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ON THE NEW CLASS OF THE NONLINEAR RATIONAL DIFFERENCE EQUATIONS

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Abstract. In this paper we study the asymptotic behavior of the solution of the new class of the nonlinear rational Difference Equations. Namely, we consider the stability, boundedness, and periodicity of the solution. Moreover we give the periodic character of solutions of these equations, which is not familiar. We do not know a similar feature for other class of the nonlinear rational Difference Equations. We also give some interesting counter examples in order to verify our results.

1. Introduction

Difference Equations describe real life situations in probability theory, statistical problems, queuing theory, electrical network, combinatorial analysis, genetics in biology, sociology, psychology, economics, etc, [16, 17]. So our study of the Difference Equations is so interesting. There has been many work about the global asymptotic of solutions of rational difference equations, [18] - [23] and references therein. It is very important to investigate the asymptotic behavior of solutions of a system of nonlinear difference equations and to discuss the boundedness, periodicity and stability (local and global) of their equilibrium points.

Cinar [3] - [5] gave the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}},\ x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}},\ x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Cinar et al. [6] studied the solutions and attractivity of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Elabbasy et al. [9] investigated the asymptotic behavior of the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

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Karatas et al. [18] acquired the solution of the difference equation

$$x_{n+1} = \frac{ax_{n-(2k+2)}}{-a + \prod_{i=0}^{2k+2} x_{n-i}}.$$

For further studying of the asymptotic behaviour of solutions of rational difference equations, one can refer to [1] - [15] and references therein.

In this paper, we are concerned with analytical investigation of the solution of the following recursive sequence

$$x_{n+1} = ax_{n-k} + \frac{bx_{n-k}}{\alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i\neq j}^{k} x_{n-i}}, \tag{1}$$

where the initial conditions $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0$ are arbitrary positive real numbers and $a, b, \alpha, \beta_j, j = 0, 1, ...k$ are positive constants.

The rest of the paper is organized as follows: In Section 2 we study the stability behaviour of the solution for equation (1) and give an interesting counter example to support our analysis. In Section 3, we prove that the positive solution of equation (1) is bounded. In Section 4 we study the periodic behaviour of the solution for the equation (1). We also give two counter examples to show how our model is so rich.

2. The stability of solutions

In this section we study the local stability character of the solutions of equation (1). The positive equilibrium points of equation (1) are given by

$$\overline{x} = 0, \ \overline{x} = \left[\frac{b-(1-a)\alpha}{(1-a)B}\right]^{\frac{1}{k}}, \text{ where } B = \sum_{j=0}^{k} \beta_j \, .$$

Now, we define the continuous function $f : (0, \infty)^3 \to (0, \infty)$, such that

$$f(u_0, u_1, ..., u_k) = au_k + \frac{bu_k}{\alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i\neq j}^{k} u_i},$$

Therefore, it follows that

$$\frac{\partial f}{\partial u_m} = \frac{-bu_k \sum_{j=0}^{k} \beta_j \prod_{i=0, i\neq j,m}^{k} u_i}{\alpha + \sum_{j=0}^{k} \beta_j \prod_{i=0, i\neq j}^{k} u_i}, m = 0, 1, 2, ..., k-1\, . \tag{2}$$

and

$$
\begin{aligned}
\frac{\partial f}{\partial u_k} &= a+\frac{b(\alpha+\sum\limits_{j=0}^{k}\beta_j\prod\limits_{i=0,i\neq j}^{k}u_i)-bu_k\sum\limits_{j=0}^{k-1}\beta_j\prod\limits_{i=0,i\neq j}^{k-1}u_i}{\left[\alpha+\sum\limits_{j=0}^{k}\beta_j\prod\limits_{i=0,i\neq j}^{k}u_i\right]^2} \\
&= a+\frac{b\alpha+b\beta_k\prod\limits_{i=0}^{k-1}u_i}{\left[\alpha+\sum\limits_{j=0}^{k}\beta_j\prod\limits_{i=0,i\neq j}^{k}u_i\right]^2} \qquad (3)
\end{aligned}
$$

$$(4)$$

Theorem 1 If $a+\frac{b}{\alpha}<1$, then the equilibrium point $\overline{x}=0$ of equation (1) is locally stable.

Proof. The linearized equation of (1) about the equilibrium point $\overline{x}$ is the linear difference equation

$$z_{n+1}=\sum_{i=0}^{k}\frac{\partial f\left(\overline{x},\overline{x},...,\overline{x}\right)}{\partial u_i}z_{n-i}\,.$$

From equations (2), (3) we have

$$\frac{\partial f}{\partial u_m}\left(\overline{x},\overline{x},...,\overline{x}\right)=0,$$

for $m=0,1,...,k-1$ and

$$\frac{\partial f}{\partial u_k}\left(\overline{x},\overline{x},...,\overline{x}\right)=a+\frac{b}{\alpha}\,.$$

It is follows by [19, Theorem 1] that, equation (1) is locally stable at $\overline{x}=0$ if

$$a+\frac{b}{\alpha}<1\,.$$

Hence, the proof is completed. □

The following counter example shows the stability of solution of equation (1) at $\overline{x}=0$.

