A STUDY FOR A CLASS OF ENTIRE DIRICHLET SERIES IN n - VARIABLES

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Abstract. Let \( L \) represents a class of entire functions represented by Dirichlet series in \( n \) - variables of the form 
\[
    f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{nmn} s} 
\]
whose coefficients belong to the set of complex numbers \( \mathbb{C} \). \( L \) which becomes a complete Banach space is thereby proved to be a complex FK-space and a Frechet space.

1. Introduction

Let
\[
f(s_1, s_2, \ldots, s_n) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} a_{m_1, m_2, \ldots, m_n} e^{(\lambda_{1m_1} s_1 + \lambda_{2m_2} s_2 + \cdots + \lambda_{nmn} s_n)} \tag{1}
\]
be a \( n \)-tuple Dirichlet series where \( s_j = \sigma_j + it_j, j \in \{1, 2, \ldots, n\} \) and \( a_{m_1, m_2, \ldots, m_n} \in \mathbb{C} \). Also
\[
0 < \lambda_{p_1} < \lambda_{p_2} < \ldots < \lambda_{p_k} \to \infty \text{ as } k \to \infty \text{ for } p = 1, 2, \ldots, n.
\]
To simplify the form of \( n \)-tuple Dirichlet series, we have the following notations
\[
s = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n, \\
m = (m_1, m_2, \ldots, m_n) \in \mathbb{C}^n \text{ and} \\
\lambda_{nmn} = (\lambda_{1m_1}, \lambda_{2m_2}, \ldots, \lambda_{nmn}) \in \mathbb{R}^n. \\
\lambda_{nmn} s = \lambda_{1m_1} s_1 + \lambda_{2m_2} s_2 + \cdots + \lambda_{nmn} s_n \\
|\lambda_{nmn}| = \lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nmn} \\
|m| = m_1 + m_2 + \cdots + m_n.
\]
Thus the series (1) can be written as
\[
f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{nmn} s}. \tag{2}
\]
Janusauskas in [4] showed that if there exists a tuple \( p > 0 = (0, 0, \ldots, 0) \) such that
\[
\limsup_{|m| \to \infty} \frac{\sum_{k=1}^{\infty} \log m_k}{p \lambda_{n,m}} = 0,
\]
then the domain of absolute convergence of (2) coincides with its domain of convergence. Sarkar in [1] proved that the necessary and sufficient condition for series (2) satisfying (3) to be entire is that
\[
\lim_{|m| \to \infty} \frac{\log |a_m|}{|\lambda_{n,m}|} = -\infty.
\]
Consider \( L \) as the set of series (2) satisfying (3) and (4) for which
\[
(|\lambda_{n,m}|)^{c_1}|\lambda_{n,m}| e^{c_2|m|(|\lambda_{n,m}|)} |a_m|
\]
is bounded. Then every element of \( L \) represents an entire function. Define the binary operations in \( L \) as
\[
\begin{align*}
    f(s) + h(s) &= \sum_{m=1}^{\infty} (a_m + b_m) e^{\lambda_{n,m} s}, \\
    \xi f(s) &= \sum_{m=1}^{\infty} (\xi a_m) e^{\lambda_{n,m} s}, \\
    f(s) h(s) &= \sum_{m=1}^{\infty} (|\lambda_{n,m}|)^{c_1}|\lambda_{n,m}| e^{c_2|m|(|\lambda_{n,m}|)} a_m b_m e^{\lambda_{n,m} s}.
\end{align*}
\]
The norm in \( L \) is defined as
\[
\|f\| = \sup_{|m| \geq 1} (|\lambda_{n,m}|)^{c_1}|\lambda_{n,m}| e^{c_2|m|(|\lambda_{n,m}|)} |a_m|.
\]

**Definition 1** A space \( L \) is called an FK-space if the following conditions are satisfied

1.a) \( L \) is a linear space over the field of complex numbers (or real numbers) and elements of \( L \) are sequences of complex numbers (or real numbers).

1.b) \( L \) is a locally convex topological linear space in which the topology is given by a countable family of semi-norms.

1.c) \( L \) is metrizable and is a complete metric space.

1.d) If \( \{\alpha_m\} \) is a base for \( L \) such that for \( l \in L \),
\[
l = \sum_{m=1}^{\infty} \theta_m(l) \alpha_m
\]
then \( \theta_m(l) (|m| \geq 1) \) are continuous linear functionals. If the field for \( L \) is complex numbers then \( L \) is called a complex FK-space.

