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DISEASE DYNAMICS FROM EXOTIC PREY TO NATIVE POPULATION: A PREY-PREDATOR MODEL

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Abstract. In this paper a prey-predator model between two communities is proposed in which disease transmits from endemic exotic prey to native prey and predator population with functional response Holling type II. Bounded region for positive solutions is found out. Trivial and non trivial (disease free and endemic in absence and presence of predator) equilibrium points are calculated. Bendixson-Dulac criteria is used to derive the conditions for stability of equilibrium points. Persistence of the model is also discussed. Transmission rate of disease, predation rate and carrying capacity of environment were taken under consideration as these parameters affect community structure. Numerical solution and graphs are illustrated to support the results.

1. Introduction

There are number of factors that play an important role in transmission of disease. Currently, serious concern has been raised about the role of endemic exotic prey and native prey- predator in the transmission of disease [1]. Different species in various ecosystems perpetuate and transmit disease to new geographical locations. Migration of species from one geographical region may introduce a new disease to other ecosystem by prey-predator relationship [2]. It is well established fact that when endemic exotic prey is consumed by the predator, if infected, may transmit its disease to the prey, if it survives. This effect is negative or unnoticeable depending upon the virulence of the disease. Sometimes, the disease can change the behavioural pattern of prey, which makes then more susceptible to predation. There are several studies that have been conducted by the researchers all over the world related to present study [3, 4, 5]. However, studies are lacking in the direction of endemic exotic prey and native prey-predator. Therefore, this study is necessary to study the interaction between the endemic exotic prey and native prey-predator.

A model of predator dommalia ecologically infected populations was studied by Mukhopadhyay and Bhattacharyya (2009) [6]. Yang, Dan and Shengqiang (2016) [7] discussed the global asymptotical behaviours of the model and optimal hunting strategies for the predators on various carrying capacities, infection rates and

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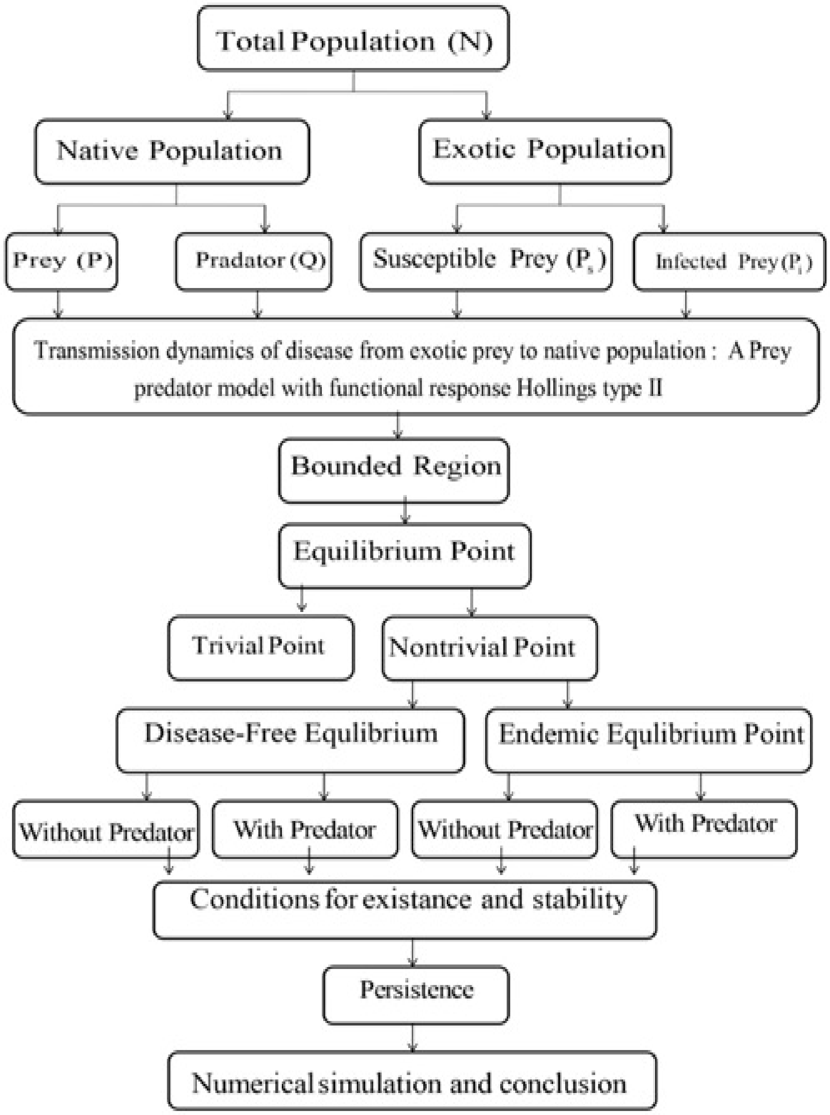

FIGURE 1. **Flowchart for preparation of this paper**

hunting rates. Santosh Biswas, Sudip Samanta and Joydev Chattopadhyay (2017) [19] suggested a cannibalistic eco-epidemiological model with disease in predator population. Preparation of model can be understood by flow chart 1 given below;

2. Basic Assumptions, Model Preliminaries and Formulation

In the proposed model, we have taken population from two linked communities as follows;
(i) Native population, consisting prey (P) and predator (Q).
(ii) Exotic population, consisting susceptible prey (P_s) and infected prey (P_i).

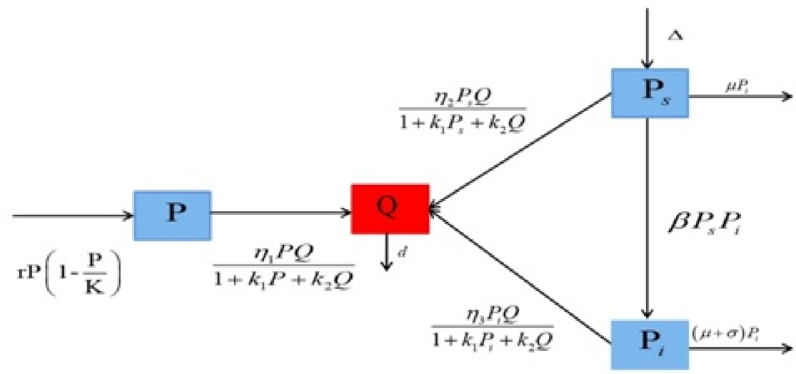

Figure 2. **Systematic diagram of the proposed model**

The following assumptions were made to form system of non linear ordinary differential equations to demonstrate some real situation;

(A1) We considered prey-predator type of interaction.

(A2) In the absence of disease and predation, native prey population grows logistically with growth rate $r(r > 0)$ and carrying capacity $K(K > 0)$.

(A3) There is certain recruitment rate Δ of exotic prey population.

(A4) It is assumed that disease transmits only from exotic infected prey to exotic susceptible prey through contact. Let the transmission rate of disease be β then the term $\beta P_s P_i$ indicates dynamics of infection. The detailed dynamics and cause of infection along with control strategies were ignored in the present model.

(A5) Predators consume both the native prey as well as exotic (susceptible and infected) prey as they attack any of the prey of their choice. We denote predation rate by η_1, η_2, η_3 on preys P, P_s, P_i, respectively, we assume that $\eta_3 > \eta_1$, η_2 because the infected prey P_i is weaker than uninfected preys P, P_s.

(A6) Holling type II functional response of the predation is assumed. k_1 is positive constant represents handling time on feeding rate and k_2 is a nonnegative constant representing the magnitude of interference among predators. $\alpha\,(0 < \alpha < 1)$ is conversion coefficient of prey into predator.

(A7) It assumes that native prey species are controlled logistically. Native predators have their natural death rate d. Exotic preys are having natural death rate μ and disease induced death rate σ.

