then the notion of $S_{\lambda_3}\left(\widetilde{\phi}\right)$-convergent sequence reduces the concept of statistical convergence for triple sequences in [20].

Now we define the $S_{\sigma_3,\lambda_3}\left(\widetilde{\phi}\right)$-convergence by using σ-means and Orlicz function $\widetilde{\phi}$.

Definition 4 Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A triple sequence $x=(x_{jkl})$ is said to be triple $S_{\sigma_3,\lambda_3}\left(\widetilde{\phi}\right)$-convergent to L if for every $\varepsilon>0$,

$$\lim_{n,m,o\to\infty}\frac{1}{\lambda_{n,m,o}}\left|\left\{j\in I_n,k\in J_m,l\in L_o:\widetilde{\phi}\left(x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}-L\right)\geq\varepsilon\right\}\right|=0$$

uniformly in $p,q,\ r=1,2,....$

In this case, $S_{\sigma_3,\lambda_3}\left(\widetilde{\phi}\right)$-$\lim x=L$ or $x_{jkl}\to L\left(S_{\sigma_3,\lambda_3}\left(\widetilde{\phi}\right)\right)$.

Definition 5 Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. The two nonnegative triple sequences $x=(x_{jkl})$ and $y=(y_{jkl})$ are S_{σ_3}-asymptotically $\widetilde{\phi}$-equivalent of multiple L provided that for every $\varepsilon>0$

$$\lim_{n,m,o\to\infty}\frac{1}{nmo}\left|\{(n,m,o)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:j\leq n,k\leq m,l\leq o,\right.$$
$$\left.\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon\right\}\Bigg|=0,$$

uniformly in $p,q,r=1,2,...,$ $(x\overset{S_{\sigma_3}(\widetilde{\phi})}{\sim}y)$ and simply σ_3-asymptotically $\widetilde{\phi}$-equivalent, if $L=1$.

Definition 6 Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. The two nonnegative triple sequences $x=(x_{jkl})$ and $y=(y_{jkl})$ are S_{σ_3,λ_3}-asymptotically $\widetilde{\phi}$-equivalent of multiple L provided that for every $\varepsilon>0$

$$\lim_{n,m,o\to\infty}\frac{1}{\lambda_{n,m,o}}\left|\left\{j\in I_n,k\in J_m,l\in L_o:\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon\right\}\right|=0,$$

uniformly in $p,q,\ r=1,2,...,$ $(x\overset{S_{\sigma_3,\lambda_3}(\widetilde{\phi})}{\sim}y)$ and simply S_{σ_3,λ_3}-asymptotically $\widetilde{\phi}$-equivalent, if $L=1$.

Definition 7 Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. The two nonnegative triple sequences $x=(x_{jkl})$ and $y=(y_{jkl})$ are strongly (σ_3,λ_3)-asymptotically $\widetilde{\phi}$-equivalent of multiple L provided that for every $\varepsilon>0$

$$\lim_{n,m,o\to\infty}\frac{1}{\lambda_{n,m,o}}\sum_{j\in I_n,k\in J_m,l\in L_o}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)=0,$$

uniformly in $p,q,r=1,2,...,$ $(x\overset{[V_{\sigma_3,\lambda_3}](\widetilde{\phi})}{\sim}y)$ and simply strongly (σ_3,λ_3)-asymptotically $\widetilde{\phi}$-equivalent, if $L=1$.

Definition 8 Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A triple sequence $\{x_{jkl}\}$ is said to be $\widetilde{\phi}$-bounded if there exists $M>0$ such that $\widetilde{\phi}(x_{jkl})\leq M$ for all $j,k,l\in\mathbb{N}$. We denote the space of all bounded triple sequences by ℓ^3_∞.

Let Λ_3 denote the set of all non-decreasing sequences $\lambda=(\lambda_n)$, $\mu=(\mu_m)$ and $\nu=(\nu_o)$ of positive numbers tending to infinity and $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$; $\mu_{m+1}\leq\mu_m+1$, $\mu_1=1$ and $\nu_{o+1}\leq\nu_o+1$, $\nu_1=1$. Also, let

$$[C,1]_3^{\widetilde{\phi}}=\left\{x=(x_{jkl}):\exists L\in\mathbb{R},\ \lim_{n,m,o}\frac{1}{nmo}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{o}\widetilde{\phi}(x_{jkl}-L)=0\right\}.$$

In this case, $x_{jkl} \to L\left([C,1]_3^{\widetilde{\phi}}\right)$.

