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A REVISIT OF THE MODIFIED CELEBIOGLU-CUADRAS COPULA

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Abstract. In recent years, efforts have been made to improve the scope of modeling the dependence of well-known copulas by modifying their mathematical structure. This was the case, among others, of the so-called Celebioglu-Cuadras copula. In this article, we make contributions to this subject by (i) significantly improving an existing result from the literature on the admissible values for a modified version of the Celebioglu-Cuadras copula and (ii) studying a generalization of this modified copula using an additional setting shape parameter. The characteristics of the introduced copulas are discussed, including the shapes of the copula-related functions, various symmetry and dependence structure types, copula inequalities, diverse correlation measures, and bivariate distribution generation. In particular, we highlight the fact that they are ideal for modeling a wide variety of negative-type dependencies and offer an interesting alternative to the Celebioglu-Cuadras and Gumbel-Barnett copulas. Several graphics are produced, and digital work is carried out as support.

1. Introduction

Since the work in [1, 2], copulas have proven to be effective mathematical tools that allow for the modeling of complex dependence structures between random variables. In the past, dependence structures were often assumed to be linear, which limited the types of relationships that could be modeled. Copulas, however, allow for a more flexible approach. They enable researchers to capture more nuanced dependence structures, better understand the relationships between the random variables, and more accurately model their behavior. Copulas are used in a variety of fields, including finance, environmental science, and engineering, among others. The majority of the theoretical and practical information on copulas can be found

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in [3, 4, 5, 6]; modern applications are given in [7, 8, 9, 10]; and current advances include [11, 12, 13, 14, 15, 16, 17, 18, 19].

Recently, the author in [20] considers a three-parameter bivariate copula of the following general form:

$$C_{op}(x, y; a, b, c) = xy \exp\left[-a(1 - x^b)(1 - y^c)\right], \quad (x, y) \in \Xi, \tag{1}$$

where $\Xi = [0, 1]^2$, and a, b and c are real numbers that aim to flexibilize the exponential structure. The optimal ranges of values for a, b and c that make $C_{op}(x, y; a, b, c)$ a valid copula are a mathematical challenge. Beyond this challenge, the motivations for considering this copula and the limitations of the existing results are described below.

- By taking $b = c = 1$ and $a \in [-1, 1]$, $C_{op}(x, y; a, b, c)$ corresponds to the Celebioglu-Cuadras (CC) copula established in [21, 22]. This copula has numerous attractive properties, including the cover of the independence copula, simple copula-related functions, modulable functionalities thanks to a, and various degrees of negative and positive-type dependence structures (see [23], [24], [25], [26], [27], and [20]).
- For the tunable cases $b > 0$, $c > 0$ and $a \in \mathbb{R}$ but with strong interdependence conditions on them, the above properties are considerably enhanced. In particular, the modulation of these parameters allows the copula-related functions to accommodate a wide panel of shapes. This implies greater modeling perspectives for $C_{op}(x, y; a, b, c)$ in comparison to the former CC copula, but at the price of more complexity in the handling of the parameters. These aspects are demonstrated in detail in [20].
- A contribution in [20, Proposition 1] is to show that, under some circumstances, the cases $b < 0$, $c < 0$ and $a \in \mathbb{R}$ can be considered. More precisely, under a strict condition involving a stringent interdependence between a, b and c, $C_{op}(x, y; a, b, c)$ is a valid copula. Furthermore, it can reach an extreme level of negative dependence structure; the rho of Spearman can attain the value -1.

For the last point, however, there is room for improvements and new discoveries. This article provides contributions in this direction. More precisely, in the first part, we revisit the case $b = c = -1$ corresponding to a copula of the following form:

$$C_{op}(x, y; a) = xy \exp\left[-a\frac{(1 - x)(1 - y)}{xy}\right], \quad (x, y) \in \Xi. \tag{2}$$

For this copula, if we apply the strict condition in [20, Proposition 1], we find $a = 1$ only. We prove that this pointwise value condition can be significantly improved; it can be replaced by $a \in [0, 1]$. In particular, this shows that it covers a plethora of intermediate cases, starting with the independence copula obtained by taking $a = 0$. The proof is based on appropriate differentiation techniques, factoring methods, and polynomial inequalities.

