Electronic Journal of Mathematical Analysis and Applications
Vol. 10(1) Jan. 2022, pp. 236-252.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

FIRST ZAGREB INDEX AND F-INDEX OF FOUR NEW CO-NORMAL PRODUCTS OF GRAPHS AND THEIR COMPLEMENTS

B. BASAVANAGOUD AND SHRUTI POLICEPATIL

Abstract. For a molecular graph G, the first Zagreb index is equal to the sum of squares of degrees of vertices, and the F-index or forgotten topological index is defined as the sum of cubes of degrees of vertices. In this paper, we introduce $\mathcal{F}-$co-normal products of graphs. Further, we obtain the first Zagreb index, F-index and their coindices of $\mathcal{F}-$co-normal products (four new co-normal products based on transformations of a graph) of graphs and their complements.

1. Introduction

Chemical graph theory is a branch of mathematics which combines graph theory and chemistry. Graph theory is used to mathematically model molecules in order to gain insight into the physico-chemical properties of these chemical compounds. The molecular graph is a simple graph, representing the carbon-atom skeleton of a hydrocarbon. The vertices of a molecular graph represent the carbon atoms, and its edges the carbon-carbon(covalent) bonds. The topological indices are graph invariants which are numerical values associated with molecular graphs. In mathematical chemistry, these are known as molecular descriptors. Topological indices play a vital role in mathematical chemistry specially, in chemical documentation, isomer discrimination, quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR) analysis. *Wiener index* is the first topological index used by H. Wiener [36] in the year 1947, to calculate boiling point of paraffins. There are various degree based topological indices which are found applicable and employed in QSPR/QSAR analysis. For chemical applications of topological indices refer [6, 18, 34].

There are several papers devoted to topological indices of graph operations. The first and second Zagreb indices of graph operations are investigated by Khalifeh et al. [23], Akhtera et al., obtained F-index in [2], Basavanagoud et al., obtained hyper-Zagreb index in [3, 4], N. De et al., obtained F-coindex in [12]. For some other topological indices of graph operations one can refer [5, 9, 15, 25, 28, 29, 30,

---

2010 *Mathematics Subject Classification.* 05C07, 05C76, 92E10.
Key words and phrases. Co-normal product, first Zagreb index, F-index, graph transformation.
Submitted Feb. 11, 2021. Revised July 26, 2021.

31, 37, 38]. For more on product related graph operations we refer a book by Imrich and Klavažar [24].

2. Definitions and Preliminaries

Let G be a finite undirected graph without loops and multiple edges on n vertices and m edges is called (n, m) graph. We denote vertex set and edge set of graph G as $V(G)$ and $E(G)$, respectively. The *neighbourhood* of a vertex $u \in V(G)$ is defined as the set $N_G(u)$ consisting of all vertices v which are adjacent to u in G. The *degree* of a vertex $u \in V(G)$, denoted by $d_G(u)$ and is equal to $|N_G(u)|$. The *complement of a graph* G is denoted by $\overline{G}$ and is defined as the graph whose vertex set is $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G. Obviously, $\overline{G}$ has n vertices and $\binom{n}{2} - m$ edges. The line graph $L(G)$ of a graph G is the graph with vertex set $E(G)$ and two vertices are adjacent in $L(G)$ if and only if the corresponding edges in G are adjacent. The line graph $L(G)$ has order $n_L = m$ and size $m_L = -m + \frac{1}{2}\sum\limits_{i=1}^{n} d_G(v_i)^2$. For undefined graph theoretic terminologies and notations refer [22].

For a molecular graph G, *first Zagreb index* was defined by Gutman and Trinajstić [20] in 1972 as

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2.$$

The second Zagreb index was defined in [19] as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) \cdot d_G(v).$$

The first Zagreb index [26] can also be expressed as

$$M_1(G) = \sum_{uv \in E(G)} \big(d_G(u) + d_G(v)\big).$$

Later, coindices were introduced to cover the contribution of the non adjacent vertices of a graph G. The *first and second Zagreb coindices* [1] were defined respectively as

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} \big(d_G(u) + d_G(v)\big), \quad \overline{M_2}(G) = \sum_{uv \notin E(G)} \big(d_G(u) \cdot d_G(v)\big).$$

For basic properties of Zagreb indices refer [17, 20] and for Zagreb indices of graph operation refer [1, 10, 23, 38]. Another degree based graph invariant called *forgotten topological index* or F- index was put forward by Furtula and Gutman [13] is defined as

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} \big(d_G(u)^2 + d_G(v)^2\big).$$

Its coindex [12] is given by

$$\overline{F}(G) = \sum_{uv \notin E(G)} \big(d_G(u)^2 + d_G(v)^2\big).$$

See [13] for basic properties and [2, 12] for F-index of graph operations.
The *hyper Zagreb index* was introduced by Shirdel et al., in [33] which is defined as

$$HM(G) = \sum_{uv \in E(G)} \big(d_G(u) + d_G(v)\big)^2$$

and hyper Zagreb coindex was introduced by Veylanki et al., in [35] as

$$\overline{HM}(G) = \sum_{uv \notin E(G)} \big(d_G(u) + d_G(v)\big)^2.$$

For basic properties of hyper Zagreb index and coindex refer [16] and for graph operations refer [3, 4, 33, 35].
The *sum-connectivity index* of a graph G was defined in [39] as

$$\chi(G) = \sum_{xy \in E(G)} \big(d_G(x) + d_G(y)\big)^{-\frac{1}{2}}.$$

Further, it has been extended to the *general sum-connectivity index* which is defined in [40] as

$$\chi_\alpha(G) = \sum_{xy \in E(G)} \big(d_G(x) + d_G(y)\big)^{\alpha}, \text{where } \alpha \text{ is any real number.}$$

For $\alpha = 3$ we have,

$$\chi_3(G) = \sum_{xy \in E(G)} \big(d_G(x) + d_G(y)\big)^3$$

For a graph G with vertex set $V(G)$ and edge set $E(G)$, there are four related transformation graphs as follows (see Figure 1):
• The *subdivision graph* $S = S(G)$ [22]; is the graph obtained by inserting a new vertex onto each edge of G.
• *Semitotal-point graph* $T_2 = T_2(G)$ [32]; $V(T_2) = V(G) \cup E(G)$ and $E(T_2) = E(S) \cup E(G)$;
• *Semitotal-line graph* $T_1 = T_1(G)$ [32]; $V(T_1) = V(G) \cup E(G)$ and $E(T_1) = E(S) \cup E(L)$;
• *Total graph* $T = T(G)$ [7]; $V(T) = V(G) \cup E(G)$ and $E(T) = E(S) \cup E(G) \cup E(L)$.
Here $L = L(G)$ is the line graph of G.