Theorem 1 Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function and $\lambda\in\Lambda_3$. Then, the following statements hold:

(i) If $x \overset{[V_{\sigma_3,\lambda_3}](\widetilde{\phi})}{\sim} y$ then $x \overset{S_{\sigma_3,\lambda_3}(\widetilde{\phi})}{\sim} y$,

(ii) If $x,y\in\ell_\infty^3$ and $x \overset{S_{\sigma_3,\lambda_3}(\widetilde{\phi})}{\sim} y$ then $x \overset{[V_{\sigma_3,\lambda_3}](\widetilde{\phi})}{\sim} y$ and hence $x \overset{[C,1]_3^{\widetilde{\phi}}}{\sim} y$,

(iii) $x \overset{S_{\sigma_3,\lambda_3}(\widetilde{\phi})}{\sim} y \cap \ell_\infty^3 = x \overset{[V_{\sigma_3,\lambda_3}](\widetilde{\phi})}{\sim} y\cap\ell_\infty^3$.

Proof. (i) Let $x \overset{[V_{\sigma_3,\lambda_3}](\widetilde{\phi})}{\sim} y$. Then, for $\varepsilon>0$, we have

$$\begin{aligned}
&\sum_{j\in I_n,k\in J_m,l\in L_o}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\\
&\geq \sum_{\substack{j\in I_n,k\in J_m,l\in L_o\\ \widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon}}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\\
&\geq \varepsilon\left|\left\{j\in I_n,k\in J_m,l\in L_o:\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon\right\}\right|
\end{aligned}$$

uniformly in $p,q,\ r=1,2,\ldots$. Hence, we have $x \overset{S_{\sigma_3,\lambda_3}(\widetilde{\phi})}{\sim} y$.

(ii) Now to prove part (ii), we suppose that $x=(x_{jkl})$ and $y=(y_{jkl})$ are in ℓ_∞^3 and $x \overset{S_{\sigma_3,\lambda_3}(\widetilde{\phi})}{\sim} y$. Then we can assume that

$$\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\leq M \text{ for all } (j,k,l) \text{ and } (p,q,r).$$

Given $\varepsilon>0$, we have

$$\begin{aligned}
&\frac{1}{\lambda_{n,m,o}}\sum_{j\in I_n,k\in J_m,l\in L_o}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\\
&=\frac{1}{\lambda_{n,m,o}}\sum_{\substack{j\in I_n,k\in J_m,l\in L_o\\ \widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon}}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\\
&+\frac{1}{\lambda_{n,m,o}}\sum_{\substack{j\in I_n,k\in J_m,l\in L_o\\ \widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)<\varepsilon}}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\\
&\leq\frac{M}{\lambda_{n,m,o}}\left|\left\{j\in I_n,k\in J_m,l\in L_o:\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\frac{\varepsilon}{2}\right\}\right|+\frac{\varepsilon}{2}.
\end{aligned}$$

Therefore, we conclude that $x \overset{[V_{\sigma_3,\lambda_3}](\widetilde{\phi})}{\sim} y$. Further, we have

$$
\begin{aligned}
&\frac{1}{nmo}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{o}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\\
&=\frac{1}{\lambda_{n,m,o}}\sum_{j=1}^{n-\lambda_n}\sum_{k=1}^{m-\mu_m}\sum_{l=1}^{o-\nu_o}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\\
&+\frac{1}{\lambda_{n,m,o}}\sum_{j\in I_n,k\in J_m,l\in L_o}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\\
&\leq\frac{2}{\lambda_{n,m,o}}\sum_{j\in I_n,k\in J_m,l\in L_o}\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right).
\end{aligned}
$$

Since $x \overset{[V_{\sigma_3,\lambda_3}](\widetilde{\phi})}{\sim} y$, so $x \overset{[C,1]_3^{\widetilde{\phi}}}{\sim} y$.

(iii) : Follows from (i) and (ii).

It is easily seen that $x \overset{S_{\sigma_3,\lambda_3}(\widetilde{\phi})}{\sim} y \subseteq x \overset{S_{\sigma_3}(\widetilde{\phi})}{\sim} y$ for all $\lambda_{n,m,o}$, since $\frac{\lambda_{n,m,o}}{nmo}$ is bounded by 1. Therefore, we prove the following relation.

Theorem 2 Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. If the two nonnegative triple sequences $x=(x_{jkl})$ and $y=(y_{jkl})$ are S_{σ_3}-asymptotically $\widetilde{\phi}$-equivalent of multiple L along with $\liminf\frac{\lambda_{n,m,o}}{nmo}>0$, then triple sequences $x=(x_{jkl})$ and $y=(y_{jkl})$ are S_{σ_3,λ_3}-asymptotically $\widetilde{\phi}$-equivalent of multiple L.