(A8) It is also assumed that exotic infected predator neither recover nor immune. A mathematical model from all these assumptions is framed as follows:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \frac{\eta_1 PQ}{1 + k_1 P + k_2 Q} \tag{1}$$

$$\frac{dQ}{dt} = \frac{\alpha\eta_1 PQ}{1 + k_1 P + k_2 Q} + \frac{\alpha\eta_2 P_S Q}{1 + k_1 P_S + k_2 Q} + \frac{\alpha\eta_3 P_i Q}{1 + k_1 P_i + k_2 Q} - dQ \tag{2}$$

$$\frac{dP_S}{dt} = \Delta - \beta P_S P_i - \frac{\eta_2 P_S Q}{1 + k_1 P_S + k_2 Q} - \mu P_S \tag{3}$$

$$\frac{dP_i}{dt} = \beta P_S P_i - \frac{\eta_3 P_i Q}{1 + k_1 P_i + k_2 Q} - (\mu + \sigma)P_i \tag{4}$$

System (1) to (4) has to be analyzed with the following initial conditions: $P(0) > 0$, $Q(0) > 0$, $P_s(0) > 0, P_i(0) > 0$. Proposed epidemic model can be easily understood by following systematic diagram given below;

Table 1: Description of variables and parameters

Parameters	**description**	**Parameter value**
P	Native prey population.	-
Q	Native predator population.	-
P_s	Exotic susceptible prey population.	-
P_i	Exotic infected prey population.	-
r	Intrinsic growth rate	3
K	The carrying capacity of the environment	45
k_1	Half saturation constant.	0.9
k_2	Magnitude of interference among predators.	0.23
α	Conversion efficiency. $(0 < \alpha < 1)$	0.4
β	Disease transmission rate	0.02
Δ	Recruitment rate of exotic susceptible prey population	400
η_1	The predation rate of native prey population	0.29
η_2	The predation rate of exotic susceptible prey population	0.52
η_3	The predation rate of exotic infected prey population	0.55
d	The constant natural death rate of native predator population	0.09
μ	The constant natural death rate of exotic prey population	0.54
σ	Disease induced death rate of infected exotic prey population	0.25

Positive invariance:
Let $Y = (P, Q, P_s, P_i)^T \in R^4$ so system (1) to (4) may express in a vector form as

$$F(Y) = (F_1(Y), F_2(Y), F_3(Y), F_4(Y))^T, \tag{5}$$

Where $F : C_+ \to R^4$ and $F \in C_\infty(R^4)$. Then equation (5) gives

$$\overset{\bullet}{Y} = F(Y) \tag{6}$$

with $Y(0) = Y_0 \in R_+^4$. It is easy to check in Eq. (5) that whenever, choosing $Y(0) \in R_+^4$ such that $\mathrm{Y}_i = 0$ then $F_i(Y)|_{Y_i=0} \geq 0$, $(\mathrm{i} = 1, 2, 3, 4)$. Now any solution of with $\mathrm{Y}_0 \in R_+^4$, say $Y(t) = \mathrm{Y}(t, \mathrm{y}_0)$, is such that $Y(t) \in R_+^4$ for all $t > 0$ (Yang and Chen 1996). [16]

3. Equilibrium points

The model system (1) to (4) possesses following feasible biological equilibrium points.

(i) The trivial equilibrium point $E_0(0, 0, 0, 0)$

(ii) Disease-free equilibrium point without Predator $E_1\left(K, 0, \frac{\Delta}{\mu}, 0\right)$.

(iii) Disease-free equilibrium point with Predator $E_2\left(\hat{P}, \hat{Q}, \hat{P}_s, 0\right)$

Where,

$$-\alpha r\left(\hat{P}\right)^2+\alpha r K \hat{P}+K\left(\alpha \Delta-\alpha \mu \hat{P}_s-d \hat{Q}\right)=0 \tag{7}$$

$$\hat{Q}=\frac{1+k_1 \hat{P}}{\frac{\eta_1}{r\left(1-\frac{\hat{P}}{K}\right)}-k_2} \tag{8}$$

$$-\mu k_1\left(\hat{P}_s\right)^2+\left(\Delta k_1-\left(\eta_2+\mu k_2\right) \hat{Q}-\mu\right) \hat{P}_s+\left(\Delta+k_2 \hat{Q}\right)=0 \tag{9}$$

It is clear from equation (7) native prey population ($\hat{P}$) survives since it has at least one positive root. Also from equation (8) if $K>\hat{P}>K\left(1-\frac{\eta_1}{r k_2}\right)$ then native predator population ($\hat{Q}$) will exist. Similarly, from equation (9) exotic susceptible preys ($\hat{P}_s$) will survive when $\hat{Q}$ exists. So under above conditions equilibrium point E_2 will exist.

(iv) Endemic equilibrium point without Predator $E_3\left(K, 0, \frac{\mu+\sigma}{\beta}, \frac{1}{\beta}\left[\frac{\Delta \beta}{\mu+\sigma}-\mu\right]\right)$.

It is clear that, the existence condition of equilibrium point E_3 is $\Delta>\frac{\mu(\mu+\sigma)}{\beta}$.

(v) Endemic equilibrium point without Predator $E_4\left(\overset{*}{P}, \overset{*}{Q}, \overset{*}{P}_s, \overset{*}{P}_i\right)$

Where,

$$-\alpha r\left(\overset{*}{P}\right)^2+\alpha r K \overset{*}{P}+K\left(\alpha \Delta-d \overset{*}{Q}-\alpha \mu \overset{*}{P}_s-\alpha(\mu+\sigma) \overset{*}{P}_i\right)=0 \tag{10}$$

$$\overset{*}{Q}=\frac{1+k_1 \overset{*}{P}}{\frac{\eta_1}{r\left(1-\frac{\overset{*}{P}}{K}\right)}-k_2} \tag{11}$$

$$\overset{*}{P}_i=\left(\frac{1+k_2 \overset{*}{Q}}{k_1}\right)\left(\frac{\eta_3 \overset{*}{Q}}{\left(\beta \overset{*}{P}_s-(\mu+\sigma)\right)\left(1+k_2 \overset{*}{Q}\right)}-1\right) \tag{12}$$

$$\begin{aligned}-k_1\left(\beta \overset{*}{P}_i+\mu\right)\left(\overset{*}{P}_s\right)^2+\left(\Delta k_1-\left(\beta \overset{*}{P}_i+\mu\right)\left(1+k_2 \overset{*}{Q}\right)-\eta_2 \overset{*}{Q}\right) \overset{*}{P}_s \\ +\Delta\left(1+k_2 \overset{*}{Q}\right)=0\end{aligned} \tag{13}$$

It is clear from equation (10) native prey population ($\overset{*}{P}$) survives since it has at least one positive root. Also from equation (11) native predator population ($\overset{*}{Q}$) will exist if $K>\overset{*}{P}>K\left(1-\frac{\eta_1}{r k_2}\right)$.

Similarly, from equation (12) exotic infected preys ($\overset{*}{P}_i$) will survive if

$\frac{1}{\beta}\left[\frac{\eta_3\overset{*}{Q}}{\left(1+k_2\overset{*}{Q}\right)}+(\mu+\sigma)\right]>\overset{*}{P_s}>\frac{(\mu+\sigma)}{\beta}$

Further, from equation (13) exotic susceptible preys population $(\overset{*}{P_s})$ will survive when $\hat{Q}$ exists. So under above conditions equilibrium point E_4 will exist.

4. Local Stability analysis

In this section, stability of the model will be discussed with the help of Jacobian matrix and Lyapunov function around equilibrium points.
The Jacobian matrix J of the model system (1) to (4) can be calculated as follows;

$$J(P,Q,P_s,P_i)=\begin{bmatrix}\frac{\partial P}{\partial P} & \frac{\partial P}{\partial Q} & \frac{\partial P}{\partial P_s} & \frac{\partial P}{\partial P_i}\\ \frac{\partial Q}{\partial P} & \frac{\partial Q}{\partial Q} & \frac{\partial Q}{\partial P_s} & \frac{\partial Q}{\partial P_i}\\ \frac{\partial P_s}{\partial P} & \frac{\partial P_s}{\partial Q} & \frac{\partial P_s}{\partial P_s} & \frac{\partial P_s}{\partial P_i}\\ \frac{\partial P_i}{\partial P} & \frac{\partial P_i}{\partial Q} & \frac{\partial P_i}{\partial P_s} & \frac{\partial P_i}{\partial P_i}\end{bmatrix}$$

Theorem 4.1 The trivial equilibrium point E_0 of model system (1) to (4) is always unstable.
Proof The Jacobian matrix of model system (1) to (4) around $E_0(0,0,0,0)$ is given by;

$$J(E_0)=\begin{bmatrix} r & 0 & 0 & 0\\ 0 & -d & 0 & 0\\ 0 & 0 & -\mu & 0\\ 0 & 0 & 0 & -(\mu+\sigma)\end{bmatrix}$$

Now from matrix $J(E_0)$, we can easily notice that one Eigen value of this matrix is positive and remaining Eigen values of $J(E_0)$ are negative, so $E_0(0,0,0,0)$ is always unstable.