In the second part, we go beyond the existing literature by proposing an extension of the copula in Equation (2). This extension consists in adding a shape parameter b to the denominator term. More precisely, we consider

$$C_{op}(x, y; a, b) = xy \exp\left[-a\frac{(1 - x)(1 - y)}{x^b y^b}\right], \quad (x, y) \in \Xi. \tag{3}$$

We thus determine the admissible values of a and b making $C_{op}(x, y; a, b)$ a valid copula. Based on our findings, we can take $a \in [0, 1]$ and $b \in (0, 1]$ without interdependence between them. This result is of particular interest for statistical purposes because the parameters can be estimated independently of each other. Appropriate differentiation techniques, factoring methods, and polynomial inequalities serve as the foundation for the proof. To the best of our knowledge, the extended modified CC copula in Equation (3) has never been considered before, and offers an alternative option to the CC, Gumbel-Barnett (GB) and extended CC copulas as described in [20]. In the third part, we illustrate this claim by investigating the shapes of the copula-related functions, various kinds of symmetry, quadrant dependence, copula bounds, tail dependence, correlation properties, and distribution generation operations.

The rest of the article is basically divided into the following sections: Section 2 gives the main theoretical results, providing the detailed proof of the admissible values of the parameters for the copulas in Equations (2) and (3). The properties of these copulas are described in Section 3. A conclusion is made in Section 4.

2. Results

Before presenting the main results of this article, it is worth recalling the precise definition of a bivariate copula. It is important to note that, here, the notion of copula will be understood in the absolutely continuous bivariate (ACB) case.

Definition 2.1. *In the ACB case, a copula is a differentiable function defined on Ξ, say $C_{op}(x, y)$, $(x, y) \in \Xi$, such that the following conditions are fulfilled:*

Boundary (Bo) condition:: *For any $(x, y) \in \Xi$, we have*

$$C_{op}(x, 0) = 0, \quad C_{op}(0, y) = 0, \quad C_{op}(x, 1) = x, \quad C_{op}(1, y) = y.$$

Positive derivative (PoDe) condition:: *For any $(x, y) \in \Xi$, we have*

$$\partial_{x,y} C_{op}(x, y) = \frac{\partial^2}{\partial x \partial y} C_{op}(x, y) \geq 0.$$

The Bo condition is often easy to demonstrate, while the PoDe condition generally requires more or less difficult developments. We are now able to prove the main results of the article, starting with the following proposition dealing with the copula in Equation (2).

Propsition 1. *The bivariate function defined by*

$$C_{op}(x, y; a) = xy \exp\left[-a\frac{(1-x)(1-y)}{xy}\right], \quad (x, y) \in \Xi, \tag{4}$$

is a valid copula for $a \in [0, 1]$.

Proof. The proof is based on checking the conditions of Definition 2.1.

Proof of the Bo condition:: For any $(x, y) \in \Xi$ and $a \in [0, 1]$, we have

$$\lim_{y \to 0} -a\frac{(1-x)(1-y)}{xy} = -a\left(\frac{1}{x} - 1\right) \lim_{y \to 0}\left(\frac{1}{y} - 1\right) = -\infty,$$

implying that

$$C_{op}(x, 0; a) = \lim_{y \to 0} xy \exp\left[-a\frac{(1-x)(1-y)}{xy}\right] = 0.$$

In a similar way, we obtain $C_{op}(0, y; a) = 0$. On the other hand, more directly, we have

$$C_{op}(x, 1; a) = x \times 1 \times \exp\left[-a\frac{(1-x)(1-1)}{x \times 1}\right] = x \times \exp(0) = x.$$

We also obtain $C_{op}(1, y; a) = y$. The Bo condition is satisfied.

Proof of the PoDe condition:: For any $(x, y) \in \Xi$, after some differential and algebraic manipulations, we establish

$$\partial_{x,y} C_{op}(x, y; a) = \frac{1}{x^2y^2} \exp\left[-a\frac{(1-x)(1-y)}{xy}\right] \left[a^2(1-x)(1-y) - axy(x+y-1) + x^2y^2\right]. \quad (5)$$

For any $(x, y) \in \Xi$, it is clear that the two first main terms are positive, i.e., $1/(x^2y^2)$ and the exponential term. Moreover, we have $a^2(1-x)(1-y) \geq 0$ and, for $a \in [0, 1]$, we find

$$\begin{aligned} &-axy(x+y-1) + x^2y^2 = -axy(x+y-1-xy) + (1-a)x^2y^2 \\ &= axy(1-x)(1-y) + (1-a)x^2y^2 \geq 0. \end{aligned}$$

As a result, the term in square brackets in Equation (5) is non-negative, implying that

$$\partial_{x,y} C_{op}(x, y; a) \geq 0.$$

The PoDe condition is fulfilled.