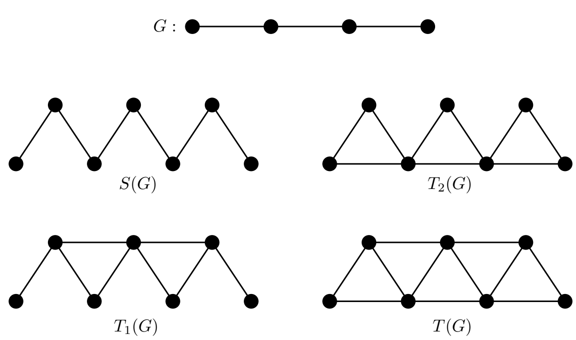

FIGURE 1. Graph G and its transformations $S(G), T_2(G), T_1(G)$ and $T(G)$.

3. New Co-normal products of graphs

Let $i = 1, 2$. For a given graph G_i, its vertex and edge sets will be denoted by $V(G_i)$ and $E(G_i)$, and their cardinalities by n_1 and m_1, respectively.

The *cartesian product* [22] $G_1 \times G_2$ of graphs G_1 and G_2 has the vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_1, v_1)(u_2, v_2)$ is an edge of $G_1 \times G_2$ if and only if $[u_1 = u_2$ and $v_1v_2 \in E(G_2)]$ or $[v_1 = v_2$ and $u_1u_2 \in E(G_1)]$. Based on the cartesian product of graphs, Eliasi and Taeri [11] introduced four new operations on graphs as follows:

Definition 1. [11] *Let $F \in \{S, T_2, T_1, T\}$. The F-sums of G_1 and G_2, denoted by $G_1 +_F G_2$, is a graph with the set of vertices $V(G_1 +_F G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(F(G_1))]$.*

Thus, authors in [11] obtained four new graph operations as $G_1 +_S G_2$, $G_1 +_{T_2} G_2$, $G_1 +_{T_1} G_2$ and $G_1 +_T G_2$ and studied the Wiener indices of these graphs. In [10], Deng et al. gave the expressions for first and second Zagreb indices of these new graphs.

In 1962, Ore [27] introduced a product graph, with the name Cartesian sum of graphs. Hammack et al. [21], named it co-normal product graph. The *co-normal product* [21] $G_1 \star G_2$ of two graphs G_1 and G_2 of order n_1 and n_2, respectively, is defined as the graph with vertex set $V_1 \times V_2$ and (u_1, v_1) is adjacent with (u_2, v_2) if and only if $u_1u_2 \in E(G_1)$ or $v_1v_2 \in E(G_2)$.

Motivated from [11], we introduce four new products of graphs by extending F-sums of graphs on cartesian product to co-normal product as follows:

Definition 2. *let $\mathcal{F}$ be the one of the symbols S, T_2, T_1 or T. The $\mathcal{F}-$co-normal product $G_1 \star_{\mathcal{F}} G_2$ is a graph with the set of vertices $V(G_1 \star_{\mathcal{F}} G_2) = (V(G_1) \cup E(G_1)) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \star_{\mathcal{F}} G_2$ are adjacent if and only if u_1 is adjacent to v_1 in $E(\mathcal{F}(G_1))$ or u_2 is adjacent to v_2 in G_2.*

We illustrate this definition in Figure 2

In this paper, we study the first Zagreb index, F-index and their coindices of $G_1 \star_S G_2$, $G_1 \star_{T_2} G_2$, $G_1 \star_{T_1} G_2$ and $G_1 \star_T G_2$.

The following results will be needed to prove our main results:

Theorem 3.1. [1, 8] *Let G be an (n, m) graph. Then*

i. $M_1(\overline{G}) = M_1(G) + n(n-1)^2 - 4m(n-1)$,
ii. $\overline{M_1}(G) = 2m(n-1) - M_1(G)$,
iii. $\overline{M_1}(\overline{G}) = 2m(n-1) - M_1(G)$.

Theorem 3.2. [16] *Let G be a graph with n vertices and m edges. Then*

(i) $F(\overline{G}) = n(n-1)^3 - 4m(n-1)^2 + 3(n-1)M_1(G) - F(G)$
(ii) $\overline{F}(G) = (n-1)M_1(G) - F(G)$
(iii) $\overline{F}(\overline{G}) = 2m(n-1)^2 - 2(n-1)M_1(G) + F(G)$.

4. First Zagreb index and coindex of $\mathcal{F}-$co-normal products of graphs and their complements

In this section, we proceed to obtain the first Zagreb index and coindex of $\mathcal{F}-$co-normal products of graphs and their complements for each $\mathcal{F} \in \{S, T_2, T_1, T\}$. We

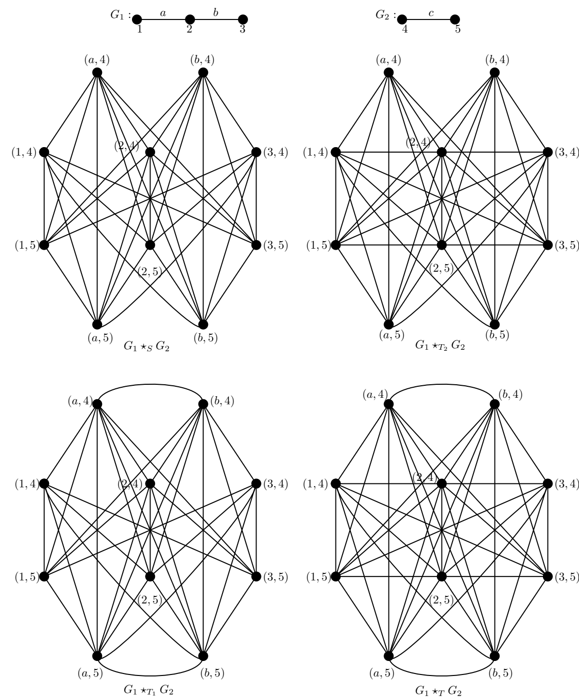

FIGURE 2. Graphs G_1, G_2 and $G_1 \star_{\mathcal{F}} G_2$

start by stating the following proposition which will be required to prove our main results:

Proposition 4.1. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively. Then*
$|V(G_1 \star_{\mathcal{F}} G_2)| = n_2(n_1 + m_1)$ *and*

(i) $E(G_1 \star_S G_2) = 2(m_1 n_2^2 + m_1 m_2 n_1 - 2m_1 m_2) + m_2(n_1^2 + m_1^2)$,
(ii) $E(G_1 \star_{T_2} G_2) = 3m_1 n_2^2 + n_1 m_2(n_1 + m_1) + m_1 m_2(n_1 + m_1 - 6)$,
(iii) $E(G_1 \star_{T_1} G_2) = 2m_1 n_2^2 + n_1 m_2(n_1 + m_1) + m_L(n_2^2 - 2m_2) + m_1 m_2(n_1 + m_1 - 4)$,
(iv) $E(G_1 \star_T G_2) = 3m_1 n_2^2 + n_1 m_2(n_1 + m_1) + m_L(n_2^2 - 2m_2) + m_1 m_2(n_1 + m_1 - 2)$.

