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TWICE DIFFERENTIABLE OSTROWSKI TYPE TENSORIAL NORM INEQUALITY FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. In this paper several tensorial norm inequalities for continuous functions of selfadjoint operators in Hilbert space have been obtained. The recent progression of the Hilbert space inequalities following the definition of the convex operator inequality has lead researchers to explore the concept of Hilbert space inequalities even further. The motivation for this paper stems from the recent development in the theory of tensorial and Hilbert space inequalities. Multiple inequalities are obtained with variations due to the convexity properties of the mapping f

$$\begin{aligned}&\Bigg\|\frac{1}{6}\left(\exp(A)\otimes 1+4\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)+1\otimes\exp(B)\right)\\&-\frac{1}{4}\Bigg(\int_0^1\exp\left(\left(\frac{1-k}{2}\right)A\otimes 1+\left(\frac{1+k}{2}\right)1\otimes B\right)k^{-\frac{1}{2}}\,dk\\&+\int_0^1\exp\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)(1-k)^{-\frac{1}{2}}\,dk\Bigg)\Bigg\|\\&\leq\frac{47}{360}\left\|1\otimes B-A\otimes 1\right\|^2(\|\exp(A)\|+\|\exp(B)\|).\end{aligned}$$

Tensorial version of a Lemma given by Hezenci is derived and utilized to obtain the desired inequalities. In the introduction section is given a brief history of the inequalities, while in the preliminary section we give necessary Lemmas and results in order to understand the paper. Structure and novelty of the paper are discussed at the end of the introduction section.

1. Introduction and preliminaries

The notion of a tensor has its origin in the 19th century, where it was formulated by Gibbs, though he didn't formally use the word tensor but a dyadic. In modern language, it can be seen as the origin of the tensor definition and its introduction to the mathematics. Interplay of inequalities in mathematics is vast, and as such it has

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applications in tensors as well. Mathematics and other scientific fields are highly influenced by inequalities. Many types of inequalities exist, but those involving Jensen, Ostrowski, Hermite–Hadamard, and Minkowski hold particular significance among them. More about inequalities and its history can be found in these books [22, 25]. Regarding the generalizations of the aforementioned inequalities, numerous studies have been published; for additional information, check the following and the references therein [1, 2, 3, 4, 5, 26, 20, 28, 7, 8, 9, 29, 30, 31, 32, 33, 34]. The first five citations are related to the inequalities concerning the possible relation which can be defined on an interval, such as center-radius(CR) relation, interval-valued relation and the inequalities related to the Hermite-Hadamard type. Citations from 7-9 are concerned with obtaining Hermite-Hadamard and Fejer type inequalities in a classical sense with addition to using fractional operators. Citation 20 is concerned with obtaining Hermite-Hadamard and Fejer type inequalities. Citation 26 is concerned with obtaining midpoint type Hermite-Hadamard inequalities with Caputo-Fabrizio fractional operators. Citations from 28-31 and 33 are concerned with obtaining variations of the modified integral inequalities as well as obtaining various Hermite-Hadamard-Fejer type inequalities. Citation 32 is concerned with obtaining refinement of the Hermite-Hadamard type inequality in the Hilbert space in tensorial sense.

Since our paper is about tensorial Ostrowski type inequalities, we give the brief introduction to the topic. In 1938, A. Ostrowski [23] , proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b) and $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)| < +\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a),$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

If we take $x=\frac{a+b}{2}$ we get the midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{1}{4}\|f'\|_\infty (b-a),$$

with $\frac{1}{4}$ as best possible constant.

In order to derive similar inequalities of the tensorial type, we need the following introduction and preliminaries.

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f: I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_k)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$ by following , we define

$$f(A_1,...,A_k) := \int_{I_1}...\int_{I_k} f(\lambda_1,...,\lambda_k)dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k)$$

as bounded selfadjoint operator on the tensorial product $H_1 \otimes ...H_k$.
If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction extends the definition of Koranyi [21] for functions of two variables and have the property that

$$f(A_1, ...A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D)$$

that holds for any $A, B, C, D \in B(H)$.
From the property we can deduce easily the following

$$A^n \otimes B^n = (A \otimes B)^n, n \geqslant 0,$$
$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B,$$

which can be extended, for two natural numbers m; n we have

$$(A \otimes 1)^n (1 \otimes B)^m = (1 \otimes B)^n (A \otimes 1)^m = A^n \otimes B^m.$$

The current research concerning tensorial inequalities can be seen in the following papers, [10, 11, 12, 13, 14]. The following Lemma which we require can be found in a paper of Dragomir [15].

Lemma 1.1. *Assume A and B are selfadjoint operators with $Sp(A) \subset I, Sp(B) \subset J$ and having the spectral resolutions . Let $f; h$ be continuous on I, g, k continuous on J and ϕ and ψ continuous on an interval K that contains the sum of the intervals $f(I) + g(J); h(I) + k(J)$,then*

$$\phi(f(A) \otimes 1 + 1 \otimes g(B))\psi(h(A) \otimes 1 + 1 \otimes k(B))$$
$$= \int_I \int_J \phi(f(t) + g(s))\psi(h(t) + k(s)) dE_t \otimes dF_s.$$

Definition of a well known Riemann-Liouville (RL) fractional integral is given.

Definition 1.1. *Let $f \in C([a, b])$. Then the left and right sided Riemann Liouville (RL) fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as*

$$I^{\alpha}_{a^+} f(z) = \frac{1}{\Gamma(\alpha)} \int_a^z (z - u)^{\alpha-1} f(u)du, z > a, \tag{1}$$

and

$$I^{\alpha}_{b^-} f(z) = \frac{1}{\Gamma(\alpha)} \int_z^b (u - z)^{\alpha-1} f(u)du, z < b, \tag{2}$$

where $\Gamma(.)$ denotes the Gamma function.

Definition 1.2. *A real valued function $f : I \to \mathbb{R}$ is called a convex function on interval I if*

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \tag{3}$$

holds for all $t \in [0, 1]$ and for all $x, y \in I$.