The desired result is demonstrated. □

For the purposes of this study, the copula described in Equation (4) is called the modified CC (MCC) copula. This is a special case of the one proposed in [20] but with a strong contribution: the proof of its validity for $a \in [0, 1]$, and not just $a = 1$. It is worth noting that the MCC and CC copulas are related in the following intriguing way: for any $(x, y) \in \Xi$ and $a \in [0, 1]$, we have

$$C_{op(MCC)}(x, y; a) = \left[C_{op(CC)}(x, y; a)\right]^{1/(xy)},$$

where $C_{op(MCC)}(x, y; a)$ is the MCC copula as defined in Equation (4) and $C_{op(CC)}(x, y; a)$ is the CC copula defined by $C_{op(CC)}(x, y; a) = xy \exp[-a(1-x)(1-y)]$. However, the CC copula allows negative values for a, i.e., from $a \in [-1, 1]$, whereas the MCC copula does not.

In order to underline the interest of the MCC copula, we can mention the scheduling properties presented below involving some major copulas from the literature.

Lemma 2.1. *For any $(x, y) \in \Xi$ and $a \in [0, 1]$, the following scheduling property holds:*

$$C_{op(MCC)}(x, y; a) \leq C_{op(GB)}(x, y; a) \leq C_{op(CC)}(x, y; a) \ (\leq C_{op(Ind)}(x, y)),$$

where $C_{op(MCC)}(x, y; a)$ is the MCC copula defined in Equation (4), $C_{op(GB)}(x, y; a)$ is the GB copula defined by $C_{op(GB)}(x, y; a) = xy \exp[-a \log(x) \log(y)]$, $C_{op(CC)}(x, y; a)$ is the CC copula defined by $C_{op(CC)}(x, y; a) = xy \exp[-a(1-x)(1-y)]$, and $C_{op(Ind)}(x, y)$ is the indepence copula, i.e., $C_{op(Ind)}(x, y) = xy$.

Proof. For any $x \in [0,1]$, the following logarithmic inequalities are well-known:

$$\frac{x-1}{x} \leq \log(x) \leq x-1 \leq 0,$$

with the limit case for $x=0$. As a result, for any $(x,y) \in \Xi$, we have

$$-\frac{(1-x)(1-y)}{xy} \leq -\log(x)\log(y) \leq -(1-x)(1-y) \leq 0$$

and, for $a \in [0,1]$, since the exponential function is increasing, we have

$$\exp\left[-a\frac{(1-x)(1-y)}{xy}\right] \leq \exp\left[-a\log(x)\log(y)\right] \leq \exp\left[-(1-x)(1-y)\right] \leq 1.$$

By multiplying all the terms by $xy \in [0,1]$, we get the stated scheduling property. The lemma is proved. □

In the sense of Lemma 2.1, the MCC copula can be considered an alternative to the GB and CC copulas. Moreover, it has other interesting properties, which will be studied later.

Let us now deal with a more general version of the MCC copula described in the result below.

Propsition 2. *The bivariate function defined by*

$$C_{op}(x,y;a,b) = xy\exp\left[-a\frac{(1-x)(1-y)}{x^b y^b}\right], \quad (x,y) \in \Xi, \tag{6}$$

is a valid copula for $a \in [0,1]$ and $b \in (0,1]$. The case $b=0$ can be included with the condition $a \in [0,1]$, but this condition on a is not optimal anymore.

Proof. The proof is based on checking the conditions of Definition 2.1. It is, however, more technical than the proof of Proposition 1 because of the presence of b.

Proof of the Bo condition:: For any $(x,y) \in \Xi$, $a \in [0,1]$ and $b \in (0,1]$, we have

$$\lim_{y\to 0} -a\frac{(1-x)(1-y)}{x^b y^b} = -a\frac{1-x}{x^b}\lim_{y\to 0}\frac{1-y}{y^b} = -\infty,$$

implying that

$$C_{op}(x,0;a,b) = \lim_{y\to 0} xy\exp\left[-a\frac{(1-x)(1-y)}{x^b y^b}\right] = 0.$$

In a similar way, we obtain $C_{op}(0,y;a,b) = 0$. On the other hand, we have

$$C_{op}(x,1;a,b) = x \times 1 \times \exp\left[-a\frac{(1-x)(1-1)}{x^b \times 1^b}\right] = x \times \exp(0) = x.$$

We also arrive at $C_{op}(1,y;a,b) = y$. The Bo condition is satisfied.