Proposition 4.2. *Let G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively. If (u, v) is a vertex of $G_1 \star_{\mathcal{F}} G_2$, then*

1. $d_{G_1 \star_S G_2}(u, v) = \begin{cases} n_2 d_{G_1}(u) + \big(n_1 + m_1 - d_{G_1}(u)\big) d_{G_2}(v), & if\ u \in V(S(G_1)) \cap V(G_1),\ v \in V(G_2) \\ 2n_2 + \big(n_1 + m_1 - 2\big) d_{G_2}(v), & if\ u \in V(S(G_1)) \cap E(G_1),\ v \in V(G_2). \end{cases}$
2. $d_{G_1 \star_{T_2} G_2}(u, v) = \begin{cases} 2n_2 d_{G_1}(u) + \big(n_1 + m_1 - 2d_{G_1}(u)\big) d_{G_2}(v), & if\ u \in V(T_2(G_1)) \cap V(G_1),\ v \in V(G_2) \\ 2n_2 + \big(n_1 + m_1 - 2\big) d_{G_2}(v), & if\ u \in V(T_2(G_1)) \cap E(G_1),\ v \in V(G_2). \end{cases}$
3. $d_{G_1 \star_{T_1} G_2}(u, v) = \begin{cases} n_2 d_{G_1}(u) + \big(n_1 + m_1 - d_{G_1}(u)\big) d_{G_2}(v), & if\ u \in V(T_1(G_1)) \cap V(G_1),\ v \in V(G_2) \\ 2n_2 + \big(n_2 - 2d_{G_2}(v)\big) d_{G_1}(u) + (n_1 + m_1 - 2) d_{G_2}(v), & if\ u \in V(T_1(G_1)) \cap E(G_1),\ v \in V(G_2). \end{cases}$
4. $d_{G_1 \star_{T_1} G_2}(u, v) = \begin{cases} 2n_2 d_{G_1}(u) + \big(n_1 + m_1 - 2d_{G_1}(u)\big) d_{G_2}(v), & if\ u \in V(T_2(G_1)) \cap V(G_1),\ v \in V(G_2) \\ 2n_2 + \big(n_2 - 2d_{G_2}(v)\big) d_{G_1}(u) + (n_1 + m_1 - 2) d_{G_2}(v), & if\ u \in V(T_1(G_1)) \cap E(G_1),\ v \in V(G_2). \end{cases}$

The following theorem gives the first Zagreb index and coindex of S-co-normal product of two graphs G_1 and G_2.

Theorem 4.3. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} M_1(G_1 \star_S G_2) &= M_1(G_1)\Big(n_2^3 + M_1(G_1) - 4n_2m_2\Big) + M_1(G_2)(n_1+m_1)\Big(n_1(n_1+m_1) \\ &\quad -4m_1\Big) + 16m_1m_2n_2(n_1+m_1-1) + 4m_1n_2^3. \end{aligned}$$

Proof. Using the definition of the first Zagreb index and Proposition 4.2(1), we have

$$\begin{aligned} M_1(G_1 \star_S G_2) &= \sum_{(u,v)\in V(G_1 \star_S G_2)} d^2_{G_1\star_S G_2}(u,v) \\ &= \sum_{u\in V(S(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} \Big(n_2 d_{G_1}(u) + \big(n_1+m_1-d_{G_1}(u)\big)d_{G_2}(v)\Big)^2 \\ &\quad + \sum_{z\in V(G_2)} \sum_{e\in V(S(G_1))\cap E(G_1)} \Big(2n_2 + \big(n_1+m_1-2\big)d_{G_2}(z)\Big)^2. \\ &= M_1(G_1)\Big(n_2^3 + M_1(G_1) - 4n_2m_2\Big) + M_1(G_2)(n_1+m_1)\Big(n_1(n_1+m_1) \\ &\quad -4m_1\Big) + 16m_1m_2n_2(n_1+m_1-1) + 4m_1n_2^3. \end{aligned}$$

□

Following corollaries give the first Zagreb index of $\overline{G_1 \star_S G_2}$, first Zagreb coindex of graph $G_1 \star_S G_2$ and its complement $\overline{M_1}(\overline{G_1 \star_S G_2})$, respectively.

Corollary 4.4. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} M_1(\overline{G_1 \star_S G_2}) &= M_1(G_1)\Big(n_2^3 + M_1(G_1) - 4n_2m_2\Big) + M_1(G_2)\Big(n_1(n_1+m_1)^2 \\ &\quad -4m_1(n_1+m_1)\Big) + 16m_1m_2n_2(n_1+m_1-1) + 4m_1n_2^3 \\ &\quad +\big(n_2(n_1+m_1)-1\big)\Big(n_2(n_1+m_1)\big(n_2(n_1+m_1)-1\big) \\ &\quad -4\big(2(m_1n_2^2 + m_1m_2n_1 - 2m_1m_2) + m_2(n_1^2+m_1^2)\big)\Big). \end{aligned}$$

Proof. Using Proposition 4.1 (i) and Theorem 4.3 in Theorem 3.1 (i) we get the desired result. □

Corollary 4.5. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} \overline{M_1}(G_1 \star_S G_2) &= 2\big(2(m_1n_2^2 + m_1m_2n_1 - 2m_1m_2) + m_2(n_1^2+m_1^2)\big)\big(n_2(n_1+m_1)-1\big) \\ &\quad -\big(n_2M_1(G_1) + (n_1+m_1)^3M_1(G_2) + 16m_1m_2(n_1+m_1) + 4m_1n_2\big). \end{aligned}$$

Proof. Using Proposition 4.1 (i) and Theorem 4.3 in Theorem 3.1 (ii) we get the desired result. □

Corollary 4.6. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} \overline{M_1}(\overline{G_1 \star_S G_2}) &= 2\big(2(m_1n_2^2 + m_1m_2n_1 - 2m_1m_2) + m_2(n_1^2+m_1^2)\big)\big(n_2(n_1+m_1)-1\big) \\ &\quad -\big(n_2M_1(G_1) + (n_1+m_1)^3M_1(G_2) + 16m_1m_2(n_1+m_1) + 4m_1n_2\big). \end{aligned}$$

Proof. Using Proposition 4.1 (i) and Theorem 4.3 in Theorem 3.1 (iii) we get the desired result. □

The following theorem gives the first Zagreb index of T_2-co-normal product of two graphs G_1 and G_2.

Theorem 4.7. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} M_1(G_1 \star_{T_2} G_2) &= 4n_2M_1(G_1)\Big(n_2^2 - 4m_2\Big) + M_1(G_2)\Big(n_1(n_1+m_1)^2 + 4M_1(G_1) \\ &\quad -8m_1(n_1+m_1) + m_1(n_1+m_1-2)^2\Big) + 8n_2m_1m_2\Big(3(n_1+m_1)-2\Big) \\ &\quad +4n_2^3m_1 \end{aligned}$$

Proof. Using the definition of the first Zagreb index and Proposition 4.2(2), we have

$$\begin{aligned} M_1(G_1 \star_{T_2} G_2) &= \sum_{(u,v)\in V(G_1\star_{T_2}G_2)} d^2_{G_1\star_{T_2}G_2}(u,v) \\ &= \sum_{u\in V(T_2(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} \Big(2n_2d_{G_1}(u) + \big(n_1+m_1-2d_{G_1}(u)\big)d_{G_2}(v)\Big)^2 \\ &\quad + \sum_{z\in V(G_2)} \sum_{e\in V(T_2(G_1))\cap E(G_1)} \Big(2n_2 + \big(n_1+m_1-2\big)d_{G_2}(z)\Big)^2. \\ &= 4n_2M_1(G_1)\Big(n_2^2 - 4m_2\Big) + M_1(G_2)\Big(n_1(n_1+m_1)^2 + 4M_1(G_1) \\ &\quad -8m_1(n_1+m_1) + m_1(n_1+m_1-2)^2\Big) + 8n_2m_1m_2\Big(3(n_1+m_1)-2\Big) \\ &\quad +4n_2^3m_1. \end{aligned}$$

□

Using Proposition 4.1 (ii) and Theorem 4.7 in Theorem 3.1, we get the desired result. we get the following results for the first Zagreb index of $\overline{G_1 \star_{T_2} G_2}$, first Zagreb coindex of graph $G_1 \star_{T_2} G_2$ and its complement $\overline{M_1}(\overline{G_1 \star_{T_2} G_2})$, respectively.