In the paper written by Hezenci et al. [18] , the authors used the following Lemma. We will utilize it to produce results in the tensorial setting.

Lemma 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) such that $f'' \in L_1([a,b])$. Then, the following equality holds*

$$\begin{aligned} &\frac{1}{6}\big(f(a) + 4f\left(\frac{a+b}{2}\right) + 2f(b)\big) \qquad (4) \\ &-\frac{2^{\alpha-1}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\left(J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right) + J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)\right) \\ &= \frac{(b-a)^2}{8(\alpha+1)}\int_0^1 \left(\frac{1-2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau - \tau^{\alpha+1}\right)\left(f''\left(\left(\frac{1+\tau}{2}\right)b + \left(\frac{1-\tau}{2}\right)a\right)\right. \\ &\left. + f''\left(\left(\frac{1+\tau}{2}\right)a + \left(\frac{1-\tau}{2}\right)b\right)\right)d\tau. \end{aligned}$$

Lemma 1.3. *We will prove the following relation for fractional integrals,*

$$J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right) = \frac{b-a}{2^{\alpha}\Gamma(\alpha)}\int_0^1 f\left(\left(\frac{1-k}{2}\right)a + \left(\frac{1+k}{2}\right)b\right)k^{\alpha-1}(b-a)^{\alpha-1}dk,$$

$$J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right) = \frac{b-a}{2^{\alpha}\Gamma(\alpha)}\int_0^1 f\left(\left(1-\frac{k}{2}\right)a + \frac{k}{2}b\right)((k-1)a + (1-k)b)^{\alpha-1}dk.$$

Proof. Use the integral definition of the fractional integrals and introduce the following substitution $\zeta = (1-k)\frac{a+b}{2} + kb$ and $\zeta = (1-k)a + k\frac{a+b}{2}$ on the first and second integral, the result follows. $\square$

Novel aspects in this work can be seen in the development of the inequalities of the Ostrowski type for the twice differentiable functions in the Hilbert space of tensorial type. This field is relatively new, therefore obtaining new bounds for various convex combinations of the functions is instrumental to the development of it.
The rest of the paper is structured as follows, main results is the section in which results concerning the novelty of the work will be given. The following section , some examples and consequences will feature examples of the obtained results by using the known fact about the exponential operator and its integral, therefore by utilizing it and choosing f to be a specific convex function, we obtain numerous examples and bounds of the Ostrowski type in the tensorial sense. In the conclusion section we conclude what has been done in the paper. In the following Theorem, we give a fundamental result which we will use in our paper to produce inequalities.

2. Main results

Theorem 2.2. *Assume that f is continuously differentiable on I, A and B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\begin{aligned} &\frac{1}{6}\left(f(A)\otimes 1 + 4f\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) + 1\otimes f(B)\right) \qquad (5) \\ &-\frac{1}{2}\alpha\left(\int_0^1 f\left(\left(\frac{1-k}{2}\right)A\otimes 1 + \left(\frac{1+k}{2}\right)1\otimes B\right)k^{\alpha-1}dk\right. \\ &\left. + \int_0^1 f\left(\left(1-\frac{k}{2}\right)A\otimes 1 + \frac{k}{2}1\otimes B\right)(1-k)^{\alpha-1}dk\right) \end{aligned}$$

$$= (1 \otimes B - A \otimes 1)^2 \frac{1}{8(\alpha+1)} \int_0^1 \left(\frac{1-2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau - \tau^{\alpha+1} \right)$$
$$\left(f'' \left(\left(\frac{1+\tau}{2} \right) 1 \otimes B + \left(\frac{1-\tau}{2} \right) A \otimes 1 \right) \right.$$
$$\left. + f'' \left(\left(\frac{1+\tau}{2} \right) A \otimes 1 + \left(\frac{1-\tau}{2} \right) 1 \otimes B \right) \right) d\tau.$$

Proof. We start with Lemma 3 (4). Rewriting the fractional integral using Lemma 4, simplifying we obtain

$$\frac{1}{6} \left(f(a) + 4f \left(\frac{a+b}{2} \right) + f(b) \right)$$
$$- \frac{1}{2}\alpha \left(\int_0^1 f \left(\left(\frac{1-k}{2} \right) a + \left(\frac{1+k}{2} \right) b \right) k^{\alpha-1} dk \right.$$
$$\left. + \int_0^1 f \left(\left(1 - \frac{k}{2} \right) a + \frac{k}{2} b \right) (1-k)^{\alpha-1} dk \right)$$
$$= (b-a)^2 \frac{1}{8(\alpha+1)} \int_0^1 \left(\frac{1-2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau - \tau^{\alpha+1} \right)$$
$$\left(f'' \left(\left(\frac{1+\tau}{2} \right) b + \left(\frac{1-\tau}{2} \right) a \right) \right.$$
$$\left. + f'' \left(\left(\frac{1+\tau}{2} \right) a + \left(\frac{1-\tau}{2} \right) b \right) \right) d\tau.$$

Assuming that A and B have the spectral resolutions

$$A = \int t dE(t) \text{ and } B = \int s dF(s).$$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\int_I \int_I \frac{1}{6} \left(f(t) + 4f \left(\frac{t+s}{2} \right) + f(s) \right) dE_t \otimes dF_s$$
$$- \int_I \int_I \frac{1}{2}\alpha \left(\int_0^1 f \left(\left(\frac{1-k}{2} \right) t + \left(\frac{1+k}{2} \right) s \right) k^{\alpha-1} dk \right.$$
$$\left. + \int_0^1 f \left(\left(1 - \frac{k}{2} \right) t + \frac{k}{2} s \right) (1-k)^{\alpha-1} dk \right) dE_t \otimes dF_s$$
$$= \int_I \int_I (s-t)^2 \frac{1}{8(\alpha+1)} \int_0^1 \left(\frac{1-2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau - \tau^{\alpha+1} \right)$$
$$\left(f'' \left(\left(\frac{1+\tau}{2} \right) s + \left(\frac{1-\tau}{2} \right) t \right) \right.$$
$$\left. + f'' \left(\left(\frac{1+\tau}{2} \right) t + \left(\frac{1-\tau}{2} \right) s \right) \right) d\tau dE_t \otimes dF_s.$$