Proof of the PoDe condition:: For any $(x, y) \in \Xi$, after some differential and algebraic manipulations, we have

$$\partial_{x,y} C_{op}(x, y; a, b) = x^{-2b} y^{-2b} \exp\left[-a\frac{(1-x)(1-y)}{x^b y^b}\right] \times$$
$$\Big\{a^2(1-x)(1-y)[b+(1-b)x][b+(1-b)y] - ax^b y^b \{(3-b)y[b+(1-b)x]$$
$$+ b[b(1-x)+3x-2] - x - y\} + x^{2b} y^{2b}\Big\}$$
$$= x^{-2b} y^{-2b} \exp\left[-a\frac{(1-x)(1-y)}{x^b y^b}\right] \times$$
$$\Big\{a^2(1-x)(1-y)[b+(1-b)x][b+(1-b)y] + ax^b y^b P(x, y; a) + Q(x, y; a, b)\Big\}, \tag{7}$$

where

$$P(x, y; a) = (b-3)y[b+(1-b)x] - b[b(1-x)+3x-2] + x + y + xy$$

and

$$Q(x, y; a, b) = -ax^{b+1} y^{b+1} + x^{2b} y^{2b}.$$

For any $(x, y) \in \Xi$, it is clear that the two first main terms in Equation (7) are positive, i.e., $x^{-2b} y^{-2b}$ and the exponential term. Moreover, we have $a^2(1-x)(1-y) \geq 0$, and, since $b \in (0, 1]$, we have $b + (1-b)x \geq 0$ and $b + (1-b)y \geq 0$. It is also clear that $ax^b y^b \geq 0$. Therefore, to conclude the proof, it is enough to prove that $P(x, y; a) \geq 0$ and $Q(x, y; a, b) \geq 0$.

Proof that $P(x, y; a) \geq 0$:: This is mainly a factoring game. After some algebraic manipulations, the following simplified expression is obtained, step by step:

$$\begin{aligned}
P(x, y; a) &= -b^2 xy + b^2 x + b^2 y - b^2 + 4bxy - 3bx - 3by + 2b - 2xy + x + y \\
&= (4b - b^2 - 2)xy + (b^2 - 3b + 1)x + (b^2 - 3b + 1)y + 2b - b^2 \\
&= (4b - 2b^2 - 2)xy + (1-b)^2 x + (1-b)^2 y + 2b - b^2 - bx - by + b^2 xy \\
&= -2(1-b)^2 xy + (1-b)^2 x + (1-b)^2 y - (1-b)^2 + (1-bx)(1-by) \\
&= (1-b)^2 x(1-y) + (1-b)^2 y(1-x) + (1-bx)(1-by) - (1-b)^2.
\end{aligned}$$

For any $(x, y) \in \Xi$, it is clear that $(1-b)^2 x(1-y) \geq 0$ and $(1-b)^2 y(1-x) \geq 0$. Since $b \in (0, 1]$, we have $1 - bx \geq 1 - b \geq 0$ and $1 - by \geq 1 - b \geq 0$, implying that $(1-bx)(1-by) \geq (1-b)^2$, so $(1-bx)(1-by) - (1-b)^2 \geq 0$. Therefore, we establish that $P(x, y; a) \geq 0$.

Proof that $Q(x, y; a, b) \geq 0$:: Let us notice that, for any $(x, y) \in \Xi$, since $b \in (0, 1]$, we have $x^{b+1} \leq x^{2b}$ and $y^{b+1} \leq y^{2b}$, implying that $-x^{b+1} y^{b+1} \geq -x^{2b} y^{2b}$. Therefore, since $a \in [0, 1]$, we have

$$Q(x, y; a, b) = -ax^{b+1} y^{b+1} + x^{2b} y^{2b} \geq (1-a)x^{2b} y^{2b} \geq 0.$$

As a result, we have

$$\partial_{x,y} C_{op}(x, y; a, b) \geq 0.$$

The PoDe condition is fulfilled.

The desired result is demonstrated. □

For the purposes of this study, the copula described in Equation (6) is called the extended MCC (EMCC) copula. By taking $b = 1$, it corresponds to the MCC copula. It is not covered in the copulas studied in [20]. When $b \to 0$, it corresponds to the CC copula. Like the MCC copula but with a higher degree of flexibility thanks to b, the EMCC copula can be considered an alternative to the GB and CC copulas. Moreover, it has other interesting properties, which will be studied in the next section.