Corollary 4.8. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} M_1(\overline{G_1 \star_{T_2} G_2}) &= 4n_2M_1(G_1)\Big(n_2^2 - 4m_2\Big) + M_1(G_2)\Big(n_1(n_1+m_1)^2 + 4M_1(G_1) \\ &\quad -8m_1(n_1+m_1) + m_1(n_1+m_1-2)^2\Big) + 8n_2m_1m_2\Big(3(n_1+m_1)-2\Big) \\ &\quad +4n_2^3m_1 + \big(n_2(n_1+m_1)-1\big)\Big(n_2(n_1+m_1)\big(n_2(n_1+m_1)-1\big) \\ &\quad -4\big(3m_1n_2^2 + n_1m_2(n_1+m_1) + m_1m_2(n_1+m_1-6)\big)\Big). \end{aligned}$$

Corollary 4.9. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} \overline{M_1}(G_1 \star_{T_2} G_2) &= 2\big(3m_1n_2^2 + n_1m_2(n_1+m_1) + m_1m_2(n_1+m_1-6)\big)\big(n_2(n_1+m_1)-1\big) \\ &\quad -\Big(4n_2M_1(G_1)\Big(n_2^2 - 4m_2\Big) + M_1(G_2)\Big(n_1(n_1+m_1)^2 + 4M_1(G_1) \\ &\quad -8m_1(n_1+m_1) + m_1(n_1+m_1-2)^2\Big) + 8n_2m_1m_2\Big(3(n_1+m_1)-2\Big) \\ &\quad +4n_2^3m_1\Big). \end{aligned}$$

Corollary 4.10. *If* G_1 *and* G_2 *are* (n_1, m_1) *and* (n_2, m_2) *graphs, respectively, then*

$$\begin{aligned}\overline{M_1}(\overline{G_1 \star_{T_2} G_2}) &= 2\big(3m_1n_2^2 + n_1m_2(n_1+m_1) + m_1m_2(n_1+m_1-6)\big)\big(n_2(n_1+m_1)-1\big)\\ &\quad -\Big(4n_2M_1(G_1)\Big(n_2^2-4m_2\Big) + M_1(G_2)\Big(n_1(n_1+m_1)^2 + 4M_1(G_1)\\ &\quad -8m_1(n_1+m_1) + m_1(n_1+m_1-2)^2\Big) + 8n_2m_1m_2\Big(3(n_1+m_1)-2\Big)\\ &\quad +4n_2^3m_1\Big).\end{aligned}$$

The following theorem gives the first Zagreb index of T_1-co-normal product of two graphs G_1 and G_2.

Theorem 4.11. *If* G_1 *and* G_2 *are* (n_1, m_1) *and* (n_2, m_2) *graphs, respectively, then*

$$\begin{aligned}M_1(G_1 \star_{T_1} G_2) &= M_1(G_1)[n_2^3 - 4n_2m_2] + M_1(G_2)(n_1+m_1)[n_1(n_1+m_1) - 4m_1]\\ &\quad +M_1(G_1)M_1(G_2) + 8n_2m_1m_2(n_1+m_1) + 4m_1n_2^3\\ &\quad +EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2]\\ &\quad +M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L]\\ &\quad +8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_1-2)(m_1+m_L).\end{aligned}$$

Proof. Using the definition of the first Zagreb index and Proposition 4.2(3) we have,

$$\begin{aligned}M_1(G_1 \star_{T_1} G_2) &= \sum_{(u,v)\in V(G_1\star_{T_1}G_2)} d^2_{G_1\star_{T_1}G_2}(u,v)\\ &= \sum_{u\in V(T_1(G_1))\cap V(G_1)}\ \sum_{v\in V(G_2)} \Big(n_2d_{G_1}(u) + (n_1+m_1-d_{G_1}(u))d_{G_2}(v)\Big)^2\\ &\quad + \sum_{z\in V(G_2)}\ \sum_{e\in V(T_1(G_1))\cap E(G_1)} \Big(2n_2 + \big(n_2 - d_{G_2}(z)\big)d_{G_1}(e) + (n_1+m_1-2)d_{G_2}(z)\Big)^2.\\ &= M_1(G_1)[n_2^3 - 4n_2m_2] + M_1(G_2)(n_1+m_1)[n_1(n_1+m_1) - 4m_1]\\ &\quad +M_1(G_1)M_1(G_2) + 8n_2m_1m_2(n_1+m_1) + 4m_1n_2^3\\ &\quad +EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2]\\ &\quad +M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L]\\ &\quad +8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_1-2)(m_1+m_L).\end{aligned}$$

□

Using Proposition 4.1 (iii) and Theorem 4.11 in Theorem 3.1, we get the following results for the first Zagreb index of $\overline{G_1 \star_{T_1} G_2}$, the first Zagreb index coindex of graph $G_1 \star_{T_1} G_2$ and its complement $\overline{M_1}(\overline{G_1 \star_{T_1} G_2})$, respectively.

Corollary 4.12. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} M_1(\overline{G_1 \star_{T_1} G_2}) &= M_1(G_1)[n_2^3 - 4n_2m_2] + M_1(G_2)(n_1+m_1)[n_1(n_1+m_1) - 4m_1] \\ &\quad + M_1(G_1)M_1(G_2) + 8n_2m_1m_2(n_1+m_1) + 4m_1n_2^3 \\ &\quad + EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2] \\ &\quad + M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L] \\ &\quad + 8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_1-2)(m_1+m_L) \\ &\quad + \big(n_2(n_1+m_1)\big)\Big(n_2(n_1+m_1)\big(n_2(n_1+m_1)-1\big) \\ &\quad - 4\big(2m_1n_2^2 + n_1m_2(n_1+m_1) + m_L(n_2^2 - 2m_2) + m_1m_2(n_1+m_1-4)\big)\Big). \end{aligned}$$

