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GEODETIC EVEN DECOMPOSITION OF GRAPHS

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Abstract. Let $G = (V(G), E(G))$ be the graph. For a non empty set S of $V(G)$ we define $I[S] = \cup I[x, y]$, for some $x, y \in S$, where $I[x, y]$ is the closed interval consists of x, y and all vertices lying on some $x - y$ geodesic of G. If G is a connected graph, then a set S of vertices is a geodetic set if $I[S] = V(G)$. The cardinality of a geodetic set is called the geodetic number and is denoted as $g(G)$. The decomposition of a graph G is a collection of edge-disjoint subgraphs $G_1, G_2, G_3, ..., G_n$ of G such that every edge of G belongs to exactly one G_i. A decomposition $(G_2, G_4, ..., G_{2n})$ of graph G admits Geodetic even decomposition if $g(G_{2i}) = 2i+1$, $i = 1, 2, ..., n$ where $g(G)$ is the geodetic number of a graph G.

1. Introduction

A graph G consist of a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set whose elements are called vertices and $E(G)$ is a set of unordered pair of distinct elements of $V(G)$. The elements of $E(G)$ are called the edges of the graph G. In graph theory the concept of geodetic set was introduced by Gary Chartrand, Frank Harary and Ping Zhang [3]. If G is a connected graph then the distance $d(x, y)$ is the length of a shortest $x - y$ path in G, where x and y are any two vertices in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. For non empty set S of $V(G)$ we define $I[S] = \cup I[x, y]$, for some $x, y \in S$, where $I[x, y]$ is the closed interval consists of x, y and all vertices lying on some $x - y$ geodesic of G. If G is a connected graph, then a set S of vertices is a geodetic set if $I[S] = V(G)$. The cardinality of a geodetic set is called the geodetic number and is denoted as $g(G)$. In [4] we introduced "Geodetic Decomposition of Graphs". In this paper we develop a new concept "Geodetic Even Decomposition of Graphs".

Definition 1.1 [2] The decomposition of a graph G is a collection of edge-disjoint subgraphs $G_1, G_2, G_3, ..., G_n$ of G such that every edge of G belongs to exactly one G_i.

Definition 1.2 [2] Caterpillar is a tree in which the removal of pendant vertices results in a path. Lobster is a tree in which the removal of pendant vertices results in a caterpillar.

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Definition 1.3 [2] In a ***lobster*** L, the vertex with degree atleast 3 is called a ***junction*** of L.

Definition 1.4 [2] An edge $e = uv$ in L such that u is adjacent to a junction and v is adjacent to another junction is said to be a ***junction-neighbor.***

Definition 1.5 [4] Let G be a any connected graph and $(G_1, G_2, G_3, ..., G_n)$ be the decomposition of G. The graph G admits Geodetic Decomposition, if the following conditions are satisfied.

(i) Each G_i is connected
(ii) Each edge of G is in exactly one G_i
(iii) $g(G_i) = i + 1$ $(i \geq 1)$, where $g(G)$ is the geodetic number of a graph G.

2. Geodetic Even Decomposition

Definition 2.1 A decomposition $(G_2, G_4, ..., G_{2n})$ of a graph G is said to be geodetic even decomposition if

(i) Each G_{2i} is connected
(ii) Each edge of G is in exactly one G_{2i}
(iii) $g(G_{2i}) = 2i + 1,\ i = 1, 2, ..., n.$

Example 2.2 The following figure illustrates geodetic even decomposition of G.

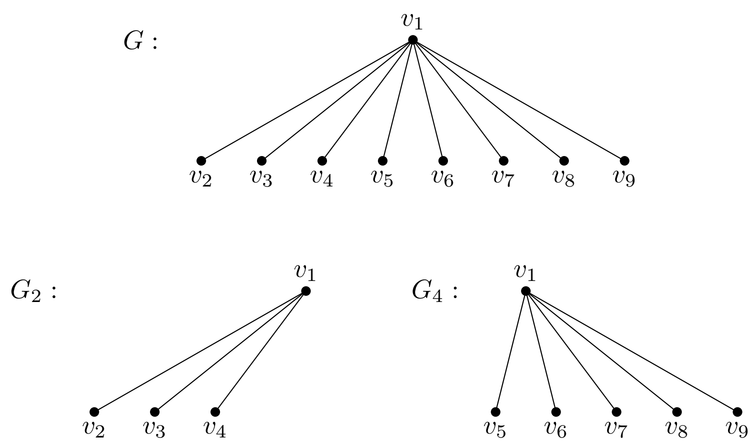

Figure 1. Geodetic Even Decomposition (G_2, G_4) of graph G

Here $g(G_2) = 3$ and $g(G_4) = 5$. Then G admits Geodetic Even Decomposition (G_2, G_4).