During the last two decades a lot of research has been carried out in the field of Dirichlet series and many important results have been proved where few of them may be found in [2] - [3]. Kumar and Manocha in [5] considered the condition \((\lambda_n)^{c_1}(\lambda_n) e^{c_2 n - c_1} (\lambda_n) \|a_n\| \) of weighted norm for a Dirichlet series in one variable and established some results on it. Recently in [6] results were established on Dirichlet series with complex frequencies. Until now a lot work has been done for the Dirichlet series in one variable. The purpose of this paper is to give a wider view to the study of Dirichlet series in \( n \)-variables. In this section main results have been proved. For the definitions of terms used refer [7, 8].
respect to the usual addition and multiplication, a topology is defined such that \( L \) becomes a complex FK-space.

**Proof.** Let for \( f(s), h(s) \in L \) define addition of \( f(s) \) and \( g(s) \) as \((f + h)(s) = f(s) + h(s)\) and scalar multiplication of \( f(s) \) as \((\tau f)(s) = \tau f(s)\).

Let us now define the zero element of \( L \) defined by \( 0^* \) as the entire function which is zero that is \( f = 0^* \) implies \( \sum_{m=1}^{\infty} a_m e^{\lambda_{nm} s} = 0 \) which further implies \( a_m = 0 \) for all \( |m| \geq 1 \) and conversely.

Clearly \( L \) forms an infinite dimensional linear space over the field of complex numbers and hence one gets basis for \( L \) namely Schauder basis as

\[
\delta_{m_1, m_2, \ldots, m_n} = \sum_{m_1, m_2, \ldots, m_n} a_{m_1} e^{\lambda_{m_1} s_1 + \lambda_{m_2} s_2 + \ldots + \lambda_{m_n} s_n}
\]

or

\[
\delta_m = e^{\lambda_{nm} s}.
\]

Also

\[
L_{m_1} = (1, 0, 0, \ldots)
\]

\[
L_{m_2} = (0, 1, 0, \ldots)
\]

\[
\vdots
\]

\[
L_{m_n} = (0, 0, \ldots, 1, 0, \ldots)
\]

where 1 in \( L_{m_n} \) is at the \( m_n \)-th place. It therefore implies that if \( x(s) \in L \) then

\[
x(s) = (a_1(x), a_2(x), \ldots, a_m(x), \ldots)
\]

where

\[
\lim_{|m| \to \infty} \frac{\log |a_m|}{|\lambda_{nm}|} = -\infty
\]

and this shows that \( L \) satisfies (1.a).

Define \( H = \{L_{m_1}, L_{m_2}, \ldots, L_{m_n}, \ldots\}. \) For each \( L_{m_n} \in H \) define the norm as

\[
\|f, L_{m_n}\| = \sup_{|m| \geq 1} (|\lambda_{nm}|)^c_1 |\lambda_{nm}| e^{c_2 |m|} (|\lambda_{nm}|) |a_m|\]

where

\[
f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{nm} s} \in L
\]

is an entire function. Then as

\[
\frac{\log |a_m|^{-1}}{|\lambda_{nm}|} > v
\]

for \( v > c_2|m| \) implies

\[
|a_m| < e^{-v|\lambda_{nm}|} \text{ for } |m| \geq |m'|,
\]

where \( m' = (m'_1, m'_2, \ldots, m'_n) \). Therefore

\[
\|f, L_{m_n}\| < \sup_{|m| < |m'|} (|\lambda_{nm}|)^{c_1} |\lambda_{nm}| e^{c_2 |m|} (|\lambda_{nm}|) |a_m| + \sup_{|m| \geq |m'|} (|\lambda_{nm}|)^{c_1} |\lambda_{nm}| e^{(c_2 |m| - v)} (|\lambda_{nm}|) |a_m|
\]