Example 1 We consider the following initial data: $x_{-2}=1.01, x_{-1}=0.99, x_0$ for equation (1) with $k=1$, Figure 1 .

Theorem 2 If

$$|T|\left((k-1)\,B+\beta_k\right)+\left|abB+T\beta_k+(1-a)^2\,\alpha B\right|<bB.$$

where $T=\left(b-(1-a)\,\alpha\right)(1-a)$, then the equilibrium point $\overline{x}$ of equation (1) is locally stable.

Proof. From equations (2), (3) we have

$$\frac{\partial f}{\partial u_m}\left(\overline{x},\overline{x},...,\overline{x}\right)=-\frac{b-(1-a)\,\alpha}{bB}\,(1-a)\,B^{[m]}=-h_m,$$

for $m=0,1,...,k-1$, where $B^{[m]}=B-\beta_m$ and

$$\frac{\partial f}{\partial u_k}\left(\overline{x},\overline{x},...,\overline{x}\right)=a+\frac{b\beta_k+(1-a)\,\alpha B^{[k]}}{bB}\,(1-a)=-h_k,$$

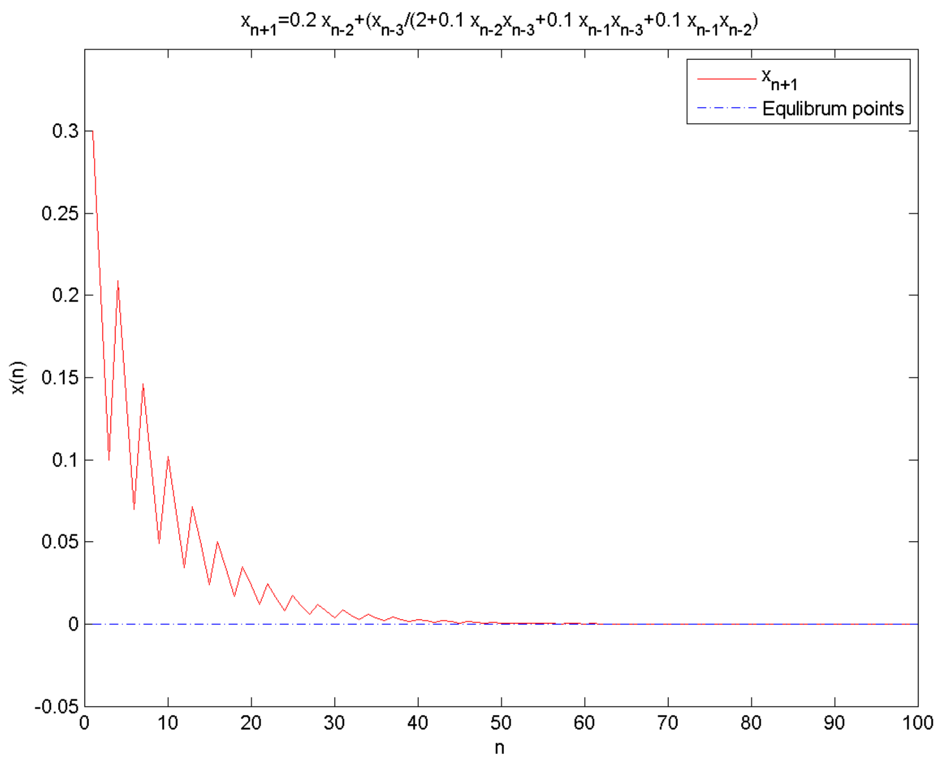

Figure 1. Stability of solution at $\overline{x} = 0$.

Then the linearized equation

$$z_{n+1} + \sum_{i=0}^{k} h_i z_{n-i} = 0 \tag{5}$$

It follows by [19, Theorem 1] that, equation (1) is locally stable at $\overline{x} = \left[\frac{b-(1-a)\alpha}{(1-a)B}\right]^{\frac{1}{k}}$ if

$$\sum_{i=0}^{k} |h_i| < 1.$$

This implies that

$$\left|\frac{b-(1-a)\alpha}{bB}(1-a)\right|\left(\sum_{m=0}^{k-1} B^{[m]}\right) + \left|a + \frac{b\beta_k + (1-a)\alpha B^{[k]}}{bB}(1-a)\right| < 1.$$

and so,

$$|T|\left((k-1)B + \beta_k\right) + \left|abB + T\beta_k + (1-a)^2 \alpha B\right| < bB.$$

Hence, the proof is completed. □

The following counter example shows the stability of solution of equation (1) at $\overline{x} = \left[\frac{b-(1-a)\alpha}{(1-a)B}\right]^{\frac{1}{k}}$.