By utilizing Lemma 3 for appropriate choices of the functions involved, we have successively

$$\int_I \int_I \frac{1}{6} f(t) dE_t \otimes dF_s = \frac{1}{6} (f(A) \otimes 1),$$
$$\int_I \int_I \frac{1}{6} f \left(\frac{s+t}{2} \right) dE_t \otimes dF_s = \frac{2}{3} f \left(\frac{A \otimes 1 + 1 \otimes B}{2} \right),$$

$$\int_I\int_I \frac{1}{6} f(s) dE_t \otimes dF_s = \frac{1}{6}(1 \otimes f(B)).$$

The right hand side can be dealt with in a similar manner. We will illustrate it with the one term, the rest follow analogously

$$\int_I\int_I\int_0^1 (s-t)^2 \tau^{\alpha+1}\Bigg(f''\left(\left(\frac{1+\tau}{2}\right)s + \left(\frac{1-\tau}{2}\right)t\right)$$

$$+f''\left(\left(\frac{1+\tau}{2}\right)t + \left(\frac{1-\tau}{2}\right)s\right)\Bigg) d\tau dE_t \otimes dF_s$$

$$= \int_0^1\int_I\int_I (s-t)^2 \tau^{\alpha+1}\Bigg(f''\left(\left(\frac{1+\tau}{2}\right)s + \left(\frac{1-\tau}{2}\right)t\right)$$

$$+f''\left(\left(\frac{1+\tau}{2}\right)t + \left(\frac{1-\tau}{2}\right)s\right)\Bigg) dE_t \otimes dF_s d\tau$$

$$= \int_0^1 (1\otimes B - A \otimes 1)^2 \tau^{\alpha+1}\Bigg(f''\left(\left(\frac{1+\tau}{2}\right)1\otimes B + \left(\frac{1-\tau}{2}\right)A\otimes 1\right)$$

$$+f''\left(\left(\frac{1+\tau}{2}\right)A\otimes 1 + \left(\frac{1-\tau}{2}\right)1\otimes B\right)\Bigg) dE_t \otimes dF_s d\tau.$$

□

We give our first inequality of the tensorial type utilizing the tensorial equality obtained in Theorem 2.2.

Theorem 2.3. *Assume that f is continuously differentiable on I with $\|f''\|_{I,+\infty} := \sup_{t\in I}|f''(t)| < +\infty$ and A, B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\Bigg\|\frac{1}{6}\left(f(A)\otimes 1 + 4f\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) + 1\otimes f(B)\right) \tag{6}$$

$$-\frac{1}{2}\alpha\Bigg(\int_0^1 f\left(\left(\frac{1-k}{2}\right)A\otimes 1 + \left(\frac{1+k}{2}\right)1\otimes B\right)k^{\alpha-1}dk$$

$$+\int_0^1 f\left(\left(1-\frac{k}{2}\right)A\otimes 1 + \frac{k}{2}1\otimes B\right)(1-k)^{\alpha-1}dk\Bigg)\Bigg\|$$

$$\leq \|1\otimes B - A\otimes 1\|^2 \frac{\|f''\|_{I,+\infty}(3\alpha^2+8\alpha+7)}{(\alpha+2)(12\alpha+12)}.$$

Setting $\alpha = \frac{1}{2}$ we obtain

$$\Bigg\|\frac{1}{6}\left(f(A)\otimes 1 + 4f\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) + 1\otimes f(B)\right) \tag{7}$$

$$-\frac{1}{4}\Bigg(\int_0^1 f\left(\left(\frac{1-k}{2}\right)A\otimes 1 + \left(\frac{1+k}{2}\right)1\otimes B\right)k^{-\frac{1}{2}}dk$$

$$+\int_0^1 f\left(\left(1-\frac{k}{2}\right)A\otimes 1 + \frac{k}{2}1\otimes B\right)(1-k)^{-\frac{1}{2}}dk\Bigg)\Bigg\|$$

$$\leq \frac{47}{180}\|1\otimes B - A\otimes 1\|^2 \|f''\|_{I,+\infty}.$$

Proof. If we take the operator norm and use the triangle inequality, we get

$$\Bigg\|\frac{1}{6}\left(f(A)\otimes 1+4f\left(\frac{A\otimes 1+1\otimes B}{2}\right)+1\otimes f(B)\right)$$

$$-\frac{1}{2}\alpha\Bigg(\int_0^1 f\left(\left(\frac{1-k}{2}\right)A\otimes 1+\left(\frac{1+k}{2}\right)1\otimes B\right)k^{\alpha-1}dk$$

$$+\int_0^1 f\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)(1-k)^{\alpha-1}dk\Bigg)\Bigg\|$$

$$\leq \|1\otimes B-A\otimes 1\|^2\frac{1}{8(\alpha+1)}\Bigg\|\int_0^1\left(\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right)$$

$$\left(f''\left(\left(\frac{1+\tau}{2}\right)1\otimes B+\left(\frac{1-\tau}{2}\right)A\otimes 1\right)\right.$$

$$\left.+f''\left(\left(\frac{1+\tau}{2}\right)A\otimes 1+\left(\frac{1-\tau}{2}\right)1\otimes B\right)\right)d\tau\Bigg\|.$$