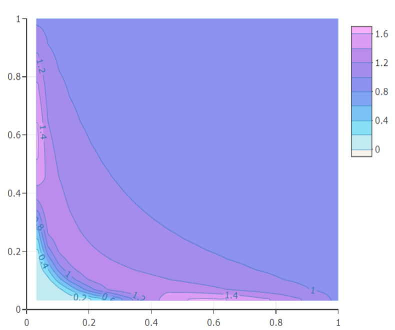
3. Properties

Some interesting properties of the proposed copulas are described below.

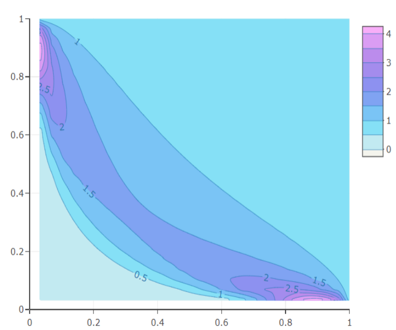
3.1. **For the MCC copula.** Let us begin by presenting MCC copula-related functions. We recall that $a \in [0, 1]$. Based on Equation (5), the MCC copula density is given by

$$\begin{aligned}
&c_{op}(x, y; a) = \partial_{x,y} C_{op}(x, y; a) \\
&= \frac{1}{x^2y^2} \exp\left[-a\frac{(1-x)(1-y)}{xy}\right] \left[a^2(1-x)(1-y) - axy(x+y-1) + x^2y^2\right], \\
&(x, y) \in \Xi.
\end{aligned}$$

The more versatile the shapes of a copula density, the more it can accommodate different dependence structures. In order to visualize the shape behavior of the MCC copula density, Figure 1 plots the corresponding contours and intensities for $a = 0.05$ and $a = 0.3$, and Figure 2 does the same for $a = 0.6$ and $a = 1$ (the values of a are arbitrarily chosen such that $a \in [0, 1]$).

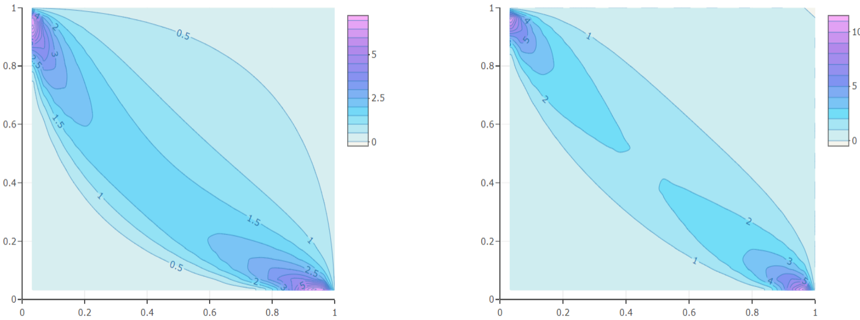


(a)



(b)

Figure 1. Plots of the MCC copula density for (a) $a = 0.05$ and (b) $a = 0.3$.



(a) (b)

FIGURE 2. Plots of the MCC copula density for (a) $a = 0.6$ and (b) $a = 1$.

From these figures, we see the flexibility of the MCC copula density with only one tuning parameter. Diverse skewness shapes and intense zones are observed, mainly in the left portion of the unit square.

As another important MCC copula-related function, we may present the MCC survival copula as

$$\begin{aligned}\hat{C}_{op}(x, y; a) &= x + y - 1 + C_{op}(1 - x, 1 - y; a)\\ &= x + y - 1 + (1 - x)(1 - y) \exp\left[-a\frac{xy}{(1 - x)(1 - y)}\right], \quad (x, y) \in \Xi.\end{aligned} \tag{8}$$

Still under the condition $a \in [0, 1]$, it offers a new one-parameter copula.