Remark 2.3 Every path does not admits geodetic even decomposition, since the geodetic number of each path is 2.

Theorem 2.4 A Lobster L admits Geodetic Even Decomposition $(G_2, G_4, ..., G_{2n-2})$ if and only if $q = n^2 - 1$ $(n \geq 2)$.

Proof. Let L be a Lobster with $n^2 - 1$ edges. To Prove L admits Geodetic Even Decomposition $(G_2, G_4, ..., G_{2n-2})$. We prove this by induction on n.
Let $n = 2$. Then L has 3 edges and $L = S_3$. Let it be G_2. Clearly $g(G_2) = 3$.

Hence the result is true, when $n = 2$.
Assume the result is true, when $n = k - 1$. Then L has $(k-1)^2 - 1$ edges and L admits geodetic even decomposition $(G_2, G_4, ..., G_{2k-6}, G_{2k-4})$ where $G_{2i} = S_{2i+1}$, $i = 1, 2, ..., k-3, k-2$.
Prove the result, when $n = k$. Let L has $k^2 - 1$ edges. Then $L = G_2 \cup G_4 \cup ... \cup G_{2k-4} \cup G_{2k-2}$.
By induction hypothesis $G_{2i} = S_{2i+1}$, $i = 1, 2, ..., k-3, k-2$ and satisfies $g(G_{2i}) = 2i+1$, $i = 1, 2, ..., k-3, k-2$. Then clearly $G_{2k-2} = S_{2k-1}$ and $g(G_{2k-2}) = 2k-1$. Thus the induction is proved and hence the theorem.
Conversely, assume that L admits Geodetic Even Decomposition $(G_2, G_4, ..., G_{2n-2})$. Then $G_i = S_{i+1}$, $i = 2, 4, 6, ..., 2n-2$. Therefore $q = n^2 - 1$.

Result 2.5 If L admits Geodetic Even Decomposition $(G_2, G_4, G_6, ..., G_{2n-2})$, then $diam(L) = 2n - 2$.

Theorem 2.6 Let L be a Lobster with $diam(L) = 2n-2$. Then L admits Geodetic Even Decomposition $(G_2, G_4, G_6, ..., G_{2n-2})$ if and only if

(i) L is a caterpillar
(ii) There are $(n-1)$ non-adjacent junction supports in L whose degrees are $3, 5, 7, ..., 2n-1$ respectively and
(iii) There is no junction-neighbor in L

Proof. Given that L is a Lobster with $diam(L) = 2n - 2$. Assume that L admits Geodetic Even Decomposition $(G_2, G_4, G_6, ..., G_{2n-2})$ where $G_i = S_{i+1}$, $i = 2, 4, 6, ..., 2n-2$. Since $diam(L) = 2n-2$, the centre of each G_i $'s$ are lie in the longest path P of L. Then L is caterpillar.
Let $u_3, u_5, u_7, ..., u_{2n-1}$ be the centres of $G_2, G_4, G_6, ..., G_{2n-2}$ respectively. Then clearly they are junctions. Also since $diam(L) = 2n-2$, all the centres are distinct and are supports. Hence there are $(n-1)$ non-adjacent junction supports whose degrees are $3, 5, 7, ..., 2n-1$ respectively.
Now to prove (iii). Suppose there is one junction-neighbor $e_1 = a_1b_1$.