Example 2 We consider the following initial data: $x_{-2} = 0.1, x_{-1} = 0.2, x_0 = 0.3$ for equation (1) with $k = 2$, Figure 2.

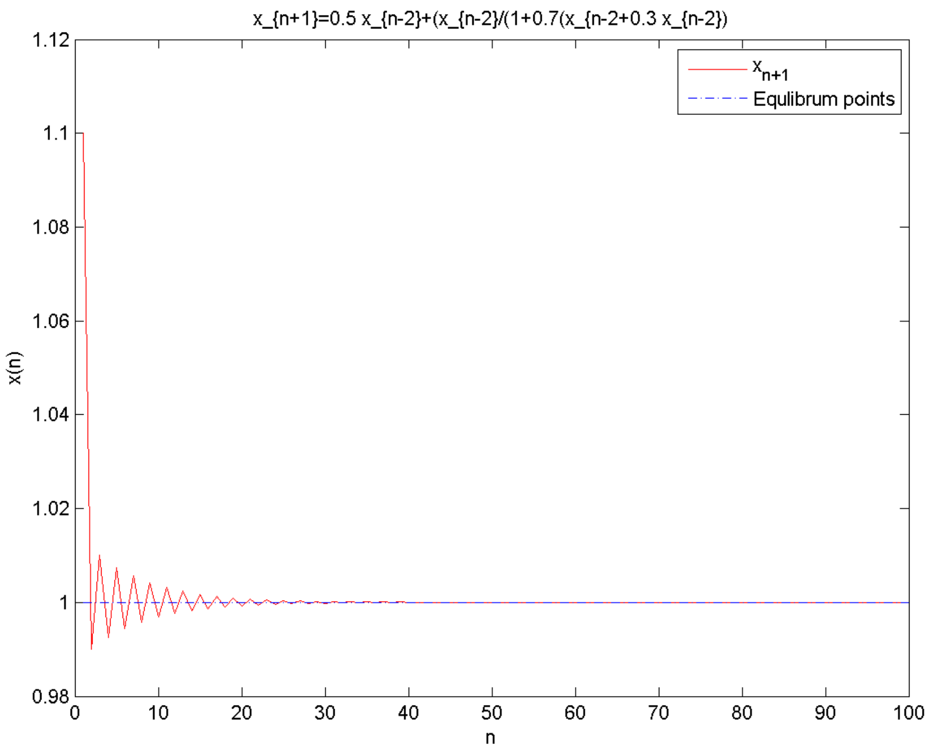

FIGURE 2. Stability of solution at $\overline{x} = \left[\frac{b-(1-a)\alpha}{(1-a)B}\right]^{\frac{1}{k}}$.

3. Boundedness of the solutions

In this section, we investigate the boundedness of the positive solutions of equation (1).

Theorem 3 If $a + \frac{b}{\alpha} \leq 1$ then the solutions of equation (1 are bounded.

Proof. Assume that $\{x_n\}_{n=-k}^{\infty}$ be a solution of equation (1). Then we have

$$\begin{aligned} x_{n+1} &= ax_{n-k} + \frac{bx_{n-k}}{\alpha + \sum\limits_{j=0}^{k} \beta_j \prod\limits_{i=0, i\neq j}^{k} x_{n-i}}, \\ &\leq ax_{n-k} + \frac{bx_{n-k}}{\alpha}, \\ &\leq \left(a + \frac{b}{\alpha}\right) x_{n-k}. \end{aligned}$$

Hence, we have $x_{n+1} \leq x_{n-k}$. Thus we can divided the sequence $\{x_n\}_{n=-k}^{\infty}$ to $k+1$ subsequence bounded above by the initial conditions as follows:

$$\begin{gathered} x_{-k} \geq x_1 \geq x_{k+2} \geq x_{2k+3} \geq ... \\ x_{-k+1} \geq x_2 \geq x_{k+3} \geq x_{2k+4} \geq ... \\ x_{-k+2} \geq x_3 \geq x_{k+4} \geq x_{2k+5} \geq ... \\ . \\ . \\ . \\ x_0 \geq x_{k+1} \geq x_{2k+2} \geq x_{3k+3} \geq ... \end{gathered}$$

Hence we chose $M = \max\{x_{-k}, x_{-k+1}, ...x_0\}$, which leads to $0 \leq x_n \leq M$. Thus, the proof is completed.

□

4. Periodic solutions

In this section we give the periodic behaviour of the solution for the non linear difference equation (1). Moreover we give the periodic character of solutions of these equations of order $k+1$, which is not familiar.