Using the properties of the integral and the norm, we get

$$\|1\otimes B-A\otimes 1\|^2\frac{1}{8(\alpha+1)}\Bigg\|\int_0^1\left(\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}\right)$$

$$\left(f''\left(\left(\frac{1+\tau}{2}\right)1\otimes B+\left(\frac{1-\tau}{2}\right)A\otimes 1\right)\right.$$

$$\left.+f''\left(\left(\frac{1+\tau}{2}\right)A\otimes 1+\left(\frac{1-\tau}{2}\right)1\otimes B\right)\right)d\tau\Bigg\|$$

$$\leq \|1\otimes B-A\otimes 1\|^2\frac{1}{8(\alpha+1)}\int_0^1\left(\frac{1+2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau+\tau^{\alpha+1}\right)$$

$$\Bigg\|\left(f''\left(\left(\frac{1+\tau}{2}\right)1\otimes B+\left(\frac{1-\tau}{2}\right)A\otimes 1\right)\right.$$

$$\left.+f''\left(\left(\frac{1+\tau}{2}\right)A\otimes 1+\left(\frac{1-\tau}{2}\right)1\otimes B\right)\right)\Bigg\|d\tau.$$

$$\leq \|1\otimes B-A\otimes 1\|^2\frac{1}{8(\alpha+1)}\int_0^1\left(\frac{1+2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau+\tau^{\alpha+1}\right)$$

$$\left(\left\|f''\left(\left(\frac{1+\tau}{2}\right)1\otimes B+\left(\frac{1-\tau}{2}\right)A\otimes 1\right)\right\|\right.$$

$$\left.+\left\|f''\left(\left(\frac{1+\tau}{2}\right)A\otimes 1+\left(\frac{1-\tau}{2}\right)1\otimes B\right)\right\|\right)d\tau.$$

Observe that by Lemma 1

$$\left|f''\left(\left(\frac{1+\tau}{2}\right)1\otimes B+\left(\frac{1-\tau}{2}\right)A\otimes 1\right)\right|=\int_I\int_I\left|f''\left(\left(\frac{1+\tau}{2}\right)s+\left(\frac{1-\tau}{2}\right)t\right)\right|dE_t\otimes dF_s.$$

Since

$$\left|f''\left(\left(\frac{1+\tau}{2}\right)s+\left(\frac{1-\tau}{2}\right)t\right)\right|\leqslant\|f''\|_{I,+\infty}$$

for all $\tau \in [0,1]$ and $t,s \in I$. If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\left| f''\left(\left(\frac{1+\tau}{2}\right) 1 \otimes B + \left(\frac{1-\tau}{2}\right) A \otimes 1\right)\right| = \int_I \int_I \left| f''\left(\left(\frac{1+\tau}{2}\right) s + \left(\frac{1-\tau}{2}\right) t\right)\right| dE_t \otimes dF_s$$
$$\leqslant \|f'\|_{I,+\infty} \int_I \int_I dE_t \otimes dF_s = \|f''\|_{I,+\infty}$$

for all $\tau \in [0,1]$ and $t,s \in I$. Similarly, we get

$$\left| f''\left(\left(\frac{1+\tau}{2}\right) A \otimes 1 + \left(\frac{1-\tau}{2}\right) 1 \otimes B\right)\right| \leq \|f''\|_{I,+\infty}.$$

Which combined gives

$$\|1 \otimes B - A \otimes 1\|^2 \frac{1}{8(\alpha+1)} \int_0^1 \left(\frac{1+2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau + \tau^{\alpha+1}\right)$$
$$\left(\left\| f''\left(\left(\frac{1+\tau}{2}\right) 1 \otimes B + \left(\frac{1-\tau}{2}\right) A \otimes 1\right)\right\|\right.$$
$$\left. + \left\| f''\left(\left(\frac{1+\tau}{2}\right) A \otimes 1 + \left(\frac{1-\tau}{2}\right) 1 \otimes B\right)\right\|\right) d\tau.$$
$$\leq \|1 \otimes B - A \otimes 1\|^2 \frac{1}{8(\alpha+1)} \int_0^1 \left(\frac{1+2\alpha}{3} + \frac{2(\alpha+1)}{3}\tau + \tau^{\alpha+1}\right)$$
$$\left(\|f''\|_{I,+\infty} + \|f''\|_{I,+\infty}\right) d\tau.$$

Solving the resulting integral and simplifying, we obtain the desired result. $\square$

Theorem 2.4. *Assume that f is continuously differentiable on I and $|f''|$ is convex and A, B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\left\| \frac{1}{6}\left(f(A) \otimes 1 + 4f\left(\frac{A \otimes 1 + 1 \otimes B}{2}\right) + 1 \otimes f(B)\right)\right. \tag{8}$$
$$-\frac{1}{2}\alpha\left(\int_0^1 f\left(\left(\frac{1-k}{2}\right) A \otimes 1 + \left(\frac{1+k}{2}\right) 1 \otimes B\right) k^{\alpha-1} dk\right.$$
$$\left.\left. + \int_0^1 f\left(\left(1 - \frac{k}{2}\right) A \otimes 1 + \frac{k}{2} 1 \otimes B\right)(1-k)^{\alpha-1} dk\right)\right\|$$
$$\leq \|1 \otimes B - A \otimes 1\|^2 \frac{\left(\|f''(A)\| + \|f''(B)\|\right)\left(3\alpha^2 + 8\alpha + 7\right)}{(\alpha+2)(24\alpha+24)}.$$

Proof. Since $|f''|$ is convex on I, then we get

$$\left| f''\left(\frac{1-\tau}{2} t + \frac{1+\tau}{2} s\right)\right| \leqslant \frac{1-\tau}{2}|f''(t)| + \frac{1+\tau}{2}|f''(s)|$$

for all $\tau \in [0,1]$ and $t,s \in I$.
If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\left| f''\left(\frac{1-\tau}{2} A \otimes 1 + \frac{1+\tau}{2} 1 \otimes B\right)\right| = \int_I \int_I \left| f''\left(\frac{1-\tau}{2} t + \frac{1+\tau}{2} s\right)\right| dE_t \otimes dF_s$$
$$\leqslant \int_I \int_I \left[\frac{1-\tau}{2}|f''(t)| + \frac{1+\tau}{2}|f''(s)|\right] dE_t \otimes dF_s$$

$$= \frac{1-\tau}{2}|f''(A)| \otimes 1 + \frac{1+\tau}{2} 1 \otimes |f''(B)|$$

for all $\tau \in [0,1]$.
If we take the norm in the inequality, we get the following

$$\left\| f''\left(\frac{1-\tau}{2} A \otimes 1 + \frac{1+\tau}{2} 1 \otimes B\right)\right\| \leqslant \left\| \frac{1-\tau}{2}|f''(A)| \otimes 1 + \frac{1+\tau}{2} 1 \otimes |f''(B)|\right\|$$
$$\leqslant \frac{1-\tau}{2} \left\| |f''(A)| \otimes 1 \right\| + \frac{1+\tau}{2} \left\| 1 \otimes |f''(B)| \right\|$$
$$= \frac{1-\tau}{2} \left\| f''(A) \right\| + \frac{1+\tau}{2} \left\| f''(B) \right\|.$$