Corollary 4.13. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} \overline{M_1}(G_1 \star_{T_1} G_2) &= 2\big(2m_1n_2^2 + n_1m_2(n_1+m_1) + m_L(n_2^2 - 2m_2) \\ &\quad + m_1m_2(n_1+m_1-4)\big)(n_2(n_1+m_1)-1) \\ &\quad - \Big(M_1(G_1)[n_2^3 - 4n_2m_2] + M_1(G_2)(n_1+m_1)[n_1(n_1+m_1) - 4m_1] \\ &\quad + M_1(G_1)M_1(G_2) + 8n_2m_1m_2(n_1+m_1) + 4m_1n_2^3 \\ &\quad + EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2] \\ &\quad + M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L] \\ &\quad + 8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_1-2)(m_1+m_L)\Big). \end{aligned}$$

Corollary 4.14. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} \overline{M_1}(\overline{G_1 \star_{T_1} G_2}) &= 2\big(2m_1n_2^2 + n_1m_2(n_1+m_1) + m_L(n_2^2 - 2m_2) \\ &\quad + m_1m_2(n_1+m_1-4)\big)\big(n_2(n_1+m_1)-1\big) \\ &\quad - \Big(M_1(G_1)[n_2^3 - 4n_2m_2] + M_1(G_2)(n_1+m_1)[n_1(n_1+m_1) - 4m_1] \\ &\quad + M_1(G_1)M_1(G_2) + 8n_2m_1m_2(n_1+m_1) + 4m_1n_2^3 \\ &\quad + EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2] \\ &\quad + M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L] \\ &\quad + 8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_1-2)(m_1+m_L)\Big). \end{aligned}$$

The following theorem gives the first Zagreb index of T-co-normal product of two graphs G_1 and G_2.

Theorem 4.15. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned} M_1(G_1 \star_T G_2) &= 4M_1(G_1)[n_2^3 - 4n_2m_2] + (n_1+m_1)M_1(G_2)[n_1(n_1+m_1) - 8m_1] \\ &\quad 16n_2m_1m_2(n_1+m_1) + 4M_1(G_1)M_1(G_2) + 4n_2^3m_1 \\ &\quad + EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2] \\ &\quad + M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L] \\ &\quad + 8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_+-2)(m_1+m_L). \end{aligned}$$

Proof. Using the definition of the first Zagreb index and Proposition 4.2(4) we have,

$$
\begin{aligned}
M_1(G_1 \star_T G_2) &= \sum_{(u,v)\in V(G_1\star_T G_2)} d^2_{G_1\star_T G_2}(u,v)\\
&= \sum_{u\in V(T(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} \Big(2n_2 d_{G_1}(u) + \big(n_1+m_1-2d_{G_1}(u)\big)d_{G_2}(v)\Big)^2\\
&\quad + \sum_{z\in V(G_2)} \sum_{e\in V(T(G_1))\cap E(G_1)} \Big(2n_2 + \big(n_2 - d_{G_2}(z)\big)d_{G_1}(e) + (n_1+m_1-2)d_{G_2}(z)\Big)^2.\\
&= 4M_1(G_1)[n_2^3 - 4n_2m_2] + (n_1+m_1)M_1(G_2)[n_1(n_1+m_1) - 8m_1]\\
&\quad 16n_2m_1m_2(n_1+m_1) + 4M_1(G_1)M_1(G_2) + 4n_2^3m_1\\
&\quad +EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2]\\
&\quad +M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L]\\
&\quad +8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_+-2)(m_1+m_L).
\end{aligned}
$$

□

Using Proposition 4.1 (iv) and Theorem 4.15 in Theorem 3.1, we get the following results for the first Zagreb index of $\overline{G_1 \star_T G_2}$, first Zagreb coindex of graph $G_1 \star_T G_2$ and its complement $\overline{M_1}(\overline{G_1 \star_T G_2})$, respectively.

Corollary 4.16. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$
\begin{aligned}
M_1(\overline{G_1 \star_T G_2}) &= 4M_1(G_1)[n_2^3 - 4n_2m_2] + (n_1+m_1)M_1(G_2)[n_1(n_1+m_1) - 8m_1]\\
&\quad 16n_2m_1m_2(n_1+m_1) + 4M_1(G_1)M_1(G_2) + 4n_2^3m_1\\
&\quad +EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2]\\
&\quad +M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L]\\
&\quad +8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_+-2)(m_1+m_L)\\
&\quad +\big(n_2(n_1+m_1)-1\big)\Big(n_2(n_1+m_1)\big(n_2(n_1+m_1)-1\big)\\
&\quad -4\Big(3m_1n_2^2 + n_1m_2(n_1+m_1) + m_L(n_2^2-2m_2) + m_1m_2(n_1+m_1-2)\Big)\Big).
\end{aligned}
$$

Corollary 4.17. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$
\begin{aligned}
\overline{M_1}(G_1 \star_T G_2) &= 2\big(3m_1n_2^2 + n_1m_2(n_1+m_1) + m_L(n_2^2-2m_2)\\
&\quad +m_1m_2(n_1+m_1-2)\big)\big(n_2(n_1+m_1)-1\big)\\
&\quad -\Big(4M_1(G_1)[n_2^3 - 4n_2m_2] + (n_1+m_1)M_1(G_2)[n_1(n_1+m_1) - 8m_1]\\
&\quad 16n_2m_1m_2(n_1+m_1) + 4M_1(G_1)M_1(G_2) + 4n_2^3m_1\\
&\quad +EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2]\\
&\quad +M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L]\\
&\quad +8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_+-2)(m_1+m_L)\Big).
\end{aligned}
$$

Corollary 4.18. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned}\overline{M_1}(\overline{G_1 \star_T G_2}) &= 2\big(3m_1n_2^2 + n_1m_2(n_1+m_1) + m_L(n_2^2 - 2m_2) \\ &\quad + m_1m_2(n_1+m_1-2)\big)\big(n_2(n_1+m_1) - 1\big) \\ &\quad -\Big(4M_1(G_1)[n_2^3 - 4n_2m_2] + (n_1+m_1)M_1(G_2)[n_1(n_1+m_1) - 8m_1] \\ &\quad 16n_2m_1m_2(n_1+m_1) + 4M_1(G_1)M_1(G_2) + 4n_2^3m_1 \\ &\quad + EM_1(G_1)[n_2^3 + M_1(G_2) - 4n_2m_2] \\ &\quad + M_1(G_2)(n_1+m_1-2)[m_1(n_1+m_1-2) - 4m_L] \\ &\quad + 8n_2m_L[n_2^2 - 2m_2] + 8n_2m_2(n_1+m_+-2)(m_1+m_L)\Big).\end{aligned}$$

5. F-index and coindex of $\mathcal{F}-$co-normal products of graphs and their complements

In this section, we obtain F-index and coindex of $\mathcal{F}-$co-normal products of graphs and their complements. From Theorem 3.2, it is clear that $M_1(G_1 \star_{\mathcal{F}} G_2)$ and $F(G_1 \star_{\mathcal{F}} G_2)$ are known then $F(\overline{G_1 \star_{\mathcal{F}} G_2})$, $\overline{F}(G_1 \star_{\mathcal{F}} G_2)$ and $\overline{F}(\overline{G_1 \star_{\mathcal{F}} G_2})$ are known, what really needs to be calculated are expressions for $F(G_1 \star_{\mathcal{F}} G_2)$.