Theorem 4 Assume that $\beta_j = 1$, for $j = 0, 1, 2, ...k$ and at least one of the initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \neq 0$. The equation (1) has period $k+1$ solution if

$$b = (1-a)(\alpha + A),$$

where $A = \sum\limits_{j=0}^{k} \prod\limits_{i=0, i\neq j}^{k} x_{-i}$.

Proof. Suppose that there exists a distinct prime period $k+1$ solutions of equation (1). Thus, we have the following algebraic system of $k+1$ equations

$$x_1 = x_{-k}, x_{-k+1} = x_2, ...x_0 = x_{k+1}\,. \tag{6}$$

Solving the algebraic system (6), we get

$$x_1 = ax_{-k} + \frac{bx_{-k}}{\alpha + \sum\limits_{j=0}^{k} \beta_j \prod\limits_{i=0, i\neq j}^{k} x_{-i}} = x_{-k}, \tag{7}$$

then

$$ax_{-k} + \frac{bx_{-k}}{\alpha + A} = x_{-k}, \tag{8}$$

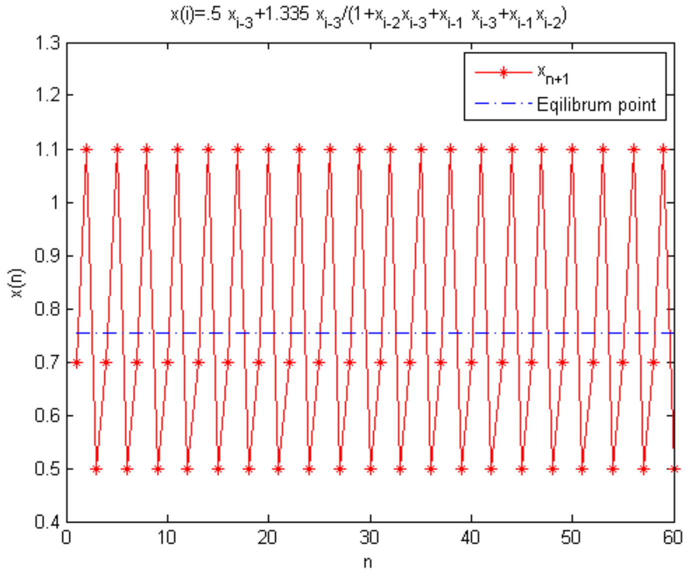

FIGURE 3. Prime period three.

i.e.,

$$[b-(1-a)(\alpha+A)]x_{-k}=0$$

Similarly by the solving the further algebraic equations we get

$$[b-(1-a)(\alpha+A)]x_{-k+1}=0,$$

$$[b-(1-a)(\alpha+A)]x_{-k+2}=0,$$

$$.$$

$$.$$

$$.$$

$$[b-(1-a)(\alpha+A)]x_0=0.$$

Hence, from assumption we get $b=(1-a)(\alpha+A)$. Thus, the proof is completed. □

Example 3 In this example We give two different initial data, one of them with period three and the other with period four:

(1) The first initial data: $x_{-2}=0.5, x_{-1}=1.1, x_0=0.7$ for equation (1) with $k=2$ gives solution of prime period three, Figure 3 .
(2) The second initial data: $x_{-3}=0.5, x_{-2}=1, x_{-1}=0.8, x_0=1.2$ for equation (1) with $k=3$ gives solution of prime period four, Figure 4 .

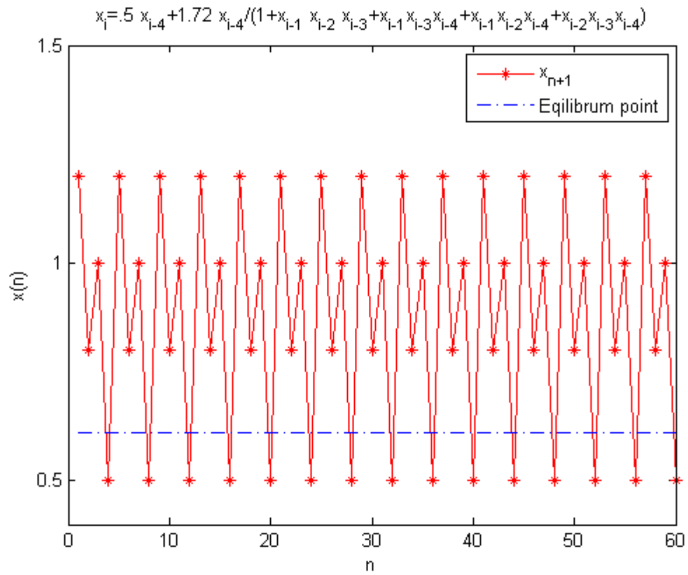

Figure 4. Prime period four.

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