Theorem 4.2 The Disease-free equilibrium point without predator E_1 of model system (1) to (4) is always stable if $\left(\frac{\alpha\eta_1 K}{1+k_1K}+\frac{\alpha\eta_2\Delta}{\mu+k_1\Delta}\right)<d$ and $\frac{\beta\Delta}{\mu}<(\mu+\sigma)$ hold, otherwise unstable.
Proof The Jacobian matrix of model system (1) to (4) around $E_1\left(K,0,\frac{\Delta}{\mu},0\right)$ is given by;

$$J(E_1)=\begin{bmatrix} -r & -\frac{\eta_1K}{1+k_1K} & 0 & 0\\ 0 & \frac{\alpha\eta_1K}{1+k_1K}+\frac{\alpha\eta_2\Delta}{\mu+k_1\Delta}-d & 0 & 0\\ 0 & -\frac{\eta_2\Delta}{\mu+k_1\Delta} & -\mu & -\frac{\beta\Delta}{\mu}\\ 0 & 0 & 0 & \frac{\beta\Delta}{\mu}-(\mu+\sigma)\end{bmatrix}$$

Now from matrix $J(E_1)$, we can easily notice that two Eigen values of this matrix are negative also remaining are positive if $\left(\frac{\alpha\eta_1K}{1+k_1K}+\frac{\alpha\eta_2\Delta}{\mu+k_1\Delta}\right)>d$ and $\frac{\beta\Delta}{\mu}>(\mu+\sigma)$ or vice versa. Hence $E_1\left(K,0,\frac{\Delta}{\mu},0\right)$ is stable if $\left(\frac{\alpha\eta_1K}{1+k_1K}+\frac{\alpha\eta_2\Delta}{\mu+k_1\Delta}\right)<d$ and $\frac{\beta\Delta}{\mu}<(\mu+\sigma)$ otherwise unstable.

Theorem 4.3 The Disease-free equilibrium point with predator E_2 of model system (1) to (4) is always stable under following conditions;

$$\frac{r}{K} > \frac{\eta_1 k_1 \hat{Q}}{A\hat{A}}, \quad \left(\frac{r}{K} - \frac{\eta_1 k_1 \hat{Q}}{A\hat{A}}\right) \cdot \left(\frac{\alpha\eta_1 k_2 \hat{P}}{A\hat{A}} + \frac{\alpha\eta_2 k_2 \hat{P_s}}{B\hat{B}}\right) > \frac{1}{4}\left(\frac{\eta_1\left(1+k_1\hat{P}\right)-\alpha\eta_1\left(1+k_2\hat{Q}\right)}{A\hat{A}}\right)^2,$$

$$\left(\frac{r}{K} - \frac{\eta_1 k_1 \hat{Q}}{A\hat{A}}\right)\left(\frac{\alpha\eta_1 k_2 \hat{P}}{A\hat{A}} + \frac{\alpha\eta_2 k_2 \hat{P_s}}{B\hat{B}}\right)\left(\frac{\Delta}{P_s\hat{P_s}} - \frac{\eta_2 k_1 \hat{Q}}{B\hat{B}}\right) > \frac{1}{4}\left(\frac{r}{K} - \frac{\eta_1 k_1 \hat{Q}}{A\hat{A}}\right)\left(\frac{\eta_2\left(1+k_1\hat{P_s}\right)-\alpha\eta_2\left(1+k_2\hat{Q}\right)}{B\hat{B}}\right)^2$$

$$+\frac{1}{4}\left(\frac{\Delta}{P_s\hat{P_s}} - \frac{\eta_2 k_1 \hat{Q}}{B\hat{B}}\right)\left(\frac{\eta_1\left(1+k_1\hat{P}\right)-\alpha\eta_1\left(1+k_2\hat{Q}\right)}{A\hat{A}}\right)^2$$

and $\frac{\Delta}{P_s\hat{P_s}} > \frac{\eta_2 k_1 \hat{Q}}{B\hat{B}}$ Otherwise unstable.

Proof To determine stability of $E_2\left(\hat{P}, \hat{Q}, \hat{P_s}, 0\right)$, we consider the following positive definite Lyapunov function of model system (1) to (4) as follows;

$$V(P,Q,P_s,0) = \left(P - \hat{P} - \hat{P}\log\frac{P}{\hat{P}}\right) + \left(Q - \hat{Q} - \hat{Q}\log\frac{Q}{\hat{Q}}\right) + \left(P_s - \hat{P_s} - \hat{P_s}\log\frac{P_s}{\hat{P_s}}\right)$$

Now,computing the time derivative of V and using model system (1) to (4), we get

$$\dot{V} = \left(P - \hat{P}\right)\left(r\left(1 - \frac{P}{K}\right) - \frac{\eta_1 Q}{1+k_1P+k_2Q}\right) + \left(Q - \hat{Q}\right)\left(\frac{\alpha\eta_1 P}{1+k_1P+k_2Q} + \frac{\alpha\eta_2 P_S}{1+k_1P_S+k_2Q} - d\right)$$
$$+\left(P_s - \hat{P_s}\right)\left(\frac{\Delta}{P_s} - \frac{\eta_2 Q}{1+k_1P_S+k_2Q} - \mu\right)$$

$$\dot{V} = \left(P - \hat{P}\right)\left(-\frac{r}{K}\left(P - \hat{P}\right) - \eta_1\left[\frac{\left(1+k_1\hat{P}\right)\left(Q-\hat{Q}\right) - k_1\hat{Q}\left(P-\hat{P}\right)}{(1+k_1P+k_2Q)\left(1+k_1\hat{P}+k_2\hat{Q}\right)}\right]\right)$$
$$+\left(Q - \hat{Q}\right)\left(\alpha\eta_1\left[\frac{\left(1+k_2\hat{Q}\right)\left(P-\hat{P}\right) - k_2\hat{P}\left(Q-\hat{Q}\right)}{(1+k_1P+k_2Q)\left(1+k_1\hat{P}+k_2\hat{Q}\right)}\right]\right.$$
$$\left.+\alpha\eta_2\left[\frac{\left(1+k_2\hat{Q}\right)\left(P_s-\hat{P_s}\right) - k_2\hat{P_s}\left(Q-\hat{Q}\right)}{(1+k_1P_s+k_2Q)\left(1+k_1\hat{P_s}+k_2\hat{Q}\right)}\right]\right)$$
$$+\left(P_s - \hat{P_s}\right)\left(-\frac{\Delta}{P_s\hat{P_s}}\left(P_s - \hat{P_s}\right) - \eta_2\left[\frac{\left(1+k_1\hat{P_s}\right)\left(Q-\hat{Q}\right) - k_1\hat{Q}\left(P_s-\hat{P_s}\right)}{(1+k_1P_s+k_2Q)\left(1+k_1\hat{P_s}+k_2\hat{Q}\right)}\right]\right)$$

Consequently, we get

$$\dot{V} = -\left[\left(\frac{r}{K} - \frac{\eta_1 k_1\hat{Q}}{A\hat{A}}\right)\left(P - \hat{P}\right)^2 + \left[\frac{\alpha\eta_1 k_2\hat{P}}{A\hat{A}} + \frac{\alpha\eta_2 k_2\hat{P_s}}{B\hat{B}}\right]\left(Q - \hat{Q}\right)^2\right.$$

$$+\left(\frac{\Delta}{P_s\,\overset{\wedge}{P_s}}-\frac{\eta_2 k_1\,\hat{Q}}{B\,\hat{B}}\right)\left(P_s-\overset{\wedge}{P_s}\right)^2+\left(\frac{\eta_1\left(1+k_1\,\hat{P}\right)-\alpha\eta_1\left(1+k_2\,\hat{Q}\right)}{A\,\hat{A}}\right)\left(Q-\hat{Q}\right)\left(P-\hat{P}\right)$$

$$+\left(\frac{\eta_2\left(1+k_1\,\overset{\wedge}{P_s}\right)-\alpha\eta_2\left(1+k_2\,\hat{Q}\right)}{B\,\hat{B}}\right)\left(Q-\hat{Q}\right)\left(P_s-\overset{\wedge}{P_s}\right)\Bigg]$$

The above expression can be written as $L^T M L$, where $L=\left(P-\hat{P},\,Q-\hat{Q},\,P_s-\overset{\wedge}{P_s}\right)$
and

$$M=\begin{bmatrix} M_{PP} & M_{PQ} & M_{PP_s}\\ M_{PQ} & M_{QQ} & M_{QP_s}\\ M_{PP_s} & M_{QP_s} & M_{P_sP_s}\end{bmatrix}$$

with
$M_{PP}=\left(\frac{r}{K}-\frac{\eta_1 k_1\,\hat{Q}}{A\,\hat{A}}\right)$, $M_{QQ}=\left(\frac{\alpha\eta_1 k_2\,\hat{P}}{A\,\hat{A}}+\frac{\alpha\eta_2 k_2\,\overset{\wedge}{P_s}}{B\,\hat{B}}\right)$, $M_{P_sP_s}=\left(\frac{\Delta}{P_s\,\overset{\wedge}{P_s}}-\frac{\eta_2 k_1\,\hat{Q}}{B\,\hat{B}}\right)$,
$M_{PP_s}=M_{P_sP}=0$, $M_{PQ}=M_{QP}=\frac{1}{2}\left(\frac{\eta_1\left(1+k_1\,\hat{P}\right)-\alpha\eta_1\left(1+k_2\,\hat{Q}\right)}{A\,\hat{A}}\right)$, $M_{P_sQ}=$
$M_{QP_s}=\frac{1}{2}\left(\frac{\eta_2\left(1+k_1\,\overset{\wedge}{P_s}\right)-\alpha\eta_2\left(1+k_2\,\hat{Q}\right)}{B\,\hat{B}}\right)$
Therefore,
$\overset{\bullet}{V}=\frac{dV}{dt}$ is negative definite if the symmetric matrix M is positive definite.
The matrix M is positive definite, if all the principal minors of M are positive.
$P_1=M_{PP}=\left(\frac{r}{K}-\frac{\eta_1 k_1\,\hat{Q}}{A\,\hat{A}}\right)$,