Thus

\[
\|f, L_{m_n}\| < \infty
\]
for any fixed $L_{m, n} \in H$. Hence $\| f, L_{m, n} \|$ is defined for each $L_{m, n} \in H$. Let $L_{m, n}$ be fixed, then

$$\| f, L_{m, n} \| = f(s)$$

$$\Leftrightarrow |a_m| = 0 \text{ for } |m| \geq 1$$

$$\Leftrightarrow f(s) = 0 \text{ for all } |s|$$

$$\Leftrightarrow f = 0^*.$$

Since

$$h(s) = \sum_{m=1}^{\infty} b_m e^{\lambda_m n_s}$$

Then

$$|a_m + b_m| \leq |a_m| + |b_m|$$

implies

$$\| f + h, L_{m, n} \| \leq \| f, L_{m, n} \| + \| h, L_{m, n} \|.$$ Again if $\nu$ is any complex number then

$$\| \nu f, L_{m, n} \| = |\nu| \| f, L_{m, n} \|.$$ Thus $\| \ldots, L_{m, n} \|$ defines a norm for each $L_{m, n} \in H$. Hence $L$ becomes a locally convex linear topological space as there exists a sequence $\{\| \ldots, L_{m, n} \| : |n| = 1, 2, 3, \ldots \}$ of enumerable number of norms on $L$. Let

$$\| f \| = \sup_{|m| \geq 1} \frac{|f, L_{m, n}|}{1 + |f, L_{m, n}|}$$

and

$$e(f, h) = \| f - h \|$$

Then $e$ is a metric on $L$. It can be easily verified that the topology induced by $e$ on $L$ is the same as induced by the sequence $\{\| \ldots, L_{m, n} \|\}$. In fact if $Y$ is open in the topology induced by the family of norms then $Y$ is also open in the $e$-metric topology of $L$. Now let $Y$ be open in the $e$-metric topology of $L$. Then for each $g(s) \in Y$ we have $\epsilon > 0$ such that

$$K = \{ g(s) : g \in B(f; \epsilon) \} \subset Y \text{ for } 0 < \epsilon < 1$$

where $B(f; \epsilon)$ is an open ball centered at $f(s)$ and is of radius $\epsilon$. We find $M$ such that

$$\sup_{|m| \geq |M| + 1} \frac{1}{2^m} \frac{1}{1 + \| k - g, L_{m, n} \|} < \frac{\epsilon}{2},$$

where $k - g$ is a vector in the neighbourhood of 0. Let

$$F = \{ x(s) : \| x, L_1 \| \leq \epsilon_1 \} \cap \ldots \cap \{ x(s) : \| x, L_M \| \leq \epsilon_M \}$$

where

$$\epsilon_m < \frac{\epsilon}{2} (|m| = 1, 2, \ldots, |M|).$$
Let \( k(s) \in g(s) + F \). Then \( k(s) = g(s) + x(s) \) where \( x(s) \in F \). Then

\[
e(k, g) = \sup_{1 \leq |m| \leq |M|} \left( 1 + 2m \right) \frac{1}{2^m} \left( \| k - g, L_{m_n} \| + \sup_{|m| \geq |M| + 1} \frac{1}{2^m} \| k - g, L_{m_n} \| \right)
\]

\[
< \frac{1}{\epsilon} \sup_{1 \leq |m| \leq |M|} \left( 1 + 2m \right) \left( 1 + \epsilon_m \right) + \epsilon
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Therefore \( e(k, g) < \epsilon \) implies that \( k(s) \in B(g; \epsilon) \) that is \( k(s) \in K \) which further implies \( k(s) \in Y \). Thus \( g(s) + F \subset Y \) which establishes that \( Y \) is open in the topology induced by the family of norms. Hence \( L \) is metrizable.

Now we show that \( L \) is complete with respect to the metric \( e \). It is known that a space is complete if and only if every nested sequence of closed balls whose radii tend to zero has non empty intersection.

Let \( \{ f_m : m \in M \} \) be a cauchy sequence in \( L \). For each \( m \in M \), let \( W_m = \{ x_k : k \geq m \} \) be \( m \)-th tail of sequence and \( s_m \) be twice the diameter of \( W_m \). Also let \( B_m \) be a closed ball centered at \( f_m \) of radius \( r_m = 2s_m \). Then

\[
W_m \subseteq B_m.
\]

Since the sequence is cauchy therefore \( \lim_{m \to \infty} s_m = 0 \). Now let \( m \in M \) be arbitrary. Therefore there exists \( k > m \) such that

\[
s_k < \frac{1}{2}s_m.
\]

Suppose \( g(s) \in B_k \) then

\[
e(g, f_m) \leq e(g, f_k) + e(f_k, f_m)
\]

\[
\leq r_k + s_m
\]

\[
= 2s_k + s_m
\]

\[
< 2s_m = r_m.
\]