Similarly, we get

$$\left\| f''\left(\frac{1-\tau}{2} 1 \otimes B + \frac{1+\tau}{2} A \otimes 1\right)\right\|$$
$$\leqslant \frac{1-\tau}{2} \left\| f''(B) \right\| + \frac{1+\tau}{2} \left\| f''(A) \right\|.$$

We have the following from the previous Theorem, which when combined with the estimates we obtained for $|f''|$, we obtain the following inequality

$$\Bigg\| \frac{1}{6}\left(f(A)\otimes 1 + 4f\left(\frac{A\otimes 1 + 1 \otimes B}{2}\right) + 1 \otimes f(B)\right)$$
$$-\frac{1}{2}\alpha \Bigg(\int_0^1 f\left(\left(\frac{1-k}{2}\right) A\otimes 1 + \left(\frac{1+k}{2}\right) 1 \otimes B\right) k^{\alpha-1} dk$$
$$+ \int_0^1 f\left(\left(1-\frac{k}{2}\right) A \otimes 1 + \frac{k}{2} 1 \otimes B\right)(1-k)^{\alpha-1} dk\Bigg)\Bigg\|$$
$$\leqslant \left\| 1 \otimes B - A \otimes 1 \right\|^2 \frac{1}{8(\alpha+1)} \int_0^1 \left(\frac{1+2\alpha}{3} + \frac{2(\alpha+1)}{3} + \tau^{\alpha+1}\right)$$
$$\Bigg(\frac{1-\tau}{2} \left\| f''(B) \right\| + \frac{1+\tau}{2} \left\| f''(A) \right\|$$
$$+ \frac{1+\tau}{2} \left\| f''(B) \right\| + \frac{1-\tau}{2} \left\| f''(A) \right\| \Bigg) d\tau.$$

Which when simplified after integrating the terms, we obtain the original inequality. □

We recall that the function $f : I \to \mathbb{R}$ is quasi-convex, if

$$f((1-\lambda)t + \lambda s) \leqslant \max(f(t), f(s)) = \frac{1}{2}(f(t) + f(s) + |f(s) - f(t)|)$$

for all $t, s \in I$ and $\lambda \in [0,1]$.

Theorem 2.5. *Assume that f is continuously differentiable on I with $|f''|$ is quasi-convex on I, A and B are selfadjoint operators with $Sp(A), Sp(B) \subset I$, then*

$$\Bigg\| \frac{1}{6}\left(f(A)\otimes 1 + 4f\left(\frac{A\otimes 1 + 1 \otimes B}{2}\right) + 1 \otimes f(B)\right) \tag{9}$$
$$-\frac{1}{2}\alpha \Bigg(\int_0^1 f\left(\left(\frac{1-k}{2}\right) A\otimes 1 + \left(\frac{1+k}{2}\right) 1 \otimes B\right) k^{\alpha-1} dk$$
$$+ \int_0^1 f\left(\left(1-\frac{k}{2}\right) A \otimes 1 + \frac{k}{2} 1 \otimes B\right)(1-k)^{\alpha-1} dk\Bigg)\Bigg\|$$

$$\leq \|1\otimes B - A\otimes 1\|^2 \frac{\left(3\alpha^2+8\alpha+7\right)}{(\alpha+2)(24\alpha+24)}$$
$$\times \left(\||f''(A)|\otimes 1 + 1\otimes |f''(B)|\| + \||f''(A)|\otimes 1 - 1\otimes |f''(B)|\|\right).$$

Proof. Since $|f''|$ is quasi-convex on I, then we get

$$\left|f''\left(\frac{1-\tau}{2}t+\frac{1+\tau}{2}s\right)\right| \leqslant \frac{1}{2}(|f''(t)|+|f''(s)|+||f''(t)|-|f''(s)||)$$

for all $\tau\in[0,1]$ and $t,s\in I$. If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}
&\left|f''\left(\frac{1-\tau}{2}A\otimes 1+\frac{1+\tau}{2}1\otimes B\right)\right| \\
&= \int_I\int_I |f''\left(\frac{1-\tau}{2}t+\frac{1+\tau}{2}s\right)| dE_t\otimes dF_s \\
&\leqslant \frac{1}{2}\int_I\int_I (|f''(t)|+|f''(s)|+||f''(t)|-|f''(s)||) dE_t\otimes dF_s \\
&= \frac{1}{2}(|f''(A)|\otimes 1 + 1\otimes |f''(B)| + ||f''(A)|\otimes 1 - 1\otimes |f''(B)||)
\end{aligned}$$

for all $\tau\in[0,1]$.
If we take the norm, then we get

$$\begin{aligned}
&\left\|f''\left(\frac{1-\tau}{2}A\otimes 1+\frac{1+\tau}{2}1\otimes B\right)\right\| \\
&\leqslant \left\|\frac{1}{2}(|f''(A)|\otimes 1 + 1\otimes |f''(B)| + ||f''(A)|\otimes 1 - 1\otimes |f''(B)||)\right\| \\
&\leqslant \frac{1}{2}\left(\||f''(A)|\otimes 1 + 1\otimes |f''(B)|\| + \||f''(A)|\otimes 1 - 1\otimes |f''(B)|\|\right)
\end{aligned}$$

for all $\tau\in[0,1]$. In a similar way, we obtain

$$\begin{aligned}
&\left\|f''\left(\frac{1+\tau}{2}A\otimes 1+\frac{1-\tau}{2}1\otimes B\right)\right\| \\
&\leqslant \left\|\frac{1}{2}(|f''(A)|\otimes 1 + 1\otimes |f''(B)| + ||f''(A)|\otimes 1 - 1\otimes |f''(B)||)\right\| \\
&\leqslant \frac{1}{2}\left(\||f''(A)|\otimes 1 + 1\otimes |f''(B)|\| + \||f''(A)|\otimes 1 - 1\otimes |f''(B)|\|\right).
\end{aligned}$$