$P_2=M_{PP}.M_{QQ}-M_{PQ}^2=\left(\frac{r}{K}-\frac{\eta_1 k_1\,\hat{Q}}{A\,\hat{A}}\right)\cdot\left(\frac{\alpha\eta_1 k_2\,\hat{P}}{A\,\hat{A}}+\frac{\alpha\eta_2 k_2\,\overset{\wedge}{P_s}}{B\,\hat{B}}\right)-\frac{1}{4}\left(\frac{\eta_1\left(1+k_1\,\hat{P}\right)-\alpha\eta_1\left(1+k_2\,\hat{Q}\right)}{A\,\hat{A}}\right)^2$,

$P_3=M_{PP}.M_{QQ}.M_{P_sP_s}+2M_{PQ}.M_{PP_s}M_{QP_s}-M_{PP}.M_{QP_s}^2-M_{QQ}.M_{PP_s}^2-M_{P_sP_s}.M_{PQ}^2$

$$=\left(\frac{r}{K}-\frac{\eta_1 k_1\,\hat{Q}}{A\,\hat{A}}\right)\left(\frac{\alpha\eta_1 k_2\,\hat{P}}{A\,\hat{A}}+\frac{\alpha\eta_2 k_2\,\overset{\wedge}{P_s}}{B\,\hat{B}}\right)\left(\frac{\Delta}{P_s\,\overset{\wedge}{P_s}}-\frac{\eta_2 k_1\,\hat{Q}}{B\,\hat{B}}\right)$$

$$-\frac{1}{4}\left(\frac{r}{K}-\frac{\eta_1 k_1\,\hat{Q}}{A\,\hat{A}}\right)\left(\frac{\eta_2\left(1+k_1\,\overset{\wedge}{P_s}\right)-\alpha\eta_2\left(1+k_2\,\hat{Q}\right)}{B\,\hat{B}}\right)^2$$

$$-\frac{1}{4}\left(\frac{\Delta}{P_s\,\overset{\wedge}{P_s}}-\frac{\eta_2 k_1\,\hat{Q}}{B\,\hat{B}}\right)\left(\frac{\eta_1\left(1+k_1\,\hat{P}\right)-\alpha\eta_1\left(1+k_2\,\hat{Q}\right)}{A\,\hat{A}}\right)^2.$$

$P_1>0$ if $\frac{r}{K}>\frac{\eta_1 k_1\,\hat{Q}}{A\,\hat{A}}$,

$P_2 > 0$ if $\left(\frac{r}{K} - \frac{\eta_1 k_1 \hat{Q}}{A\hat{A}}\right) \cdot \left(\frac{\alpha\eta_1 k_2 \hat{P}}{A\hat{A}} + \frac{\alpha\eta_2 k_2 \hat{P_s}}{B\hat{B}}\right) > \frac{1}{4}\left(\frac{\eta_1\left(1+k_1\hat{P}\right)-\alpha\eta_1\left(1+k_2\hat{Q}\right)}{A\hat{A}}\right)^2$ and

$P_3 > 0$ if $\left(\frac{r}{K} - \frac{\eta_1 k_1 \hat{Q}}{A\hat{A}}\right)\left(\frac{\alpha\eta_1 k_2 \hat{P}}{A\hat{A}} + \frac{\alpha\eta_2 k_2 \hat{P_s}}{B\hat{B}}\right)\left(\frac{\Delta}{P_s\hat{P_s}} - \frac{\eta_2 k_1 \hat{Q}}{B\hat{B}}\right) > \frac{1}{4}\left(\frac{r}{K} - \frac{\eta_1 k_1 \hat{Q}}{A\hat{A}}\right)$
$\left(\frac{\eta_2\left(1+k_1\hat{P_s}\right)-\alpha\eta_2\left(1+k_2\hat{Q}\right)}{B\hat{B}}\right)^2 + \frac{1}{4}\left(\frac{\Delta}{P_s\hat{P_s}} - \frac{\eta_2 k_1 \hat{Q}}{B\hat{B}}\right)\left(\frac{\eta_1\left(1+k_1\hat{P}\right)-\alpha\eta_1\left(1+k_2\hat{Q}\right)}{A\hat{A}}\right)^2$

Thus if previous conditions hold then $E_2\left(\hat{P}, \hat{Q}, \hat{P_s}, 0\right)$ is stable, otherwise unstable.

Theorem 4.4 The endemic equilibrium point without predator E_3 of model system (1) to (4) is always stable if $\frac{\alpha\eta_1 K}{1+k_1 K} + \frac{\alpha\eta_2(\mu+\sigma)}{\beta+k_1(\mu+\sigma)} + \frac{\alpha\eta_3[\beta\Delta-\mu(\mu+\sigma)]}{\beta(\mu+\sigma)+k_1[\beta\Delta-\mu(\mu+\sigma)]} < d$ and $\frac{\beta\Delta}{\mu} > (\mu+\sigma)$ hold, otherwise unstable.

Proof The Jacobian matrix around $E_3\left(K, 0, \frac{\mu+\sigma}{\beta}, \frac{1}{\beta}\left[\frac{\Delta\beta}{\mu+\sigma} - \mu\right]\right)$ of the model system (1) to (4) is given by,

$$J(E_3) = \begin{bmatrix} -r & -\frac{\eta_1 K}{1+k_1 K} & 0 & 0 \\ 0 & \frac{\alpha\eta_1 K}{1+k_1 K} + \frac{\alpha\eta_2(\mu+\sigma)}{\beta+k_1(\mu+\sigma)} + \frac{\alpha\eta_3[\beta\Delta-\mu(\mu+\sigma)]}{\beta(\mu+\sigma)+k_1[\beta\Delta-\mu(\mu+\sigma)]} - d & 0 & 0 \\ 0 & \frac{-\eta_2(\mu+\sigma)}{\beta+k_1(\mu+\sigma)} & -\frac{\beta\Delta}{(\mu+\sigma)} & -(\mu+\sigma) \\ 0 & \frac{-\eta_3[\beta\Delta-\mu(\mu+\sigma)]}{\beta(\mu+\sigma)+k_1[\beta\Delta-\mu(\mu+\sigma)]} & \frac{\beta\Delta-\mu(\mu+\sigma)}{(\mu+\sigma)} & 0 \end{bmatrix}$$

Now from matrix $J(E_3)$, we can easily notice that one Eigen value of this matrix is negative also remaining Eigen values are negative if $\frac{\alpha\eta_1 K}{1+k_1 K} + \frac{\alpha\eta_2(\mu+\sigma)}{\beta+k_1(\mu+\sigma)} + \frac{\alpha\eta_3[\beta\Delta-\mu(\mu+\sigma)]}{\beta(\mu+\sigma)+k_1[\beta\Delta-\mu(\mu+\sigma)]} < d$ and $\frac{\beta\Delta}{\mu} > (\mu+\sigma)$so in this case E_3 is stable otherwise unstable.