Theorem 5.1. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned}F(G_1 \star_S G_2) &= F(G_1)\Big[n_2^4 - 6n_2^2m_2 + 3n_2M_1(G_2)\Big] \\ &\quad + F(G_2)\Big[(n_1+m_1)^3n_1 - F(G_1) - 6m_1(n_1+m_1)^2 \\ &\quad + 3(n_1+m_1)M_1(G_1) + m_1(n_1+m_1-2)^3\Big] + 6n_2^2m_2(n_1+m_1) \\ &\quad + M_1(G_2)\Big[6n_2m_1(n_1+m_1)^2 - 6n_2(n_1+m_1)M_1(G_1) \\ &\quad + 6n_2m_1(n_1+m_1-2)^2\Big] + 8n_2^4m_1 + 24n_2^2m_1m_2(n_1+m_1-2).\end{aligned}$$

Proof. By the definition of $F-$ index and Proposition 4.2(1), we have

$$\begin{aligned}F(G_1 \star_S G_2) &= \sum_{(u,v)\in V(G_1\star_S G_2)} d^3_{G_1\star_S G_2}(u,v) \\ &= \sum_{u\in V(S(G_1))\cap V(G_1)} \sum_{v\in V(G_2)} \Big(n_2d_{G_1}(u) + [n_1+m_1-d_{G_1}(u)]d_{G_2}(v)\Big)^3 \\ &\quad + \sum_{z\in V(G_2)} \sum_{e\in V(S(G_1))\cap E(G_1)} \Big(2n_2 + [n_1+m_1-2]d_{G_2}(z)\Big)^3 \\ &= J_1 + J_2\end{aligned}$$

Where J_1, J_2 are the sums of the above terms, in order

$$\begin{aligned}
J_1 &= \sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\Big(n_2 d_{G_1}(u)+[n_1+m_1-d_{G_1}(u)]d_{G_2}(v)\Big)^3\\
&= F(G_1)\Big[n_2^4-6n_2^2m_2+3n_2M_1(G_2)\Big]\\
&\quad +F(G_2)\Big[(n_1+m_1)^3n_1-F(G_1)-6m_1(n_1+m_1)^2\\
&\quad +3(n_1+m_1)M_1(G_1)\Big]+6n_2^2m_2(n_1+m_1)\\
&\quad +M_1(G_2)\Big[6n_2m_1(n_1+m_1)^2-6n_2(n_1+m_1)M_1(G_1)\Big].
\end{aligned}$$

$$\begin{aligned}
J_2 &= \sum_{z\in V(G_2)}\sum_{e\in E(G_1)}\Big(2n_2+[n_1+m_1-2]d_{G_2}(z)\Big)^3\\
&= 8n_2^4m_1+m_1(n_1+m_1-2)^3F(G_2)+24n_2^2m_1m_2(n_1+m_1-2)\\
&\quad +6n_2m_1(n_1+m_1-2)^2M_1(G_2).
\end{aligned}$$

Adding J_1, J_2, we get the desired result. □

The following theorem gives the $F-$ index of T_2- co-normal product of two graphs G_1 and G_2.

Theorem 5.2. *If G_1 and G_2 are (n_1,m_1) and (n_2,m_2) graphs, respectively, then*

$$\begin{aligned}
F(G_1\star_{T_2}G_2) &= F(G_1)\Big[8n_2^4-24n_2^3+24n_2^2-F(G_2)\Big]+F(G_2)\Big[n_1(n_1+m_1)^3\\
&\quad -12(n_1+m_1)^2m_1+m_1(n_1+m_1-2)^3\Big]+6n_2^2m_2(n_1+m_1)\\
&\quad +12(n_1+m_1)M_1(G_1)\Big[n_2^3-2n_2^2+F(G_2)\Big]+12m_1n_2^2(n_1+m_1)^2\\
&\quad +6n_2m_1(n_1+m_1-2)^2M_1(G_2)+8n_2^4m_1+24n_2^2m_1m_2(n_1+m_1-2).
\end{aligned}$$

Proof. By the definition of $F-$ index and Proposition 4.2(2), we have

$$\begin{aligned}
F(G_1\star_{T_2}G_2) &= \sum_{(u,v)\in V(G_1\star_{T_2}G_2)} d^3_{G_1\star_{T_2}G_2}(u,v)\\
&= \sum_{u\in V(T_2(G_1))\cap V(G_1)}\sum_{v\in V(G_2)}\Big(2n_2d_{G_1}(u)+[n_1+m_1-2d_{G_1}(u)]d_{G_2}(v)\Big)^3\\
&\quad +\sum_{z\in V(G_2)}\sum_{e\in V(T_2(G_1))\cap E(G_1)}\Big(2n_2+[n_1+m_1-2]d_{G_2}(z)\Big)^3\\
&= K_1+K_2
\end{aligned}$$

Where K_1, K_2 are the sums of the above terms, in order

$$\begin{aligned}
K_1 &= \sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\Big(2n_2d_{G_1}(u)+[n_1+m_1-2d_{G_1}(u)]d_{G_2}(v)\Big)^3\\
&= F(G_1)\Big[8n_2^4-24n_2^3+24n_2^2-F(G_2)\Big]\\
&\quad +F(G_2)\Big[n_1(n_1+m_1)^3-12(n_1+m_1)^2m_1\Big]+6n_2^2m_2(n_1+m_1)\\
&\quad +12(n_1+m_1)M_1(G_1)\Big[n_2^3-2n_2^2+F(G_2)\Big]+12m_1n_2^2(n_1+m_1)^2.
\end{aligned}$$

$$\begin{aligned}
K_2 &= \sum_{z\in V(G_2)}\sum_{e\in E(G_1)}\Big(2n_2+[n_1+m_1-2]d_{G_2}(z)\Big)^3\\
&= 8n_2^4m_1+m_1(n_1+m_1-2)^3F(G_2)+24n_2^2m_1m_2(n_1+m_1-2)\\
&\quad +6n_2m_1(n_1+m_1-2)^2M_1(G_2).
\end{aligned}$$

Adding K_1, K_2, we obtain the required result. □

The following theorem gives the F- index of T_1- co-normal product of two graphs G_1 and G_2.