Using these inequalities in the inequality obtained during Theorem 5, we obtain the following

$$\begin{aligned}
&\left\|\frac{1}{6}\left(f(A)\otimes 1 + 4f\left(\frac{A\otimes 1+1\otimes B}{2}\right)+1\otimes f(B)\right)\right. \\
&-\frac{1}{2}\alpha\Bigg(\int_0^1 f\left(\left(\frac{1-k}{2}\right)A\otimes 1+\left(\frac{1+k}{2}\right)1\otimes B\right)k^{\alpha-1}dk \\
&\left.+\int_0^1 f\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)(1-k)^{\alpha-1}dk\Bigg)\right\| \\
&\leqslant \|1\otimes B - A\otimes 1\|^2 \frac{1}{8(\alpha+1)}\int_0^1\left(\frac{1+2\alpha}{3}+\frac{2(\alpha+1)}{3}+\tau^{\alpha+1}\right) \\
&\Bigg(\frac{1}{2}\left(\||f''(A)|\otimes 1 + 1\otimes |f''(B)|\| + \||f''(A)|\otimes 1 - 1\otimes |f''(B)|\|\right)
\end{aligned}$$

$$+\frac{1}{2}\left(\| |f''(A)|\otimes 1+1\otimes |f''(B)|\| + \| |f''(A)|\otimes 1-1\otimes |f''(B)|\|\right)\Bigg)d\tau.$$

Which when simplified, we obtain the desired inequality. □

3. Some examples and consequences

It is known that if U and V are commuting, that is $UV = VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V) = \exp(V)\exp(U) = \exp(U+V).$$

Also, if U is invertible and $a, b \in \mathbb{R}$ and $a < b$ then

$$\int_a^b \exp(tU)dt = U^{-1}[\exp(bU) - \exp(aU)].$$

Moreover, if U and V are commuting and $V - U$ is invertible, then

$$\int_0^1 \exp((1-k)U + kV)dk = \int_0^1 \exp(k(V-U))\exp(U)dk$$
$$= (\exp(k(V-U))dk)\exp(U)$$
$$= (V-U)^{-1}[\exp(V-U) - I]\exp(U) = (V-U)^{-1}[\exp(V) - \exp(U)].$$

Since the operators $U = A\otimes 1$ and $V = 1\otimes B$ are commutative and if $1\otimes B - A\otimes 1$ is invertible, then

$$\int_0^1 \exp((1-k)A\otimes 1 + k1\otimes B)dk$$
$$= (1\otimes B - A\otimes 1)^{-1}[\exp(1\otimes B) - \exp(A\otimes 1)].$$

Corollary 3.0. *If A, B are selfadjoint operators with $Sp(A), Sp(B) \subset [m, M]$ and $1\otimes B - A\otimes 1$ is invertible, then by Theorem 3, we get*

$$\Bigg\|\frac{1}{6}\left(\exp(A)\otimes 1 + 4\exp\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) + 1\otimes\exp(B)\right) \tag{10}$$
$$-\frac{1}{2}\alpha\Bigg(\int_0^1 \exp\left(\left(\frac{1-k}{2}\right)A\otimes 1 + \left(\frac{1+k}{2}\right)1\otimes B\right)k^{\alpha-1}dk$$
$$+\int_0^1 \exp\left(\left(1-\frac{k}{2}\right)A\otimes 1 + \frac{k}{2}1\otimes B\right)(1-k)^{\alpha-1}dk\Bigg)\Bigg\|$$
$$\le \|1\otimes B - A\otimes 1\|^2\,\frac{2\exp(M)(3\alpha^2+8\alpha+7)}{(\alpha+2)(12\alpha+12)}.$$

Setting $\alpha = \frac{1}{2}$ we obtain

$$\Bigg\|\frac{1}{6}\left(\exp(A)\otimes 1 + 4\exp\left(\frac{A\otimes 1 + 1\otimes B}{2}\right) + 1\otimes\exp(B)\right) \tag{11}$$
$$-\frac{1}{4}\Bigg(\int_0^1 \exp\left(\left(\frac{1-k}{2}\right)A\otimes 1 + \left(\frac{1+k}{2}\right)1\otimes B\right)k^{-\frac{1}{2}}dk$$
$$+\int_0^1 \exp\left(\left(1-\frac{k}{2}\right)A\otimes 1 + \frac{k}{2}1\otimes B\right)(1-k)^{-\frac{1}{2}}dk\Bigg)\Bigg\|$$
$$\le \frac{47}{180}\|1\otimes B - A\otimes 1\|^2\exp(M).$$

Corollary 3.0. *Since for $f(t)=\exp(t), t\in\mathbb{R}$, $|f''|$ is convex, then by Theorem 4*

$$\left\|\frac{1}{6}\left(\exp(A)\otimes 1+4\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)+1\otimes\exp(B)\right)\right. \tag{12}$$
$$-\frac{1}{2}\alpha\Bigg(\int_0^1\exp\left(\left(\frac{1-k}{2}\right)A\otimes 1+\left(\frac{1+k}{2}\right)1\otimes B\right)k^{\alpha-1}dk$$
$$\left.+\int_0^1\exp\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)(1-k)^{\alpha-1}dk\Bigg)\right\|$$
$$\leq \|1\otimes B-A\otimes 1\|^2\,\frac{\left(\|\exp(A)\|+\|\exp(B)\|\right)\left(3\alpha^2+8\alpha+7\right)}{(\alpha+2)(24\alpha+24)}.$$