Theorem 4.5 The Endemic equilibrium point with predator E_4 of model system (1) to (4) is always stable if following conditions hold;

$$\left(\frac{r}{K} - \frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right) \cdot \left(\frac{r}{K} - \frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right) \cdot \left(\frac{\alpha\eta_1 k_2 \overset{*}{P}}{A\overset{*}{A}} + \frac{\alpha\eta_2 k_2 \overset{*}{P_s}}{B\overset{*}{B}} + \frac{\alpha\eta_3 k_2 \overset{*}{P_i}}{C\overset{*}{C}}\right) > \frac{1}{4}\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)^2,$$

$$\left(\frac{r}{K} - \frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\left(\frac{\alpha\eta_1 k_2 \overset{*}{P}}{A\overset{*}{A}} + \frac{\alpha\eta_2 k_2 \overset{*}{P_s}}{B\overset{*}{B}} + \frac{\alpha\eta_3 k_2 \overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\Delta}{P_s\overset{*}{P_s}} - \frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right) > \left(\frac{1}{4}\left(\frac{r}{K} - \frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\right.$$

$$\left.\left(\frac{\eta_2\left(1+k_1\overset{*}{P_s}\right)-\alpha\eta_2\left(1+k_2\overset{*}{Q}\right)}{B\overset{*}{B}}\right)^2 + \frac{1}{4}\left(\frac{\Delta}{P_s\overset{*}{P_s}} - \frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right)\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)^2\right) \textit{and}$$

$$\left(\frac{r}{K} - \frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\left(\frac{\alpha\eta_1 k_2 \overset{*}{P}}{A\overset{*}{A}} + \frac{\alpha\eta_2 k_2 \overset{*}{P_s}}{B\overset{*}{B}} + \frac{\alpha\eta_3 k_2 \overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\Delta}{P_s\overset{*}{P_s}} - \frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right)\left(-\frac{\eta_3 k_1 \overset{*}{P_i}}{C\overset{*}{C}}\right) > \left\{\frac{1}{4}\left(\frac{r}{K} - \frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\right.$$

$$\left(\frac{\Delta}{P_s\overset{*}{P_s}} - \frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right)\left(\frac{\eta_3\left(1+k_1\overset{*}{P_i}\right)-\alpha\eta_3\left(1+k_2\overset{*}{Q}\right)}{C\overset{*}{C}}\right)^2 + \frac{1}{4}\left(\frac{r}{K} - \frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\left(-\frac{\eta_3 k_1 \overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\eta_2\left(1+k_1\overset{*}{P_s}\right)-\alpha\eta_2\left(1+k_2\overset{*}{Q}\right)}{B\overset{*}{B}}\right)^2$$

$$+\frac{1}{4}\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2 k_1\overset{*}{Q}}{B\overset{*}{B}}\right)\left(-\frac{\eta_3 k_1\overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)^2\Bigg\}$$

Otherwise unstable.
where
$A=\frac{1}{1+k_1P+k_2Q}$, $\overset{*}{A}=\frac{1}{1+k_1\overset{*}{P}+k_2\overset{*}{Q}}$, $B=\frac{1}{1+k_1P_s+k_2Q}$, $\overset{*}{B}=\frac{1}{1+k_1\overset{*}{P_s}+k_2\overset{*}{Q}}$,
$C=\frac{1}{1+k_1P_i+k_2Q}$ and $\overset{*}{C}=\frac{1}{1+k_1\overset{*}{P_i}+k_2\overset{*}{Q}}$

Proof To determine stability of $E_4\left(\overset{*}{P},\overset{*}{Q},\overset{*}{P_s},\overset{*}{P_i}\right)$, we consider the following positive definite Lyapunov function of the model system (1) to (4) is given by,

$$W(P,Q,P_s,P_i)=\left(P-\overset{*}{P}-\overset{*}{P}\log\frac{P}{\overset{*}{P}}\right)+\left(Q-\overset{*}{Q}-\overset{*}{Q}\log\frac{Q}{\overset{*}{Q}}\right)+\left(P_s-\overset{*}{P_s}-\overset{*}{P_s}\log\frac{P_s}{\overset{*}{P_s}}\right)$$

$$+\left(P_i-\overset{*}{P_i}-\overset{*}{P_i}\log\frac{P_i}{\overset{*}{P_i}}\right)$$

Now, computing the time derivative of $W$and using model system (1) to (4), we get;

$$\overset{\bullet}{W}=\left(P-\overset{*}{P}\right)\left(r\left(1-\frac{P}{K}\right)-\frac{\eta_1 Q}{1+k_1P+k_2Q}\right)+\left(Q-\overset{*}{Q}\right)\left(\frac{\alpha\eta_1 P}{1+k_1P+k_2Q}+\frac{\alpha\eta_2 P_S}{1+k_1P_S+k_2Q}+\right.$$
$$\left.\frac{\alpha\eta_3 P_i}{1+k_1P_i+k_2Q}-d\right)+\left(P_s-\overset{*}{P_s}\right)\left(\frac{\Delta}{P_s}-\beta P_i-\frac{\eta_2 Q}{1+k_1P_S+k_2Q}-\mu\right)+\left(P_i-\overset{*}{P_i}\right)(\beta P_S$$
$$\left.-\frac{\eta_3 Q}{1+k_1P_i+k_2Q}-(\mu+\sigma)\right)$$

$$\overset{\bullet}{W}=\left(P-\overset{*}{P}\right)\left(-\frac{r}{K}\left(P-\overset{*}{P}\right)-\eta_1\left[\frac{\left(1+k_1\overset{*}{P}\right)\left(Q-\overset{*}{Q}\right)-k_1\overset{*}{Q}\left(P-\overset{*}{P}\right)}{(1+k_1P+k_2Q)\left(1+k_1\overset{*}{P}+k_2\overset{*}{Q}\right)}\right]\right)$$
$$+\left(Q-\overset{*}{Q}\right)\left[\alpha\eta_1\left(\frac{\left(1+k_2\overset{*}{Q}\right)\left(P-\overset{*}{P}\right)-k_2\overset{*}{P}\left(Q-\overset{*}{Q}\right)}{(1+k_1P+k_2Q)\left(1+k_1\overset{*}{P}+k_2\overset{*}{Q}\right)}\right)+\alpha\eta_2\left(\frac{\left(1+k_2\overset{*}{Q}\right)\left(P_s-\overset{*}{P_s}\right)-k_2\overset{v}{P_s}\left(Q-\overset{*}{Q}\right)}{(1+k_1P_s+k_2Q)\left(1+k_1\overset{*}{P_s}+k_2\overset{*}{Q}\right)}\right)+\right.$$
$$\left.\alpha\eta_3\left(\frac{\left(1+k_2\overset{*}{Q}\right)\left(P_i-\overset{*}{P_i}\right)-k_2\overset{*}{P_i}\left(Q-\overset{*}{Q}\right)}{(1+k_1P_i+k_2Q)\left(1+k_1\overset{*}{P_i}+k_2\overset{*}{Q}\right)}\right)\right]$$
$$+\left(P_s-\overset{*}{P_s}\right)\left(-\frac{\Delta}{P_s\overset{*}{P_s}}\left(P_s-\overset{*}{P_s}\right)-\eta_2\left[\frac{\left(1+k_1\overset{*}{P_s}\right)\left(Q-\overset{*}{Q}\right)-k_1\overset{*}{Q}\left(P_s-\overset{*}{P_s}\right)}{(1+k_1P_s+k_2Q)\left(1+k_1\overset{*}{P_s}+k_2\overset{*}{Q}\right)}\right]\right)$$
$$+\left(P_i-\overset{*}{P_i}\right)\left(-\eta_3\left[\frac{\left(1+k_1\overset{*}{P_i}\right)\left(Q-\overset{*}{Q}\right)-k_1\overset{*}{Q}\left(P_i-\overset{*}{P_i}\right)}{(1+k_1P_i+k_2Q)\left(1+k_1\overset{*}{P_i}+k_2\overset{*}{Q}\right)}\right]\right)$$

consequently, we get

$$\overset{\bullet}{W}=-\left[\left(\frac{r}{K}-\frac{\eta_1 k_1\overset{*}{Q}}{A\overset{*}{A}}\right)\left(P-\overset{*}{P}\right)^2+\left[\frac{\alpha\eta_1 k_2\overset{*}{P}}{A\overset{*}{A}}+\frac{\alpha\eta_2 k_2\overset{*}{P_s}}{B\overset{*}{B}}+\frac{\alpha\eta_3 k_2\overset{*}{P_i}}{C\overset{*}{C}}\right]\left(Q-\overset{*}{Q}\right)^2\right.$$
$$+\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2 k_1\overset{*}{Q}}{B\overset{*}{B}}\right)\left(P_s-\overset{*}{P_s}\right)^2+\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)\left(Q-\overset{*}{Q}\right)\left(P-\overset{*}{P}\right)$$

$$+\left[\frac{\eta_2\left(1+k_1\overset{*}{P_s}\right)-\alpha\eta_2\left(1+k_2\overset{*}{Q}\right)}{B\overset{*}{B}}\right]\left(Q-\overset{*}{Q}\right)\left(P_s-\overset{*}{P_s}\right)$$
$$+\left(\frac{\eta_3\left(1+k_1\overset{*}{P_i}\right)-\alpha\eta_3\left(1+k_2\overset{*}{Q}\right)}{C\overset{*}{C}}\right)\left(Q-\overset{*}{Q}\right)\left(P_i-\overset{*}{P_i}\right)-\left(\frac{\eta_3 k_1\overset{*}{Q}}{C\overset{*}{C}}\right)\left(P_i-\overset{*}{P_i}\right)^2\Bigg]$$