Theorem 5.3. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned}
F(G_1\star_{T_1}G_2) &= F(G_1)\Big[n_2^4-6n_2^2m_2+3n_2M_1(G_2)\Big]+F(G_2)\Big[(n_1+m_1)^3n_1\\
&\quad -F(G_1)-6m_1(n_1+m_1)^2+3(n_1+m_1)M_1(G_1)\\
&\quad +(n_1+m_1-2)\Big(m_1-6m_L(n_1+m_1-2)+3EM_1(G_1)\Big)\Big]\\
&\quad +6n_2^2m_2(n_1+m_1)+M_1(G_2)\Big[6n_2m_1(n_1+m_1)^2-6n_2(n_1+m_1)M_1(G_1)\\
&\quad +6n_2(n_1+m_1-2)^2(m_1+m_L)\Big]+6n_2\Big[n_2^3+M_1(G_2)-4n_2m_2\\
&\quad +n_2m_2(n_1+m_1-2)-2m_2(n_1+m_1-2)\Big]\\
&\quad +EF(G_1)\Big[n_2^4-F(G_2)-6n_2^2m_2+3n_2M_1(G_2)\Big]+8n_2^4m_1.
\end{aligned}$$

Proof. By the definition of $F-$ index and Proposition 4.2(3) we have,

$$\begin{aligned}
F(G_1\star_{T_1}G_2) &= \sum_{(u,v)\in V(G_1\star_{T_1}G_2)} d^3_{G_1\star_{T_1}G_2}(u,v)\\
&= \sum_{u\in V(T_1(G_1))\cap V(G_1)}\sum_{v\in V(G_2)}\Big(n_2d_{G_1}(u)+[n_1+m_1-d_{G_1}(u)]d_{G_2}(v)\Big)^3\\
&\quad +\sum_{z\in V(G_2)}\sum_{e\in V(T_1(G_1))\cap E(G_1)}\Big(2n_2+[n_2-d_{G_2}(z)]d_{G_1}(e)+(n_1+m_1-2)d_{G_2}(z)\Big)^3\\
&= L_1+L_2
\end{aligned}$$

Where L_1, L_2 are the sums of the above terms, in order

$$\begin{aligned}
L_1 &= \sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\Big(n_2 d_{G_1}(u)+[n_1+m_1-d_{G_1}(u)]d_{G_2}(v)\Big)^3\\
&= F(G_1)\Big[n_2^4-6n_2^2m_2+3n_2M_1(G_2)\Big]\\
&\quad +F(G_2)\Big[(n_1+m_1)^3n_1-F(G_1)-6m_1(n_1+m_1)^2\\
&\quad +3(n_1+m_1)M_1(G_1)\Big]+6n_2^2m_2(n_1+m_1)\\
&\quad +M_1(G_2)\Big[6n_2m_1(n_1+m_1)^2-6n_2(n_1+m_1)M_1(G_1)\Big].
\end{aligned}$$

$$\begin{aligned}
L_2 &= \sum_{z\in V(G_2)}\sum_{e\in E(G_1)}\Big(2n_2+[n_2-d_{G_2}(z)]d_{G_1}(e)+(n_1+m_1-2)d_{G_2}(z)\Big)^3\\
&= 8n_2^4m_1+F(G_2)(n_1+m_1-2)\Big[m_1-6m_L(n_1+m_1-2)\\
&\quad +3EM_1(G_1)\Big]+6n_2(n_1+m_1-2)^2M_1(G_2)\Big[m_1+m_L\Big]\\
&\quad +6n_2\Big[n_2^3+M_1(G_2)-4n_2m_2+n_2m_2(n_1+m_1-2)-2m_2(n_1+m_1-2)\Big]\\
&\quad +EF(G_1)\Big[n_2^4-F(G_2)-6n_2^2m_2+3n_2M_1(G_2)\Big].
\end{aligned}$$

Combining L_1, L_2, we obtain the desired result. □

The following theorem gives the F- index of $T-$ co-normal product of two graphs G_1 and G_2.

Theorem 5.4. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\begin{aligned}
F(G_1\star_T G_2) &= F(G_1)\Big[8n_2^4-24n_2^3+24n_2^2-F(G_2)\Big]+F(G_2)\Big[n_1(n_1+m_1)^3\\
&\quad -12(n_1+m_1)^2m_1+(n_1+m_1-2)\Big(m_1-6m_L(n_1+m_1-2)\\
&\quad +3EM_1(G_1)\Big)\Big]+6n_2^2m_2(n_1+m_1)+12(n_1+m_1)M_1(G_1)\Big[n_2^3-2n_2^2+F(G_2)\Big]\\
&\quad +12m_1n_2^2(n_1+m_1)^2+8n_2^4m_1+6n_2(n_1+m_1-2)^2M_1(G_2)\Big[m_1+m_L\Big]\\
&\quad +6n_2\Big[n_2^3+M_1(G_2)-4n_2m_2+n_2m_2(n_1+m_1-2)-2m_2(n_1+m_1-2)\Big]\\
&\quad +EF(G_1)\Big[n_2^4-F(G_2)-6n_2^2m_2+3n_2M_1(G_2)\Big].
\end{aligned}$$

Proof. By the definition of $F-$ index and Proposition 4.2(4) we have,

$$\begin{aligned}
F(G_1\star_T G_2) &= \sum_{(u,v)\in V(G_1\star_T G_2)} d^3_{G_1\star_T G_2}(u,v)\\
&= \sum_{u\in V(T_2(G_1))\cap V(G_1)}\sum_{v\in V(G_2)}\Big(2n_2d_{G_1}(u)+[n_1+m_1-2d_{G_1}(u)]d_{G_2}(v)\Big)^3\\
&\quad +\sum_{z\in V(G_2)}\sum_{e\in V(T_2(G_1))\cap E(G_1)}\Big(2n_2+[n_2-d_{G_2}(z)]d_{G_1}(e)+(n_1+m_1-2)d_{G_2}(z)\Big)^3\\
&= M_1+M_2
\end{aligned}$$

Where M_1, M_2 are the sums of the above terms, in order

$$\begin{aligned} M_1 &= \sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\Big(2n_2 d_{G_1}(u) + [n_1+m_1-2d_{G_1}(u)]d_{G_2}(v)\Big)^3 \\ &= F(G_1)\Big[8n_2^4 - 24n_2^3 + 24n_2^2 - F(G_2)\Big] \\ &\quad + F(G_2)\Big[n_1(n_1+m_1)^3 - 12(n_1+m_1)^2 m_1\Big] + 6n_2^2 m_2(n_1+m_1) \\ &\quad + 12(n_1+m_1)M_1(G_1)\Big[n_2^3 - 2n_2^2 + F(G_2)\Big] + 12m_1 n_2^2(n_1+m_1)^2. \end{aligned}$$

$$\begin{aligned} M_2 &= \sum_{z\in V(G_2)}\sum_{e\in E(G_1)}\Big(2n_2 + [n_2 - d_{G_2}(z)]d_{G_1}(e) + (n_1+m_1-2)d_{G_2}(z)\Big)^3 \\ &= 8n_2^4 m_1 + F(G_2)(n_1+m_1-2)\Big[m_1 - 6m_L(n_1+m_1-2) \\ &\quad + 3EM_1(G_1)\Big] + 6n_2(n_1+m_1-2)^2 M_1(G_2)\Big[m_1+m_L\Big] \\ &\quad + 6n_2\Big[n_2^3 + M_1(G_2) - 4n_2m_2 + n_2m_2(n_1+m_1-2) - 2m_2(n_1+m_1-2)\Big] \\ &\quad + EF(G_1)\Big[n_2^4 - F(G_2) - 6n_2^2m_2 + 3n_2M_1(G_2)\Big]. \end{aligned}$$

Combining M_1, M_2, we get the required result. □

6. Conclusion

In this paper, we have obtained the first Zagreb index and $F-$ index of $\mathcal{F}-$ co-normal products of graphs and their complements. The first Zagreb index and $F-$ index are calculated explicitly for each case $\mathcal{F} \in \{S, T_2, T_1, T\}$.