Clearly, the MCC copula is diagonally symmetric since $C_{op}(x, y; a) = C_{op}(y, x; a)$ for any $(x, y) \in \Xi$. By taking $a = 0.15$, after calculations, we find that

$$C_{op}(0.2, C_{op}(0.5, 0.8; a); a) = 0.02958326$$

and

$$C_{op}(C_{op}(0.2, 0.5; a), 0.8; a) = 0.02301698.$$

Since these values differ, the MCC copula is not associative, and consequently, it is not Archimedean. The MCC copula is not radially symmetric because, based on Equation (8), it is easy to find a $(x, y) \in \Xi$ such that $\hat{C}_{op}(x, y; a) \neq C_{op}(x, y; a)$. The MCC copula satisfies the following product geometric property: for any $(x, y) \in \Xi$, $\beta \in [0, 1]$, $a_1 \in [0, 1]$ and $a_2 \in [0, 1]$, we have

$$C_{op}(x, y; a_1)^{\beta} C_{op}(x, y; a_2)^{1-\beta} = C_{op}\left(x, y; \beta a_1 + (1 - \beta) a_2\right).$$

Hence, the MCC copula benefits from a kind of product geometric stability.

The MCC copula is a decreasing function with respect to a, implying that it is negatively quadrant dependent: for any $(x, y) \in \Xi$ and $a \in [0, 1]$, we have $C_{op}(x, y; a) \leq xy$.

As one of main ingredients of the copula theory, the Fréchet-Hoeffding bounds are satisfied. Hence, for any $(x, y) \in \Xi$ and $a \in [0, 1]$, we have $\max(x + y - 1, 0) \leq C_{op}(x, y; a) \leq \min(x, y)$, i.e.,

$$\max(x + y - 1, 0) \leq xy \exp\left[-a\frac{(1 - x)(1 - y)}{xy}\right] \leq \min(x, y).$$

The lower and upper tail dependence parameters are computed using typical limit methods as follows:

$$\sigma_{Low} = \lim_{x\to 0} \frac{C_{op}(x,x;a)}{x} = \lim_{x\to 0} x \exp\left[-a\frac{(1-x)^2}{x^2}\right] = 0$$

and

$$\sigma_{Upp} = \lim_{x\to 1} \frac{1-2x+C_{op}(x,x;a)}{1-x} = \lim_{x\to 1} \frac{1-2x+x^2\exp\left[-a(1-x)^2/x^2\right]}{1-x} = 0,$$

respectively. Hence, $\sigma_{Low} = \sigma_{Upp} = 0$ and the MCC copula is free of tail dependence. The medial correlation coefficient of the MCC copula is expressed as follows:

$$M_{ed} = 4C_{op}(0.5, 0.5; a) - 1 = \exp(-a) - 1.$$

It is obvious that this coefficient is non-positive, highlighting the negative dependence feature of the MCC copula.

The basic definition of the rho of Spearman related to the MCC copula is

$$\begin{aligned}\rho_{Spear} &= 12\int_{\Xi} [C_{op}(x,y;a) - xy]\, dxdy \\ &= 12\int_{\Xi} xy\left\{\exp\left[-a\frac{(1-x)(1-y)}{xy}\right] - 1\right\} dxdy.\end{aligned}$$

This correlation measure cannot be expressed in a simple way, but a numerical study is possible. With the use of the software R and the package `pracma` along with the function `integral2` in particular (see [28]), Table 1 presents its numerical values (four decimals are retained).

Table 1. Numerical study on ρ_{Spear} for $a = 0$, $a = 0.1$, ..., $a = 1$

a	0.0	0.1	0.2	0.3	0.4	0.5
ρ_{Spear}	0	−0.1875	−0.3124	−0.4115	−0.4949	−0.5672

a	0.6	0.7	0.8	0.9	1.0
ρ_{Spear}	−0.6312	−0.6887	−0.741	−0.789	−0.8333

This table illustrates the wide range of amplitudes that the rho of Spearman of the MCC copula can have, which is shown here to be between −0.8333 and 0. The MCC copula is therefore perfect for modeling different types of negative dependence structure. Compared to the GB and CC copulas, it provides a wider perspective on this aspect.

Naturally, the MCC copula can be used as a generator of bivariate distributions. Indeed, for any cumulative distribution functions of absolutely continuous distributions, say $F(x)$ and $G(x)$, the following bivariate function defines a new bivariate cumulative distribution function:

$$H(x,y;\xi) = C_{op}(F(x), G(y); a) = F(x)G(y)\exp\left[-a\frac{[1-F(x)][1-G(y)]}{F(x)G(y)}\right],$$
$$(x,y)\in\mathbb{R}^2,$$

where ξ symbolizes the vector of all the distributional parameters, including a. As a result, this function might be used to create an infinite number of new bivariate distributions (see, for instance, [29] for motivated choices of baseline lifetime distributions).