The above expression can be written as $L^{'T}M^{'}L^{'}$, where $L^{'}=\left(P-\overset{*}{P},\,Q-\overset{*}{Q},\,P_s-\overset{*}{P_s},\,P_i-\overset{*}{P_i}\right)$

and$M^{'}=\begin{bmatrix} M^{'}_{PP} & M^{'}_{PQ} & M^{'}_{PP_s} & M^{'}_{PP_i}\\ M^{'}_{PQ} & M^{'}_{QQ} & M^{'}_{QP_s} & M^{'}_{QP_i}\\ M^{'}_{PP_s} & M^{'}_{QP_s} & M^{'}_{P_sP_s} & M^{'}_{P_iP_s}\\ M^{'}_{PP_i} & M^{'}_{QP_i} & M^{'}_{P_iP_s} & M^{'}_{P_iP_i}\end{bmatrix}$

With

$M^{'}_{PP}=\left(\frac{r}{K}-\frac{\eta_1 k_1\overset{*}{Q}}{A\overset{*}{A}}\right)$, $M^{'}_{QQ}=\left[\frac{\alpha\eta_1k_2\overset{*}{P}}{A\overset{*}{A}}+\frac{\alpha\eta_2k_2\overset{*}{P_s}}{B\overset{*}{B}}+\frac{\alpha\eta_3k_2\overset{*}{P_i}}{C\overset{*}{C}}\right]$, $M^{'}_{P_sP_s}=\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2k_1\overset{*}{Q}}{B\overset{*}{B}}\right)$,

$M^{'}_{P_iP_i}=\left(-\frac{\eta_3k_1\overset{*}{Q}}{C\overset{*}{C}}\right)$, $M^{'}_{P_sQ}=M^{'}_{QP_s}=\frac{1}{2}\left(\frac{\eta_2\left(1+k_1\overset{*}{P_s}\right)-\alpha\eta_2\left(1+k_2\overset{*}{Q}\right)}{B\overset{*}{B}}\right)$,

$M^{'}_{PQ}=M^{'}_{QP}=\frac{1}{2}\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)$, $M^{'}_{PP_s}=M^{'}_{P_sP}=0$,

$M^{'}_{P_sP_i}=M^{'}_{P_iP_s}=0$, $M^{'}_{PP_i}=M^{'}_{P_iP}=0$, $M^{'}_{P_iQ}=M^{'}_{QP_i}=\frac{1}{2}\left(\frac{\eta_3\left(1+k_1\overset{*}{P_i}\right)-\alpha\eta_3\left(1+k_2\overset{*}{Q}\right)}{C\overset{*}{C}}\right)$

Therefore,

$\overset{\bullet}{W}=\frac{dW}{dt}$ is negative definite if the symmetric matrix M is positive definite.

The matrix M is positive definite, if all the principal minors of M are positive.

$P^{'}_1=M^{'}_{PP}=\left(\frac{r}{K}-\frac{\eta_1k_1\overset{*}{Q}}{A\overset{*}{A}}\right)$,

$P^{'}_2=M^{'}_{PP}.M^{'}_{QQ}-{M^{'}_{PQ}}^2$

$$=\left(\frac{r}{K}-\frac{\eta_1k_1\overset{*}{Q}}{A\overset{*}{A}}\right)\cdot\left(\frac{\alpha\eta_1k_2\overset{*}{P}}{A\overset{*}{A}}+\frac{\alpha\eta_2k_2\overset{*}{P_s}}{B\overset{*}{B}}+\frac{\alpha\eta_3k_2\overset{*}{P_i}}{C\overset{*}{C}}\right)-\frac{1}{4}\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)^2,$$

$P^{'}_3=M^{'}_{PP}.M^{'}_{QQ}.M^{'}_{P_sP_s}-M^{'}_{PP}.{M^{'}_{QP_s}}^2-M^{'}_{P_sP_s}.{M^{'}_{PQ}}^2$

$$=\left(\frac{r}{K}-\frac{\eta_1k_1\overset{*}{Q}}{A\overset{*}{A}}\right)\left(\frac{\alpha\eta_1k_2\overset{*}{P}}{A\overset{*}{A}}+\frac{\alpha\eta_2k_2\overset{*}{P_s}}{B\overset{*}{B}}+\frac{\alpha\eta_3k_2\overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2k_1\overset{*}{Q}}{B\overset{*}{B}}\right)$$
$$-\frac{1}{4}\left(\frac{r}{K}-\frac{\eta_1k_1\overset{*}{Q}}{A\overset{*}{A}}\right)\left(\frac{\eta_2\left(1+k_1\overset{*}{P_s}\right)-\alpha\eta_2\left(1+k_2\overset{*}{Q}\right)}{B\overset{*}{B}}\right)^2-\frac{1}{4}\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2k_1\overset{*}{Q}}{B\overset{*}{B}}\right)\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)^2$$

and

$P^{'}_4=M^{'}_{PP}.M^{'}_{QQ}.M^{'}_{P_sP_s}.M^{'}_{P_iP_i}-M^{'}_{PP}.M^{'}_{P_sP_s}{M^{'}_{QP_i}}^2-M^{'}_{PP}.M^{'}_{P_iP_i}.{M^{'}_{QP_s}}^2-M^{'}_{P_sP_s}.M^{'}_{P_iP_i}.{M^{'}_{PQ}}^2$

$$=\left(\frac{r}{K}-\frac{\eta_1k_1\overset{*}{Q}}{A\overset{*}{A}}\right)\left(\frac{\alpha\eta_1k_2\overset{*}{P}}{A\overset{*}{A}}+\frac{\alpha\eta_2k_2\overset{*}{P_s}}{B\overset{*}{B}}+\frac{\alpha\eta_3k_2\overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2k_1\overset{*}{Q}}{B\overset{*}{B}}\right)\left(-\frac{\eta_3k_1\overset{*}{P_i}}{C\overset{*}{C}}\right)$$
$$-\frac{1}{4}\left(\frac{r}{K}-\frac{\eta_1k_1\overset{*}{Q}}{A\overset{*}{A}}\right)\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2k_1\overset{*}{Q}}{B\overset{*}{B}}\right)\left(\frac{\eta_3\left(1+k_1\overset{*}{P_i}\right)-\alpha\eta_3\left(1+k_2\overset{*}{Q}\right)}{C\overset{*}{C}}\right)^2$$

$$-\frac{1}{4}\left(\frac{r}{K}-\frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\left(-\frac{\eta_3 k_1 \overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\eta_2\left(1+k_1\overset{*}{P_s}\right)-\alpha\eta_2\left(1+k_2\overset{*}{Q}\right)}{B\overset{*}{B}}\right)^2$$

$$-\frac{1}{4}\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right)\left(-\frac{\eta_3 k_1 \overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)^2.$$

$P_1^{'}>0$ if $\left(\frac{r}{K}-\frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)$,

$P_2^{'}>0$ if $\left(\frac{r}{K}-\frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right).\left(\frac{\alpha\eta_1 k_2 \overset{*}{P}}{A\overset{*}{A}}+\frac{\alpha\eta_2 k_2 \overset{*}{P_s}}{B\overset{*}{B}}+\frac{\alpha\eta_3 k_2 \overset{*}{P_i}}{C\overset{*}{C}}\right)>\frac{1}{4}\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)^2$,

$P_3^{'}>0$ if $\left(\frac{r}{K}-\frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\left(\frac{\alpha\eta_1 k_2 \overset{*}{P}}{A\overset{*}{A}}+\frac{\alpha\eta_2 k_2 \overset{*}{P_s}}{B\overset{*}{B}}+\frac{\alpha\eta_3 k_2 \overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right)>\left(\frac{1}{4}\left(\frac{r}{K}-\frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\right.$
$\left.\left(\frac{\eta_2\left(1+k_1\overset{*}{P_s}\right)-\alpha\eta_2\left(1+k_2\overset{*}{Q}\right)}{B\overset{*}{B}}\right)^2+\frac{1}{4}\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right)\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)^2\right)$

and

$P_4^{'}>0\,if$ $\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right)\left(-\frac{\eta_3 k_1 \overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{r}{K}-\frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\left(\frac{\alpha\eta_1 k_2 \overset{*}{P}}{A\overset{*}{A}}+\frac{\alpha\eta_2 k_2 \overset{*}{P_s}}{B\overset{*}{B}}+\frac{\alpha\eta_3 k_2 \overset{*}{P_i}}{C\overset{*}{C}}\right)>$

$$\left\{\frac{1}{4}\left(\frac{r}{K}-\frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right)\left(\frac{\eta_3\left(1+k_1\overset{*}{P_i}\right)-\alpha\eta_3\left(1+k_2\overset{*}{Q}\right)}{C\overset{*}{C}}\right)^2\right.$$

$$+\frac{1}{4}\left(\frac{r}{K}-\frac{\eta_1 k_1 \overset{*}{Q}}{A\overset{*}{A}}\right)\left(-\frac{\eta_3 k_1 \overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\eta_2\left(1+k_1\overset{*}{P_s}\right)-\alpha\eta_2\left(1+k_2\overset{*}{Q}\right)}{B\overset{*}{B}}\right)^2$$

$$\left.+\frac{1}{4}\left(\frac{\Delta}{P_s\overset{*}{P_s}}-\frac{\eta_2 k_1 \overset{*}{Q}}{B\overset{*}{B}}\right)\left(-\frac{\eta_3 k_1 \overset{*}{P_i}}{C\overset{*}{C}}\right)\left(\frac{\eta_1\left(1+k_1\overset{*}{P}\right)-\alpha\eta_1\left(1+k_2\overset{*}{Q}\right)}{A\overset{*}{A}}\right)^2\right\}$$

Thus if previous conditions hold then $J(E_4)$ is stable, otherwise unstable.