Therefore \( g(s) \in B_m \) and hence \( B_k \subseteq B_m \). In the like manner we construct a nested sequence of the closed balls \( \{ B_m : m \in M \} \). Then from hypothesis nested sequence of closed balls has a non empty intersection say \( f \). Let \( \{ f_{r_1} \} \) be a cauchy sequence in \( L \) where

\[
f_{r_1}(s) = \sum_{i=1}^{\infty} a_i^{(r_1)} e^{\lambda_{nt_n}s}.
\]

Now

\[
e(f_{r_1}, f_{r_2}) < \epsilon \text{ for all } r_1, r_2 \geq |M|
\]

implies

\[
\sup_{|m| \geq 1} \frac{1}{2^m} \left( 1 + \frac{1}{2^m} \right) \left( 1 + 2m \right) \left( 1 + \epsilon_m \right) + \epsilon < \epsilon \text{ for } r_1, r_2 \geq |M|.
\]

Thus

\[
(1 - 2^m \epsilon) \| f_{r_1} - f_{r_2}, L_{m_n} \| < 2^m \epsilon \text{ for } r_1, r_2 \geq |M|, |m| = 1, 2, \ldots
\]

\[
(1 - 2^m \epsilon) \sup_{|t| \geq 1} \| \lambda_{n_{t_n}} \| e^{t|\lambda_{n_{t_n}}|} e^{r_1|\lambda_{n_{t_n}}|} - e^{r_2|\lambda_{n_{t_n}}|} < 2^m \epsilon \text{ for } r_1, r_2 \geq |M|, |m| = 1, 2, \ldots
\]

\[
(1 - 2^m \epsilon) |a_t^{(r_1)} - a_t^{(r_2)}| < 2^m \epsilon \text{ for } r_1, r_2 \geq |M|, |t| \geq 1, |m| = 1, 2, \ldots
\]
and 
\[ \lim_{r_2 \to \infty} a_t^{(r_2)} = a_t, \quad |t| = 1, 2, \ldots \]
implies
\[ (1 - 2^m \epsilon) |a_t^{(r_1)} - a_t| < 2^m \epsilon \left\{ \left( |\lambda_{n_1}| \right)^{c_1 |\lambda_{n_1}|} e^{c_2 |t| (|\lambda_{n_1}|)} \right\}^{-1} \]
for \( r_1 \geq |M|, |t|, |m| = 1, 2, \ldots \)
If \( 2^m \epsilon < \theta < 1 \) then
\[ |a_t^{(r_1)} - a_t| < \frac{\theta}{1 - \theta} \left\{ \left( |\lambda_{n_1}| \right)^{c_1 |\lambda_{n_1}|} e^{c_2 |t| (|\lambda_{n_1}|)} \right\}^{-1} \]
that is
\[ |a_t| < |a_t^{(r_1)}| + \frac{\theta}{1 - \theta} \left\{ \left( |\lambda_{n_1}| \right)^{c_1 |\lambda_{n_1}|} e^{c_2 |t| (|\lambda_{n_1}|)} \right\}^{-1} \]
and since
\[ \lim_{|t| \to \infty} \frac{\log |a_t^{(r_1)}|}{|\lambda_{n_1}|} = -\infty. \]
Hence it follows
\[ \lim_{|t| \to \infty} \frac{\log |a_t|}{|\lambda_{n_1}|} = -\infty. \]
Thus
\[ f(s) = \sum_{t=1}^{\infty} a_t e^{\lambda_{n_1} s} \]
represents an entire function such that
\[ \| f_{r_1} - f, L_{m_n} \| < \epsilon \quad \text{where} \quad r_1 \geq |M|, |m| = 1, 2, \ldots \]
Therefore
\[ \| f_{r_1} - f, L_{m_n} \| \to 0 \text{ as } r_1 \to \infty \]
or
\[ e(f_{r_1}, f) \to 0 \text{ as } r_1 \to \infty \]
This proves (1.c) of Definition (??).
Next we need to prove the condition (1.d). Let therefore
\[ \beta = \sum_{m=1}^{\infty} \theta_m(\beta) \beta_m \quad ; \beta \in L \]
\[ \beta_m \equiv \gamma_m \text{ and } \gamma_m = e^{\lambda_{m_n} s}. \]
Then we show \( \theta_m(\beta) \) is a continuous linear functional of \( \beta \) in \( L \) for each \( |m| \geq 1 \). Clearly \( \theta_m \) is linear and since \( L \) is endowed with the topology given by the metric \( e \) and is a topological vector space. Therefore it is sufficient to prove that \( (\theta_m(\beta)) \) is continuous.
Let \( \{\mu_s\} \subset L \) and suppose \( e(\mu_s, 0) < \epsilon \) for \( |s| \geq |s_0| \) where \( |s| \geq 1 \), then
\[ \mu_s = \sum_{m=1}^{\infty} \theta_m(\mu_s) \beta_m. \]
Again if
\[ \mu_s^{(M)} = \sum_{m=1}^{M} \theta_m(\mu_s) \beta_m. \]
then \( e(\mu^M_s, \mu_s) < \epsilon \) for \(|M| \geq |M_o|\). Hence
\[
e(\mu^M_s, 0) < e(\mu^M_s, \mu_s) + e(\mu_s, 0) \leq 2\epsilon \quad \text{for all} \ |M| \geq |M_o|, \ |s| \geq |s_o|.
\]