Proof. For given $\varepsilon>0$, we see that

$$
\begin{aligned}
&\left\{j\in I_n,k\in J_m,l\in L_o:\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon\right\}\\
&\subset\left\{(n,m,o)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:j\leq n,k\leq m,l\leq o,\ \widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon\right\}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\frac{1}{nmo}\left|\left\{(n,m,o)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:j\leq n,k\leq m,l\leq o,\ \widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon\right\}\right|\\
&\geq\frac{1}{nmo}\left|\left\{j\in I_n,k\in J_m,l\in L_o:\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon\right\}\right|\\
&\geq\frac{\lambda_{n,m,o}}{nmo}.\frac{1}{\lambda_{n,m,o}}\left|\left\{j\in I_n,k\in J_m,l\in L_o:\widetilde{\phi}\left(\frac{x_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}{y_{\sigma^j(p),\sigma^k(q),\sigma^l(r)}}-L\right)\geq\varepsilon\right\}\right|.
\end{aligned}
$$

Taking the limits as $n,m,o\to\infty$ and employ the reality that $\liminf\frac{\lambda_{n,m,o}}{nmo}>0$, we get

$$x \overset{S_{\sigma_3}(\widetilde{\phi})}{\sim} y \Rightarrow x \overset{S_{\sigma_3,\lambda_3}(\widetilde{\phi})}{\sim} y.$$

If we take $(\sigma(n),\sigma(m),\sigma(o))=(n+1,m+1,o+1)$ the above Definitions 4-7 reduce the following definitions :

Definition 9 Let $\widetilde{\phi}:\mathbb{R}\to\mathbb{R}$ be an Orlicz function. A triple sequence $x=(x_{jkl})$

is said to be triple almost $S_{\lambda_3}\left(\widetilde{\phi}\right)$-convergent to L if for every $\varepsilon > 0$,

$$\lim_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left|\left\{j \in I_n, k \in J_m, l \in L_o : \widetilde{\phi}\left(x_{j+p,k+q,l+r} - L\right) \geq \varepsilon\right\}\right| = 0$$

uniformly in $p, q,\ r = 1, 2, \ldots$. In this case we write $\widehat{S}_{\lambda_3}\left(\widetilde{\phi}\right)$-$\lim x = L$ or $x_{jkl} \to L\left(\widehat{S}_{\lambda_3}\left(\widetilde{\phi}\right)\right)$.

Definition 10 Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. The two nonnegative triple sequences $x = (x_{jkl})$ and $y = (y_{jkl})$ are almost S-asymptotically $\widetilde{\phi}$-equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_{n,m,o\to\infty} \frac{1}{nmo} \left|\left\{(n, m, o) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : j \leq n, k \leq m, l \leq o,\right.\right.$$
$$\left.\left.\widetilde{\phi}\left(\frac{x_{j+p,k+q,l+r}}{y_{j+p,k+q,l+r}} - L\right) \geq \varepsilon\right\}\right| = 0,$$

uniformly in $p, q,\ r = 1, 2, \ldots$, $\left(x \overset{\widehat{S}\left(\widetilde{\phi}\right)}{\sim} y\right)$ and simply almost asymptotically $\widetilde{\phi}$-equivalent, if $L = 1$.

Definition 11 Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. The two nonnegative triple sequences $x = (x_{jkl})$ and $y = (y_{jkl})$ are almost S_{λ_3}-asymptotically $\widetilde{\phi}$-equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \left|\left\{j \in I_n, k \in J_m, l \in L_o : \widetilde{\phi}\left(\frac{x_{j+p,k+q,l+r}}{y_{j+p,k+q,l+r}} - L\right) \geq \varepsilon\right\}\right| = 0,$$

uniformly in $p, q,\ r = 1, 2, \ldots$, $\left(x \overset{\widehat{S}_{\lambda_3}\left(\widetilde{\phi}\right)}{\sim} y\right)$ and simply almost S_{λ_3}-asymptotically $\widetilde{\phi}$-equivalent, if $L = 1$.

Definition 12 Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. The two nonnegative triple sequences $x = (x_{jkl})$ and $y = (y_{jkl})$ are strongly almost λ_3-asymptotically $\widetilde{\phi}$-equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_{n,m,o\to\infty} \frac{1}{\lambda_{n,m,o}} \sum_{j\in I_n, k\in J_m, l\in L_o} \widetilde{\phi}\left(\frac{x_{j+p,k+q,l+r}}{y_{j+p,k+q,l+r}} - L\right) = 0,$$

uniformly in $p, q, r = 1, 2, \ldots$, $\left(x \overset{\left[\widehat{V}_{\lambda_3}\right]\left(\widetilde{\phi}\right)}{\sim} y\right)$ and simply strongly almost λ_3-asymptotically $\widetilde{\phi}$-equivalent, if $L = 1$.

In case $(\sigma(n), \sigma(m), \sigma(o)) = (n + 1, m + 1, o + 1)$, we have the following result.

Remark 2 Similar inclusions to Theorems 1 and 2 hold between strongly almost λ_3-asymptotically $\widetilde{\phi}$-equivalent and almost S_{λ_3}-asymptotically $\widetilde{\phi}$-equivalent.

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