5. Persistence

The long term stability of a particular population without caring for the initial population is known as persistence. Mathematically, A population x(t) is said to be uniformly persistent if there exists a $\delta>0$, independent of $x(0)>0$, such that $\lim_{t\to\infty}\inf\ x(t)>\delta$ and $\lim_{t\to\infty}\sup\ x(t)<\delta$. (Mukherjee 2014)

5.1. **Dissipativeness:** The terms dissipativeness refers to condition in which different species of a particular population change with respect to time i.e., instability. We claim that solution of model system (1) to (4) always exits and bounded.
Suppose $M=\max\{K,P(0)\}$
Now, taking equation (1)

$$\frac{dP}{dt}=P\left(r\left(1-\frac{P}{K}\right)-\frac{\eta_1 Q}{1+k_1P+k_2Q}\right)$$

This implies $\frac{dP}{dt}\le rP\left(1-\frac{P}{K}\right)$

it follows that $\lim_{t\to\infty} \sup P(t) \leq M$ *i.e.*,$P(t)$ is bounded and defined on the interval $[0,\infty)\,,\forall t \geq 0$.
Again, we consider the function $V(t) = P_s(t) + P_i(t)$

$$\frac{dV}{dt} = \Delta - \frac{\eta_2 P_S Q}{1 + k_1 P_S + k_2 Q} - \frac{\eta_3 P_i Q}{1 + k_1 P_i + k_2 Q} - \mu P_S - (\mu + \sigma) P_i$$

$$\frac{dV}{dt} \leq \Delta - \mu V$$

which implies

$$\frac{dV}{dt} + \mu V \leq \Delta$$

on solving we get

$$V(t) \leq \frac{\Delta}{\mu} + \frac{V(0)}{e^{\mu t}}, \forall t \geq 0$$

consequently

$$\lim_{t\to\infty} \sup V(t) = \lim_{t\to\infty} \sup(P_s(t) + P_i(t)) \leq \frac{\Delta}{\mu}$$

Now we assume

$$\frac{dQ}{dt} = \left(\frac{\alpha\eta_1 P}{1 + k_1 P + k_2 Q} + \frac{\alpha\eta_2 P_S}{1 + k_1 P_S + k_2 Q} + \frac{\alpha\eta_3 P_i}{1 + k_1 P_i + k_2 Q} - d\right) Q$$

$$\frac{dQ}{dt} \leq (\alpha\eta_1 P + \alpha\eta_2 P_S + \alpha\eta_3 P_i - d)\, Q$$

$$\frac{dQ}{dt} \leq \left(\alpha\eta_1 M + \alpha\eta_2 \frac{\Delta}{\mu} + \alpha\eta_3 \frac{\Delta}{\mu} - d\right) Q$$

$$\frac{dQ}{dt} \leq (\omega - d)\, Q$$

where $\omega = \left(\alpha\eta_1 M + \alpha\eta_2 \frac{\Delta}{\mu} + \alpha\eta_3 \frac{\Delta}{\mu}\right)$
On solving we get

$$Q \leq e^{(\omega-d)t} + Q(0)$$

Consequently when $\omega < d$, $\lim_{t\to\infty} \sup Q(t) = \lim_{t\to\infty} (e^{(\omega-d)t} + Q(0)) = Q(0)$
So all the solutions of model system (1) to (4) that initiate in R_+^4 can be confined in the region $E = \left\{(P, Q, P_s, P_i) \in R_+^4 : P \leq M,\ Q \leq Q(0),\ P_s + P_i \leq \frac{\Delta}{\mu}\right\}$ as $t \to \infty$
Therefore, the solutions of the model system (1) to (4) with positive initial conditions are dissipative.

5.2. **Permanence.** Under suitable conditions all the populations of a particular population survive in future as well. Mathematically, permanence of the system means that strictly positive solutions having no limit points on the bounded region. From the equation (1); we have

$$\frac{dP}{dt} \geq \left(r\left(1 - \frac{P}{K}\right) - \eta_1 Q\right) P$$

$$\frac{dQ}{dt} \geq Q\left(\frac{\alpha\eta_1 P}{1 + k_1 P + k_2 Q} + \frac{\alpha\eta_2 P_S}{1 + k_1 P_S + k_2 Q} - d\right)$$

$$\frac{dP_S}{dt} \geq (\Delta - \beta P_S P_i - \eta_2 P_S Q - \mu P_S)$$

$$\frac{dP_i}{dt} \geq P_i\left(\beta P_S - \eta_3 Q - (\mu+\sigma)\right)$$

if$(y_1, y_2, y_3, y_4)$is the positive root of the system of equalities

$$r\left(1-\frac{P}{K}\right) - \eta_1 Q = 0 \tag{14}$$

$$\frac{\alpha\eta_1 P}{1+k_1P+k_2Q} + \frac{\alpha\eta_2 P_S}{1+k_1P_S+k_2Q} - d = 0 \tag{15}$$

$$\Delta - \beta P_S P_i - \eta_2 P_S Q - \mu P_S = 0 \tag{16}$$

$$\beta P_S - \eta_3 Q - (\mu+\sigma) = 0 \tag{17}$$

By standard comparison theorem (Xiao and Chen 2001)
$\lim\limits_{t\to\infty} \inf\, P(t) \geq y_1,\ \lim\limits_{t\to\infty} \inf\, Q(t) \geq y_2,\ \lim\limits_{t\to\infty} \inf\, P_s(t) \geq y_3$ and $\lim\limits_{t\to\infty} \inf\, P_i(t) \geq y_4$.
where y_2 is the positive root of the equation $AQ^2 + BQ + C = 0$
where
$A = \frac{\alpha\eta_2\eta_3}{\beta}\left(k_2 - \frac{k_1\eta_1 K}{r}\right) + \left(\frac{dk_1\eta_1 K}{r} - \frac{\alpha\eta_1^2 K}{r} - dk_2\right)\left(\frac{k_1\eta_3}{\beta} + k_2\right)$,
$B = \left(\alpha\eta_1 K - d\left(1+k_1K\right)\right)\left(\frac{k_1\eta_3}{\beta} + k_2\right) + \left(\frac{dk_1\eta_1 K}{r} - \frac{\alpha\eta_1^2 K}{r} - dk_2\right)\left(1 + \frac{k_1(\mu+\sigma)}{\beta}\right)$
$+\frac{\alpha\eta_2}{\beta}\left(\left(1+k_1K\right)\eta_3 + (\mu+\sigma)\left(k_2 - \frac{k_1\eta_1 K}{r}\right)\right)$,
$C = \frac{\alpha\eta_2(\mu+\sigma)}{\beta}\left(1+k_1K\right) + \left(\alpha\eta_1 K - d\left(1+k_1K\right)\right)\left(1 + \frac{k_1(\mu+\sigma)}{\beta}\right)$,

$$y_1 = K\left(1 - \frac{\eta_1}{r}y_2\right), \quad y_3 = \frac{\eta_3 y_2 + (\mu+\sigma)}{\beta}, \quad y_4 = \frac{\Delta\beta - \left(\eta_2 y_2 + \mu\right)\left(\eta_3 y_2 + \mu + \sigma\right)}{\left(\eta_3 y_2 + \mu + \sigma\right)\beta}$$

(y_1, y_2, y_3, y_4) is feasible solution of model system (14) to (17) if the following conditions hold. Using descarte's rule of sign Equation has at least one positive root if $A < 0$, $B > 0$ or $A > 0$, $\ B < 0$, $\ 1 > \frac{\eta_1}{r}y_2$ and $\Delta\beta > \left(\eta_2 y_2 + \mu\right)\left(\eta_3 y_2 + \mu + \sigma\right)$ thus under these conditions model system (1) to (4) is uniformly persistent.