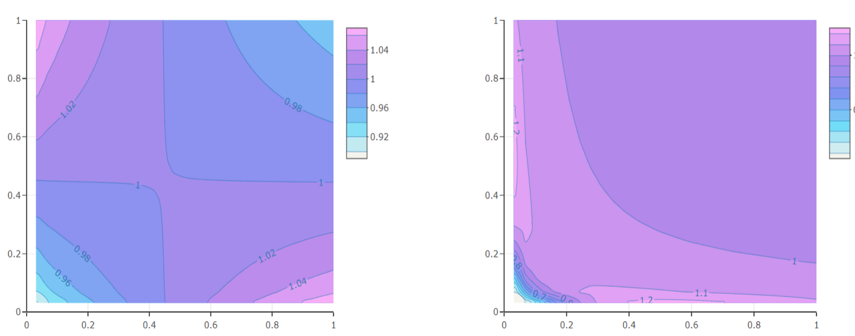
3.2. **For the EMCC copula.** Logically, the EMCC copula shares most of the properties of the MCC copula, but with more flexibility on some of them. We focus on these ones below.

Let us begin by presenting EMCC copula-related functions. We recall that $a \in [0,1]$ and $b \in (0,1]$. Based on Equation (7), the EMCC copula density is given by

$$\begin{aligned}
&c_{op}(x,y;a,b) = \partial_{x,y} C_{op}(x,y;a,b)\\
&= x^{-2b} y^{-2b} \exp\left[-a\frac{(1-x)(1-y)}{x^b y^b}\right] \times\\
&\Big\{a^2(1-x)(1-y)[b+(1-b)x][b+(1-b)y] - a x^b y^b \{(3-b)y[b+(1-b)x]\\
&+ b[b(1-x)+3x-2] - x - y\} + x^{2b} y^{2b}\Big\}, \quad (x,y) \in \Xi.
\end{aligned}$$

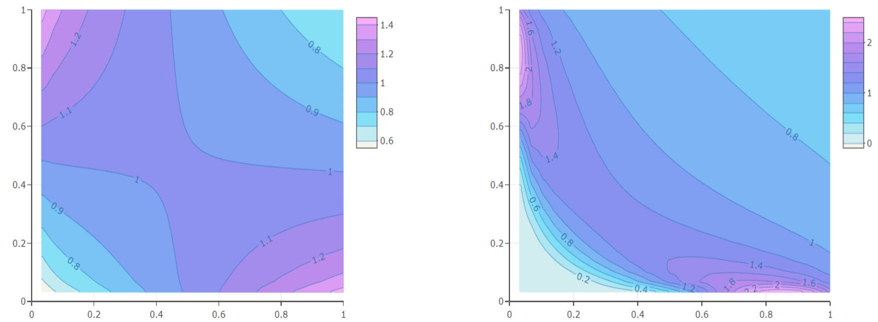
Clearly, the effect of b is real, and makes this copula density more flexible than that of the MCC copula. In order to highlight that, we take the same values of a chosen for Figure 1, and we take two values of b, here $b = 0.2$ and $b = 0.8$ to fix the idea.

Hence, Figure 3 displays the EMCC copula density for $a = 0.05$, and $b = 0.2$ and $b = 0.8$, Figure 4 does the same for $a = 0.3$, and $b = 0.2$ and $b = 0.8$, Figure 5 does the same for $a = 0.6$, and $b = 0.2$ and $b = 0.8$, and, finally, Figure 6 does the same for $a = 1$, and $b = 0.2$ and $b = 0.8$.



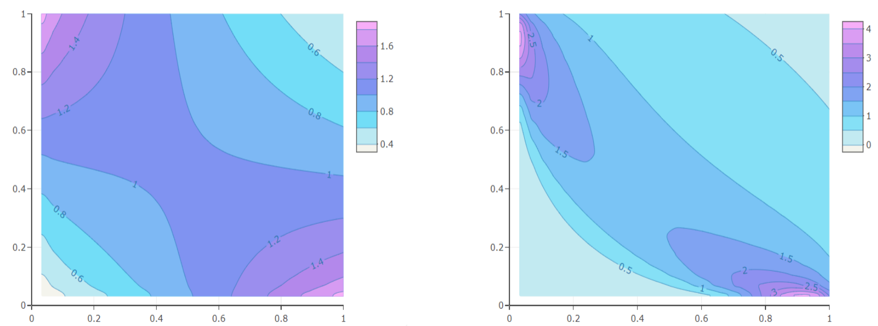
(a) (b)

FIGURE 3. Plots of the EMCC copula density for $a = 0.05$, and (a) $b = 0.2$ and (b) $b = 0.8$.