Acknowledgement B. Basavanagoud is supported by the University Grants Commission (UGC), Government of India, New Delhi, through UGC-SAP-DRS-III for 2016-2021: F.510/3/DRS-III/2016(SAP-I). Shruti Policepatil is supported by Karnatak University, Dharwad, Karnataka, India, through University Research Studentship (URS), No.KU/Sch/URS/2020/1107, dated 21.12.2020.

References

[1] A. R. Ashrafi, T. Došlić, A. Hamzeh, The Zagreb coindices of graph operations, *Discrete Appl. Math.,* **158(15)** (2010) 1571–1578.

[2] S. Akhtera, M. Imran, Computing the forgotten topological index of four operations on graphs, *AKCE Int. J. Graphs Comb.,* **14(1)** (2017) 70–79.

[3] B. Basavanagoud, S. Patil, A Note on Hyper-Zagreb Index of Graph Operations, *Iranian J. Math. Chem.,* **7(1)** (2016) 89–92.

[4] B. Basavanagoud, S. Patil, A Note on hyper-Zagreb coindex of Graph Operations, *J. Appl. Math. Comput.,* **53(1)** (2017) 647 – 655.

[5] B. Basavanagoud, Praveen Jakkannavar, Kulli-Basava indices of graphs, *Int. J. Appl. Eng. Res.,* **14(1)** (2019) 325–342.

[6] B. Basavanagoud, S. Policepatil, Chemical applicability of Gourava and hyper-Gourava indices, *Nanosystems: Physics, Chemistry, Mathematics*, **12(2)** (2021) 142–150.

[7] M. Behzad, A criterion for the planarity of a total graph, *Pro. Cambridge Philos. Soc.,* **63 (3)** (1967) 697–681.

[8] K. C. Das, I. Gutman, Some properties of the second Zagreb index, *MATCH Commun. Math. Comput. Chem.*, **52** (2004) 103–112.
[9] K. C. Das, A. Yurttas, M. Togan, A. S. Cevik, I. N. Cangul, The multiplicative Zagreb indices of graph operations, *J. Inequal. Appl.,* **90** (2013) 1–14.
[10] H. Deng, D. Sarala, S. K. Ayyaswamy, S. Balachandran, The Zagreb indices of four operations on graphs, *Appl. Math. Comput.,* **275** (2016) 422–431,
[11] M. Eliasi, B. Taeri, Four new sums of graphs and their Wiener indices, *Discrete Appl. Math.,* **157(4)** (2009) 794–803.
[12] N. De, S. M. A. Nayeem, A. Pal, The F-coindex of some graph operations, *Springer Plus*, (2016) 5 (1):221. DOI: 10.1186/s40064-016-1864-7.
[13] B. Furtula, I. Gutman, A forgotten topological index, *J. Math. Chem.,* **53(4)** (2015) 1184–1190.
[14] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.,* **15** (2008).
[15] A. Gravoć, T. Pisanski, On Weiner index of a graph, *J. Math. Chem.,* **8(1)** (1991) 53–62.
[16] I. Gutman, On hyper Zagreb index and coindex, *Bulletin T. CL de l'Académie serbe des sciences et des arts,* **42** (2017) 1–8.
[17] I. Gutman, K. C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.,* **50** (2004) 83–92.
[18] I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin (1986).
[19] I. Gutman, B. Ruščić, N. Trinajstić, C. F. Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes, *J. Chem. Phys.,* **62** (1975) 3399–3405.
[20] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals, Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.,* **17(4)** (1972) 535–538.
[21] R. Hammack, W. Imrich and S. Klavžar, Handbook of product graphs (second edition), *Taylor & Francis group*, (2011).
[22] F. Harary, *Graph Theory*, Addison-Wesely, Reading, (1969).
[23] M. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Discrete Appl. Math.,* **157(4)** (2009) 804–811.
[24] W. Imrich, S. Klavžar, *Product graphs, structure and recognition*, John Wiley and Sons, New York, USA, (2000).
[25] M. J. Nadjafi-Arani, H. Khodashenas, Distance-based topological indices of tensor product of graphs, *Iranian J. Math. Chem.,* **3(1)** (2012) 45–53.
[26] S. Nikolić, G. Kovačević, A. Milićević, N. Trinajstić, The Zagreb indices 30 years after, *Croat. Chem. Acta.,* **76(2)** (2003) 113–124.
[27] O. Ore, Theory of Graphs, *Amer. Math. Soc.*, (1962).
[28] K. Pattabiraman, P. Paulraja, On some topological indices of the tensor products of graphs, *Discret. Appl. Math.,* **160(3)** (2012) 267–279.
[29] K. Pattabiraman and P. Kandan, Weighted PI index of corona product of graphs, *Discrete Math. Algorithms Appl.*, **6(4)** (2014) 1450055(9 pages).
[30] K. Pattabiraman and P. Kandan, Generalization of the degree distance of the tensor product of graphs, Australas. J. Combin., **62(3)** (2015) 211–227.
[31] P. Paulraja, V. S. Agnes, Degree distance of product graphs, *Discrete Math. Algorithm. Appl.,* **6(1)** 1450003(19 pages) (2014) DOI: 10.1142/S1793830914500037.
[32] E. Sampathkumar, S. B. Chikkodimath, Semitotal graphs of a graph-I, *J. Karnatak Univ. Sci.,* **18** (1973) 274–280.
[33] G. H. Shirdel, H. Rezapour, A. M. Sayadi, The hyper-Zagreb index of graph operations, *Iranian J. Math. Chem.,* **4(2)** (2013) 213–220.
[34] N. Trinajstić, *Chemical Graph Theory*, CRC Press, Boca Raton, FL (1992).
[35] M. Veylaki, M. J. Nikmehr, H. A. Tavallaee, The third and hyper-Zagreb coindices of graph operations, *J. Appl. Math. Comput.,* **50(1-2)** (2015) 315–325.
[36] H. Wiener, Strucural determination of paraffin boiling points, *J. Amer. Chem. Soc.,* **69(1)** (1947) 17–20.
[37] Z. Yarahmadi, Computing Some topological Indices of Tensor product of graphs, *Iranian J. Math. Chem.,* **2(1)** (2011) 109–118.
[38] Z. Yarahmadi, A. R. Ashrafi, The Szeged, vertex PI, first and second Zagreb indices of corona product of graphs, *Filomat,* **26(3)** (2012) 467–472.

[39] B. Zhou, N. Trinajstić, On a novel connectivity index, *J. Math. Chem.,* **46(4)** (2009) 1252–1270.

[40] B. Zhou, N. Trinajstić, On general sum-connectivity index, *J. Math. Chem.,* **47(1)** (2010) 210–218.

B. Basavanagoud

Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India.

E-mail address: b.basavanagoud@gmail.com

Shruti Policepatil

Department of Mathematics, Karnatak University, Dharwad - 580 003, Karnataka, India.

E-mail address: shrutipatil300@gmail.com