6. Numerical simulations and conclusion

Predation is an important factor that controls the infection in prey. The fatal disease can harm predator population that decreases the growth rate or increasing the death rate. In this paper, a non linear mathematical model with endemic exotic prey and native prey-predator with functional response Holling type II was framed to study the transmission of disease. We have put set of biologically feasible parameter values in Table 1 which are taken from some of the reference papers also observed the dynamics of model system (1) to (4) for the same values. The obtained equilibrium point is (27.4216, 1470.55, 151.38, 24.7039). Further, we have analyzed the model in three parts. In first part; we focused on effect of transmission rate of disease (β), in second part; we noted effect of predation rate of exotic infected preys (η_3), in third part; we saw effect of carrying capacity on the environment (K) on survival of native and exotic populations, respectively, as they all are responsible for transmission of disease.

Whole mathematical analysis can be summarized in following Table-2, given below;

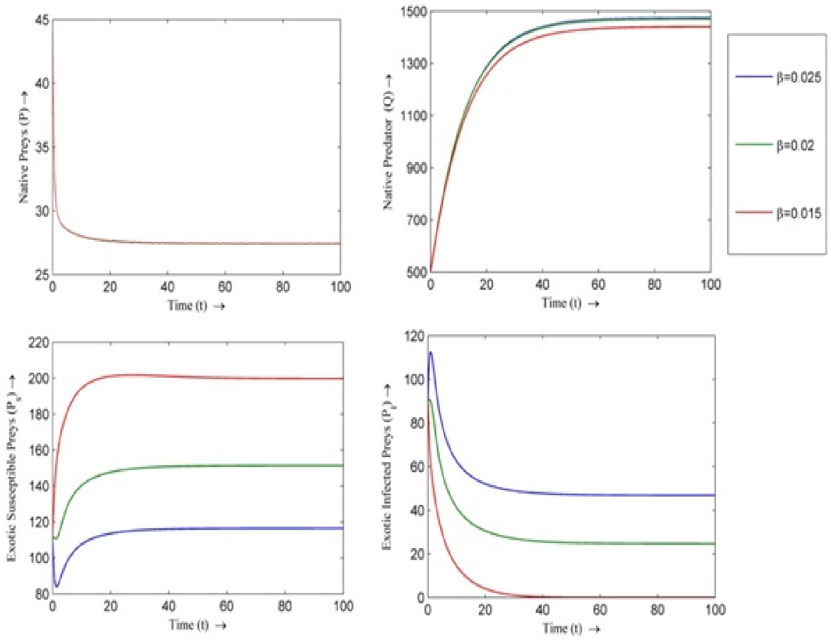

FIGURE 3. **Plot between Time and all species for various values of** β

Table 2: Summery of analysis

Equilibrium points	**Existence conditions**	**Stability conditions**
$E_0(0,0,0,0)$	Always	Always unstable in theorem 4.1
$E_1\left(K,0,\frac{\Delta}{\mu},0\right)$	Always	Stable under conditions of theorem 4.2
$E_2\left(\hat{P},\hat{Q},\hat{P}_s,0\right)$	$K>\hat{P}>K\left(1-\frac{\eta_1}{rk_2}\right)$	Stable under conditions of theorem 4.3
$E_3\left(K,0,\frac{\mu+\sigma}{\beta},\frac{1}{\beta}\left[\frac{\Delta\beta}{\mu+\sigma}-\mu\right]\right)$	$\Delta>\frac{\mu(\mu+\sigma)}{\beta}$	Stable under conditions of theorem 4.4
$E_4\left(\overset{*}{P},\overset{*}{Q},\overset{*}{P}_s,\overset{*}{P}_i\right)$	$K>\overset{*}{P}>K\left(1-\frac{\eta_1}{rk_2}\right)$ and $\frac{1}{\beta}\left[\frac{\eta_3\overset{*}{Q}}{\left(1+k_2\overset{*}{Q}\right)}+(\mu+\sigma)\right]>\overset{*}{P}_s>\frac{(\mu+\sigma)}{\beta}$	Stable under conditions of theorem 4.5

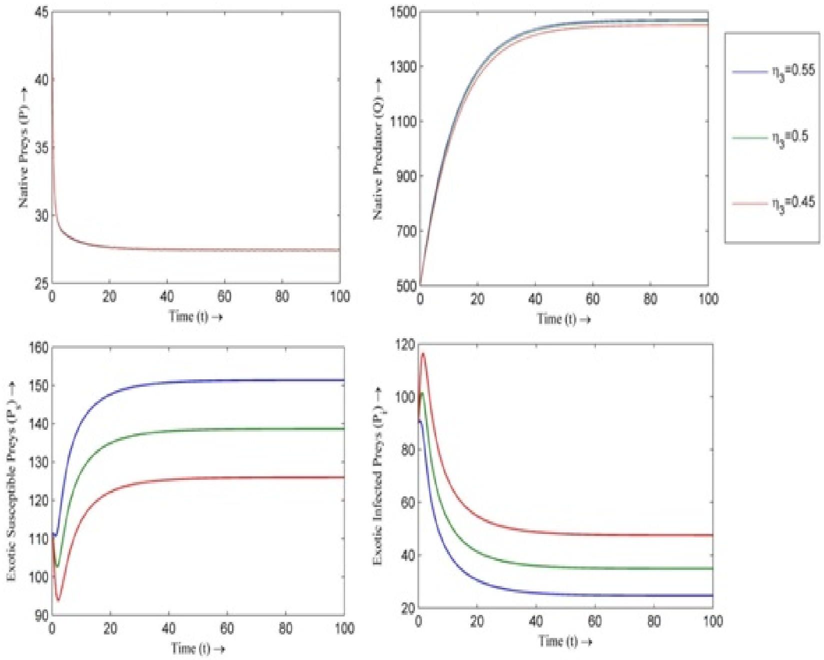

FIGURE 4. **Plot between Time and all species for various values of η_3**

Role of disease transmission rate (β)
Keeping $\eta_3 = 0.55$ and $K = 45$fixed, it was concluded that as disease transmission rate (β)decreases, then predator population decreases, exotic susceptible prey population increases, exotic infected prey population decreases and there is no effect on native preys (Fig.3).

Role of the predation rate of exotic infected prey (η_3)
Keeping$\beta = 0.02$and $K = 45$fixed, it was concluded that as the predation rate of exotic infected prey(η_3) decreases, then predator population decreases, exotic susceptible prey population decreases, exotic infected prey population increases and there is no effect on native preys (Fig4).

Role of carrying capacity of the environment (K)
Keeping $\eta_3 = 0.55$ and $\beta = 0.02$ fixed, it was concluded that as carrying capacity (K) decreases then native prey population, native predator population decreases and there is no effect on exotic susceptible and infected preys (Fig.5). However, it is also argued that consumption of prey that is infected may be harmful or beneficial, depends on the virulence of its infection severity

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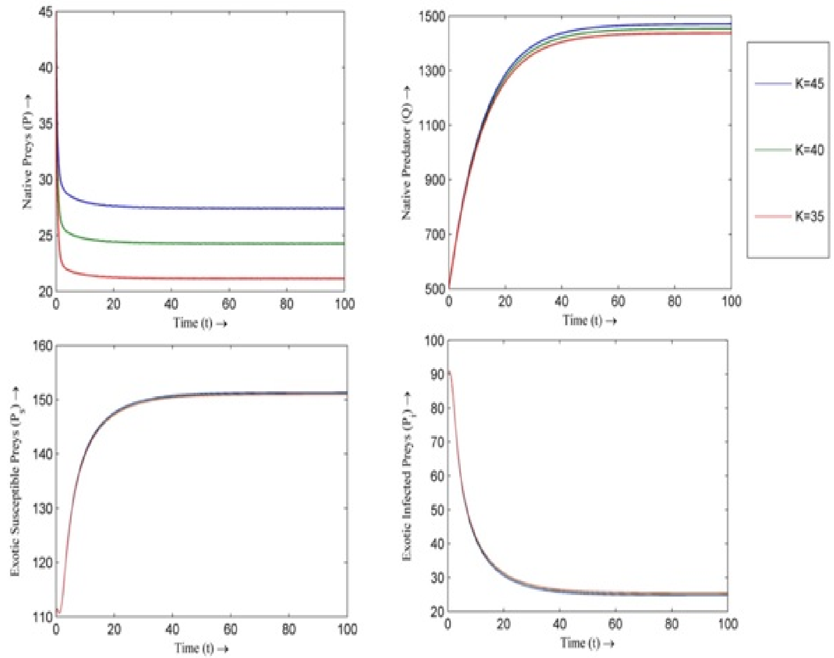

FIGURE 5. **Plot between Time and all species for various values of K**

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