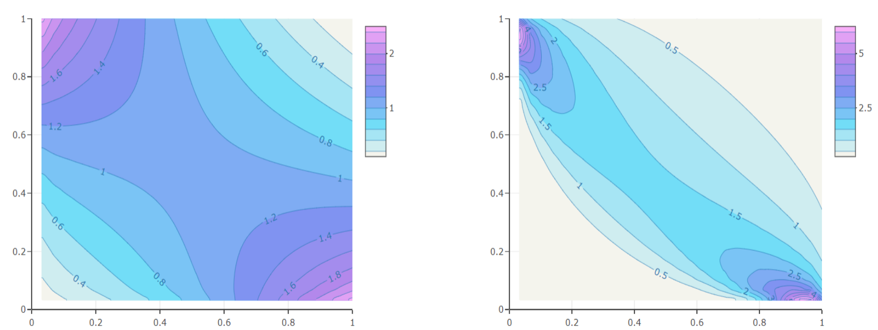
(a) (b)

FIGURE 4. Plots of the EMCC copula density for $a = 0.3$, and (a) $b = 0.2$ and (b) $b = 0.8$.



(a) (b)

FIGURE 5. Plots of the EMCC copula density for $a = 0.6$, and (a) $b = 0.2$ and (b) $b = 0.8$.



(a) (b)

FIGURE 6. Plots of the EMCC copula density for $a = 1$, and (a) $b = 0.2$ and (b) $b = 0.8$.

Based on these figures, the effects of a and b are clearly important, and demonstrate the interest of the EMCC copula in the negative dependence modeling. It is logically more flexible than the MCC copula in this regard.

As another important EMCC copula-related function, we may present the EMCC survival copula as

$$\begin{aligned}\hat{C}_{op}(x,y;a,b) &= x+y-1+C_{op}(1-x,1-y;a,b)\\ &= x+y-1+(1-x)(1-y)\exp\left[-a\frac{xy}{(1-x)^b(1-y)^b}\right], \quad (x,y)\in\Xi.\end{aligned}$$

Still under the conditions $a \in [0,1]$ and $b \in (0,1]$, it offers a new two-parameter copula.

Among the common properties between the MCC and EMCC copulas are the diagonal symmetry property, the non-radial symmetry property, the non-Archimedean property, the product geometric property, the negative quadrant dependence property, the Fréchet-Hoeffding bounds property, the no-tail dependence property, and the generation of bivariate distributions property.

The medial correlation coefficient of the EMCC copula is expressed as follows:

$$M_{ed} = 4C_{op}(0.5,0.5;a,b)-1 = \exp\left[-a2^{2(b-1)}\right]-1.$$

As expected in view of the medial correlation coefficient of the MCC copula, that of the EMCC copula is also non-positive.

The basic definition of the rho of Spearman related with the EMCC copula is

$$\begin{aligned}\rho_{Spear} &= 12\int_{\Xi}\left[C_{op}(x,y;a,b)-xy\right]dxdy\\ &= 12\int_{\Xi} xy\left\{\exp\left[-a\frac{(1-x)(1-y)}{x^by^b}\right]-1\right\}dxdy.\end{aligned}$$

As with the MCC copula, it cannot be expressed in a simple way, but a numerical study is possible. Since $C_{op}(x,y;a,b)$ is a decreasing function with respect to b, it is greater than the rho of Spearman of the MCC copula and still negative, and thus, is included into the set $[-0.8333;0]$. However, from a modeling perspective, in view of the diversity in shapes illustrated in Figures 3, 4, 5, and 6, the EMCC copula is clearly the more interesting option.

4. Conclusion

In this article, we have revisited and extended a modified version of the CC copula, significantly improving an existing result on its mathematical validity. A condition on the involved parameter is drastically relaxed. Then, based on this result, we proposed an extension of it with the addition of a shape parameter. We thus introduced a new two-parameter modified CC copula, which reveals itself to be ideal for modeling diverse kinds of negative dependence structures. A wide panel of graphics illustrated the versatility of the corresponding copula density, and a numerical study on the related rho of Spearman supports the attractive negative dependence feature. Clearly, the proposed copulas offer an alternative to the former CC and GB copulas and are recommended to be used for the same practical area when bivariate data needs to be analyzed (like finance, insurance, biology, etc.).

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