Also
\[
\|\mu^M_s, L_m\| - \|\mu^{(M-1)}_s, L_m\| = (|\lambda_n M_n|)^c_1|\lambda_n M_n| e^{c_2|M|(|\lambda_n M_n|)} \|\theta M(\mu_s)\|
\]
where
\[
\|\mu^M_s, L_m\| = \sup_{|m| \geq 1} (|\lambda_n M_n|)^c_1|\lambda_n M_n| e^{c_2|m|(|\lambda_n M_n|)} \|\theta M(\mu_s)\|.
\]

But
\[
\|\mu^M_s, L_m\| < \epsilon \quad \text{for} \ |M| \geq |M_o|, \ |s| \geq |s_o|, \ |m| \geq 1
\]

Therefore
\[
|\theta M(\mu_s)| < \epsilon \quad \text{for} \ |s| \geq |s_o|, \ |m| \geq 1.
\]

Hence the theorem.

Linear Functionals: In this section continuous linear functionals on the space \( L \) have been characterized when \( L \) is endowed with the topology given by the norms \( \{L, \ldots, L_m\} : n = 1, 2, \ldots \) \textbf{Theorem 2} Every continuous linear functional \( \theta \) on the normed linear space \((L, \ldots, L_m) : n = 1, 2, \ldots \) is of the form
\[
\theta(f) = \sum_{m=1}^\infty a_m \mu_m : f(s) = \sum_{m=1}^\infty a_m e^{\lambda n m \ s}
\]
where
\[
\{\|\mu_m|/(|\lambda_n M_n|)^c_1|\lambda_n M_n| e^{c_2|m|(|\lambda_n M_n|)}\}
\]
is bounded.

\textbf{Proof.} Let \( \theta \) be a continuous linear functional on the normed linear space \((L, \ldots, L_m) : n = 1, 2, \ldots \) and so there exists a positive constant \( G \) such that
\[
|\theta(f)| \leq G\|f, L_m\| \quad \text{for all} \ f(s) \in L.
\]

Let
\[
f M(s) = \sum_{m=1}^M a_m e^{\lambda n m \ s}
\]
then
\[
\|f - f M, L_m\| = \sup_{|m| \geq |M| + 1} (|\lambda_n M_n|)^c_1|\lambda_n M_n| e^{c_2|m|(|\lambda_n M_n|)}|a_m|.
\]
The above expression can be made as small as we want by making \( M \) large enough, one gets
\[
\|f - f M, L_m\| \to 0 \quad \text{as} \ |M| \to \infty.
\]

Thus
\[
\theta(f) = \lim_{M \to \infty} \theta(f M) = \lim_{M \to \infty} \left(\sum_{m=1}^M a_m \mu_m\right)
\]
where \( \mu_m = \theta(e^{\lambda n m \ s}) \). Now
\[
|\mu_m| = |\theta(e^{\lambda n m \ s})| \leq G\|e^{\lambda n m \ s}, L_m\|
\]
that is
\[
|\mu_m| \leq G(|\lambda_n M_n|)^c_1|\lambda_n M_n| e^{c_2|m|(|\lambda_n M_n|)}
\]
Therefore
\[
\frac{|\mu_m|}{(|\lambda_{n,n}|)^{c_1|\lambda_{n,n}|} e^{c_2 |\lambda_{n,n}|}} \leq G. \tag{6}
\]
Hence
\[
\theta(f) = \sum_{m=1}^{\infty} a_m \mu_m \tag{7}
\]
is convergent where \(\mu_m\) is given by (6). This completes the proof of the theorem.