If $\alpha=\frac{1}{2}$, then

$$\left\|\frac{1}{6}\left(\exp(A)\otimes 1+4\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)+1\otimes\exp(B)\right)\right. \tag{13}$$
$$-\frac{1}{4}\Bigg(\int_0^1\exp\left(\left(\frac{1-k}{2}\right)A\otimes 1+\left(\frac{1+k}{2}\right)1\otimes B\right)k^{-\frac{1}{2}}dk$$
$$\left.+\int_0^1\exp\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)(1-k)^{-\frac{1}{2}}dk\Bigg)\right\|$$
$$\leq \frac{47}{360}\|1\otimes B-A\otimes 1\|^2\left(\|\exp(A)\|+\|\exp(B)\|\right)$$

Corollary 3.0. *Choosing $f(t)=\exp(t); t\in\mathbb{R}$, and since $|f''|$ is convex, we get by Theorem 7*

$$\left\|\frac{1}{6}\left(\exp(A)\otimes 1+4\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)+1\otimes\exp(B)\right)\right. \tag{14}$$
$$-\frac{1}{2}\alpha\Bigg(\int_0^1\exp\left(\left(\frac{1-k}{2}\right)A\otimes 1+\left(\frac{1+k}{2}\right)1\otimes B\right)k^{\alpha-1}dk$$
$$\left.+\int_0^1\exp\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)(1-k)^{\alpha-1}dk\Bigg)\right\|$$
$$\leq \|1\otimes B-A\otimes 1\|^2\,\frac{\left(3\alpha^2+8\alpha+7\right)}{(\alpha+2)(24\alpha+24)}$$
$$\times\left(\||\exp(A)|\otimes 1+1\otimes|\exp(B)|\|+\||\exp(A)|\otimes 1-1\otimes|\exp(B)|\|\right).$$

Setting $\alpha=\frac{1}{2}$, we obtain

$$\left\|\frac{1}{6}\left(\exp(A)\otimes 1+4\exp\left(\frac{A\otimes 1+1\otimes B}{2}\right)+1\otimes\exp(B)\right)\right. \tag{15}$$
$$-\frac{1}{4}\Bigg(\int_0^1\exp\left(\left(\frac{1-k}{2}\right)A\otimes 1+\left(\frac{1+k}{2}\right)1\otimes B\right)k^{-\frac{1}{2}}dk$$
$$\left.+\int_0^1\exp\left(\left(1-\frac{k}{2}\right)A\otimes 1+\frac{k}{2}1\otimes B\right)(1-k)^{-\frac{1}{2}}dk\Bigg)\right\|$$
$$\leq \frac{47}{360}\|1\otimes B-A\otimes 1\|^2$$
$$\times\left(\||\exp(A)|\otimes 1+1\otimes|\exp(B)|\|+\||\exp(A)|\otimes 1-1\otimes|\exp(B)|\|\right).$$

4. Conclusion

Tensors have become important in various fields, for example in physics because they provide a concise mathematical framework for formulating and solving physical problems in fields such as mechanics, electromagnetism, quantum mechanics, and many others. As such inequalities are crucial in numerical aspects. Reflected in this work is the Lemma and the new Ostrowski type inequality for twice differentiable functions. Using that Lemma enabled us to obtain new Ostrowski type tensorial inequalities. Examples of specific convex functions and their inequalities using our results are given in the section some examples and consequences. Plans for future research can be reflected in the fact that the obtained inequalities in this work can be sharpened or generalized by using other methods. An interesting perspective can be seen in incorporating other techniques for Hilbert space inequalities with the techniques shown in this paper. One direction is the technique of the Mond-Pecaric inequality, on which we will work on.

References

[1] W. Afzal, M. Abbas, J.E Macías-Díaz, S. Treanţă. Some H-Godunova–Levin Function Inequalities Using Center Radius (Cr) Order Relation. Fractal Fract. 2022, 6, 518. https://doi.org/10.3390/fractalfract6090518

[2] W. Afzal.; Alb Lupa¸s, A.; Shabbir, K. Hermite–Hadamard and Jensen-Type Inequalities for Harmonical (h1, h2)-Godunova–Levin Interval-Valued Functions. Mathematics 2022, 10, 2970. https://doi.org/10.3390/math10162970

[3] W. Afzal., Khurram Shabbir, Savin Treant¸˘a, Kamsing Nonlaopon. Jensen and Hermite-Hadamard type inclusions for harmonical h-Godunova-Levin functions[J]. Aims Mathematics, 2023, 8(2): 3303-3321. doi: 10.3934/math.2023170

[4] W. Afzal., Khurram Shabbir, Thongchai Botmart. Generalized version of Jensen and Hermite-Hadamard inequalities for interval-valued (h_1, h_2)- Godunova-Levin functions[J]. AIMS Mathematics, 2022, 7(10): 19372-19387. doi: 10.3934/math.20221064

[5] W. Afzal., Waqas Nazeer, Thongchai Botmart, Savin Treant¸˘a. Some properties and inequalities for generalized class of harmonical Godunova-Levin function via center radius order relation[J]. AIMS Mathematics, 2023, 8(1): 1696-1712. doi: 10.3934/math.2023087

[6] H. Araki and F. Hansen, Jenseńs operator inequality for functions of several variables, Proc. Amer. Math. Soc. 128 (2000), No. 7, 20

[7] S.I. Butt; Tariq, M.; Aslam, A.; Ahmad, H.; Nofal, T.A. Hermite–Hadamard type inequalities via generalized harmonic exponential convexity and applications. J. Funct. Spaces 2021, 2021, 5533491.

[8] A. Chandola, Agarwal R., Pandey M. R., Some New Hermite–Hadamard, Hermite–Hadamard Fejer and Weighted Hardy Type Inequalities Involving (k-p) Riemann– Liouville Fractional Integral Operator, Appl. Math. Inf. Sci. 16, No. 2, 287–297 (2022).