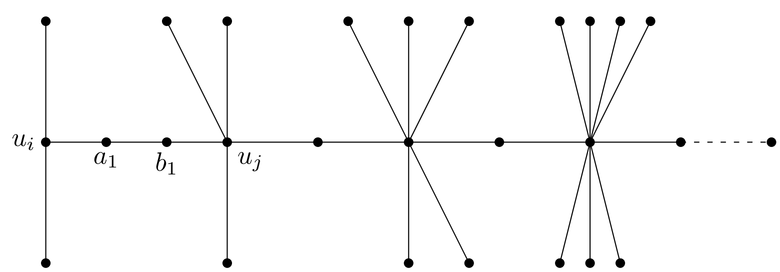

FIGURE 2. Geodetic Even Decomposition of Lobster

From figure 2, their exits junction supports u_i, u_j such that $d(u_i, u_j) = 3$.
Thus $E(L) - E(G_2 \cup G_4 \cup G_6 \cup ... \cup G_{2n-2}) = 1$, which is a contradiction to $q = n^2 - 1$.
Hence there is no junction-neighbor in L.
Conversely, assume (i),(ii),(iii).
Clearly L admits Geodetic Even Decomposition $(G_2, G_4, G_6, ..., G_{2n-2})$.

Theorem 2.7 Let L be a Lobster with $diam(L) = 2n-4$ and $n-2$ distinct supports with no junction neighbor in the longest path P of L and $N_2 \neq \emptyset$. Then L admits

Geodetic Even Decomposition $(G_2, G_4, ..., G_{2n-2})$ with distinct centres if and only if

(i) No vertex of exactly one G_{2n-2} $(n \geq 3)$ is in the longest path P
(ii) All the vertices of N_2 are adjacent to exactly one vertex of N_1

Proof. Assume that L admits Geodetic Even Decomposition $(G_2, G_4, ..., G_{2n-2})$ where $G_i = S_{i+1},\ i = 2, 4, 6, ..., 2n-2$.
To prove (i). Suppose not, atleast one vertex of each G_{2n-2} $(n \geq 3)$ is in P. Then there exist $(n-1)$ junction supports in L not all them are distinct, which is a contradiction. Hence no vertex of exactly one G_{2n-2} $(n \geq 3)$ is in the longest path P.
Suppose $|N_2| = 3$. Therefore no vertex of G_2 is in P. It is enough to prove all the vertices of N_2 are adjacent to exactly one vertex of N_1.
Suppose the vertices of N_2 are adjacent to two distinct vertices v_i and v_{i+1} of N_1.

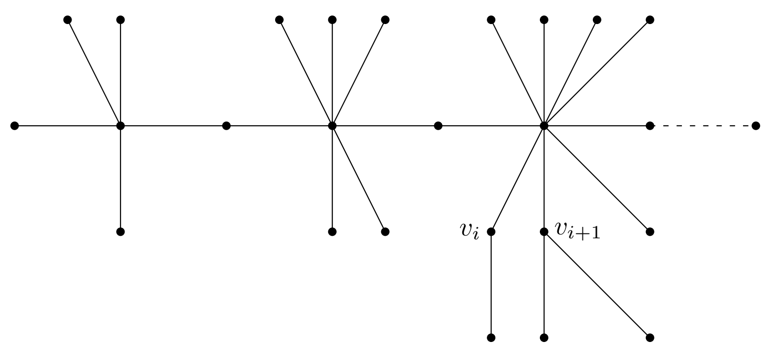

Figure 3. Geodetic Even Decomposition of Lobster

From figure 3, there exist G_i and G_{i+1} $(i = 3, 4, 5, ..., 2n-2)$ such that v_i and v_{i+1} are the centres of G_i and G_{i+1} respectively. Since L admits Geodetic Even Decomposition $(G_2, G_4, ..., G_{2n-2})$, the existence of G_i and G_{i+1} are not possible. Hence our assumption is wrong.
Therefore all the vertices of N_2 are adjacent to exactly one vertex of N_1.
Continuing in this way, no vertex of G_{2n-2} is in P and vertices of N_2 are adjacent to exactly one vertex of N_1, if $|N_2| = 2n-2$.
Hence $|N_2| \geq 2n-2$, $(n \geq 3)$ and all the vertices of N_2 are adjacent to exactly one vertex of N_1.
Conversely assume (i) and (ii).
To prove L admits Geodetic Even Decomposition. Since $diam(L) = 2n-4$ and $N_2 = \emptyset$, then there exists atleast one G_{2n-2} (say) such that the centre of G_{2n-2} is not in P. Since there are $n-2$ distinct junction supports with no junction neighbor in P, $n-3$ subgraphs exist in L. Since $q = n^2 - 1$ and all the vertices of N_2 are adjacent to exactly one vertex of N_1, then $n-2$ subgraphs exists and satisfies the condition $g(G_i) = i+1$ $(i = 2, 4, ..., 2n-2)$. Hence the proof.

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