**Theorem 3** If \(\{\gamma_m\}\) forms a base for \(L\) that is for \(\gamma \in L\),
\[
\gamma = \sum_{m=1}^{\infty} \theta_m(\gamma) \gamma_m.
\]

Let us define a metric \(\zeta(\gamma, \gamma')\) as follows
\[
\zeta(\gamma, \gamma') = \sup \| (\theta_1(\gamma) - \theta_1(\gamma')) \gamma_1 + \ldots + (\theta_m(\gamma) - \theta_m(\gamma')) \gamma_m \|.
\]
Then \(L\) is complete with respect to the metric \(\zeta\).

**Proof.** Let \(\{\lambda_r\}\) be a sequence of entire functions in \(L\) such that \(\zeta(\lambda_r, \lambda_s) < \epsilon\) for \(|r|, |s| \geq |r_0|\). That is \(\{\lambda_r\}\) is a \(\zeta\) - cauchy sequence in \(L\). Hence for each given \(\epsilon > 0\) there exists \(r_o = \epsilon(\epsilon)\) such that
\[
\sup \| \sum_{i=1}^{m} (\phi_i(\lambda_r) - \phi_i(\lambda_s)) \gamma_{i} \| \leq \epsilon \text{ for } |r|, |s| \geq |r_o|. \]

This implies \(\| (\phi_i(\lambda_r) - \phi_i(\lambda_s)) \gamma_{i} \| < \epsilon \) for \(|r|, |s| \geq |r_o|, |i| \geq 1.\)

Since \(\gamma_i \neq 0\) for \(|i| \geq 1,\)
\[
\| \phi_i(\lambda_r) - \phi_i(\lambda_s) \| < \epsilon \text{ for } |r|, |s| \geq |r_o|. \]

Therefore \(\{\phi_i(\lambda_r)\}\) being a cauchy sequence in the usual topology of the complex plane tends to \(\phi_i\) as \(|r| \to \infty.\)
\[
\| \sum_{i=1}^{m} (\phi_i(\lambda_r) - \phi_i) \gamma_{i} \| \leq \epsilon \text{ for } |r| \geq |r_o|. \]

Now for \(|r| = |r_o|\) and \(\gamma = \lambda_{r_o},\)
\[
\| \sum_{i=1}^{m} \phi_i(\lambda_{r_o}) \gamma_{i} - \sum_{i=1}^{n} \phi_i(\lambda_{r_o}) \gamma_{i} \| < \epsilon \text{ for } |n|, |m| \geq |n_o|. \]

Therefore
\[
\| \sum_{i=1}^{m} \phi_i \gamma_{i} - \sum_{i=1}^{n} \phi_i \gamma_{i} \| \leq \| \sum_{i=1}^{m} (\phi_i - \phi_i(\lambda_{r_o})) \gamma_{i} \| + \| \sum_{i=1}^{n} (\phi_i - \phi_i(\lambda_{r_o})) \gamma_{i} \| + \| \sum_{i=1}^{n} \phi_i(\lambda_{r_o}) \gamma_{i} - \sum_{i=1}^{n} \phi_i(\lambda_{r_o}) \gamma_{i} \|
\]
This implies
\[
\| \sum_{i=1}^{m} \phi_i \gamma_{i} - \sum_{i=1}^{n} \phi_i \gamma_{i} \| < 3 \epsilon \text{ for } |n|, |m| \geq |n_o|. \]
Hence $\{ \sum_{i=1}^{m} \phi_i \gamma_i \}$ converges to $\lambda$ as $L$ is complete with respect to the metric $e$. Thus $\phi_i = \phi_i(\lambda)$. Therefore $\zeta(\lambda, r, \lambda) < \epsilon$, $|r| \geq |r_0|$. Hence $\{ \lambda_r \}$ converges to $\lambda$ where $\lambda \in L$ which proves the theorem. **Theorem 4** The space $L_e$ is a Frechet space where $e$ is the metric defined on $L$.

**Proof.** $L_e$ is a normed linear metric space. In above theorem it has been proved that $L_e$ is complete with respect to the metric $e$. Thus $L_e$ is a Frechet space.

**References**