[9] H. Chen, Katugampola, U.N. Hermite–Hadamard and Hermite–Hadamard–Fejr type inequalities for generalized fractional integrals. J. Math. Anal. Appl. 2017, 446, 1274–1291

[10] S. S. Dragomir, An inequality improving the First Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. Journal of Inequalities in Pure Applied Mathematics, Volume: 3 (2002), Issue: 2, Paper No. 31, 8 p.

[11] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. Journal of Inequalities in Pure Applied Mathematics, Volume: 3 (2002), Issue: 3, Paper No.

[12] S. S. Dragomir, Bounds for the normalized Jensen functional, Bull. Austral. Math. Soc. 74(3)(2006), 417-478.

[13] S. S. Dragomir, A note on Youngís inequality, Revista de la Real Academia de Ciencias Exactas, FÌsicas y Naturales. Serie A. Matematicas 111 (2017), no. 2, 349-354.

[14] S. S. Dragomir, P. Cerone and A. Sofo, Some remarks on the trapezoid rule in numerical integration, Indian J. Pure Appl. Math. 31 (2000)

[15] S.S Dragomir, Tensorial Norm Inequalities For Taylor's Expansions Of Functions Of Selfadjoint Operators In Hilbert Spaces , ResearchGate, November 2022.

[16] S.S Dragomir, An Ostrowski Type Tensorial Norm Inequality For Continuous Functions Of Selfadjoint Operators In Hilbert Spaces, Researchgate, November 2022.

[17] S.S. Dragomir, C.E.M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, in: RGMIA Monographs, Victoria University, 2000.

[18] H. Guo , What Are Tensors Exactly?, World Scientific, June 2021, https://doi.org/10.1142/12388

[19] F. Hezenci, Budak, H. Kara, H. New version of fractional Simpson type inequalities for twice differentiable functions. Adv Differ Equ 2021, 460 (2021). https://doi.org/10.1186/s13662-021-03615-2

[20] A. Kashuri, ; Sahoo, S.K.; Mohammed, P.O.; Al-Sarairah, E.; Hamed, Y.S. Some New Hermite-Hadamard Type Inequalities Pertaining to Fractional Integrals with an Exponential Kernel for Subadditive Functions. Symmetry 2023, 15, 748. https://doi.org/10.3390/sym15030748

[21] A. Koranyi. On some classes of analytic functions of several variables. Trans. Amer. Math. Soc., 101 (1961), 520-554.

[22] D. S. Mitrinović ,Analytic Inequalities, Springer-Verlag, Berlin, 1970.

[23] A. Ostrowski, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, Comment. Math. Hel, 10 (1938), 226-227.

[24] M. E. Ozdemir, Ardic, A. A., Some companions of Ostrowski type inequality for functions whose second derivatives are convex and concave with applications, Arab J Math Sci 21(1) (2015), 53–66, https://doi.org/10.1016/j.ajmsc.2013.12.002

[25] J. Pečarić , Proschan F., Tong Y., Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, INC, United States of America, 1992.

[26] S. K. Sahoo, Y.S. Hamed, Pshtiwan Othman Mohammed, Bibhakar Kodamasingh, Kamsing Nonlaopon, New midpoint type Hermite-Hadamard-Mercer inequalities pertaining to Caputo-Fabrizio fractional operators, Alexandria Engineering Journal, Volume 65, 2023, Pages 689-698, ISSN 1110-0168, https://doi.org/10.1016/j.aej.2022.10.019. (https://www.sciencedirect.com/science/article/pii/S1110016822006755)

[27] M. Z. Sarikaya, E. Set, M.E. Özdemir, On new inequalities of Simpson's type for convex functions, RGMIA Res. Rep. Coll. 13 (2) (2010) Article2.

[28] H.M. Srivastava, Tariq, M.; Mohammed, P.O.; Alrweili, H.; Al-Sarairah, E.; De La Sen, M. On Modified Integral Inequalities for a Generalized Class of Convexity and Applications. Axioms 2023, 12, 162. https://doi.org/10.3390/axioms12020162

[29] V. Stojiljković.; Ramaswamy, R.; Abdelnaby, O.A.A.; Radenović, S. Some Novel Inequalities for LR-(k,h-m)-p Convex Interval Valued Functions by Means of Pseudo Order Relation. Fractal Fract. 2022, 6, 726. https://doi.org/10.3390/fractalfract6120726

[30] V. Stojiljković.; Ramaswamy, R.; Alshammari, F.; Ashour, O.A.; Alghazwani, M.L.H.; Radenović, S. Hermite–Hadamard Type Inequalities Involving (k-p) Fractional Operator for Various Types of Convex Functions. Fractal Fract. 2022, 6, 376. https://doi.org/10.3390/fractalfract6070376

[31] V. Stojiljković.; Ramaswamy, R.; Ashour Abdelnaby, O.A.; Radenović, S. Riemann-Liouville Fractional Inclusions for Convex Functions Using Interval Valued Setting. Mathematics 2022, 10, 3491. https://doi.org/10.3390/math10193491

[32] V. Stojiljković.; Ramaswamy, R.; Abdelnaby, O.A.A.; Radenović, S. Some Refinements of the Tensorial Inequalities in Hilbert Spaces. Symmetry 2023, 15, 925. https://doi.org/10.3390/sym15040925

[33] V. Stojiljković.; Hermite–Hadamard–type fractional–integral inequalities for (p, h)-convex fuzzy-interval-valued mappings, Electron. J. Math. 5 (2023) 18–28, DOI: 10.47443/ejm.2023.004

[34] V. Stojiljkovic, (2023). Twice Differentiable Ostrowski Type Tensorial Norm Inequality for Continuous Functions of Selfadjoint Operators in Hilbert Spaces. European Journal of Pure and Applied Mathematics, 16(3), 1421–1433. https://doi.org/10.29020/nybg.ejpam.v16i3.4843

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