UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING PRODUCT OF DIFFERENCE POLYNOMIALS

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ABSTRACT. In this paper, we deal with distribution of zeros of certain types of difference polynomial and in addition to this we investigate the uniqueness of product of difference polynomials $f^n \prod_{j=1}^d f(z + c_j)^{s_j}$ and $g^n \prod_{j=1}^d g(z + c_j)^{s_j}$ which are sharing a fixed point $z$ and $f, g$ share $\infty$ IM. I obtained some results which extends some recent results of Renukadevi S. Dyavnal and Ashwini M. Hattikal.

1. INTRODUCTION AND MAIN RESULTS

A meromorphic function $f(z)$ means meromorphic in the whole complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [3]. As usual, the abbreviation CM stands for counting multiplicities, while IM means ignoring multiplicities. We use $\rho(f)$ to denote the order of $f(z)$ and $N_p(r, \frac{1}{f - a})$ to denote the counting function of the zeros of $f - a,$ where an $m$-fold zero is counted $m$ times if $m \leq p$ and $p$ times if $m > p.$

A meromorphic function $a$ is called small function with respect to $f$ if $T(r, a) = S(r, f)$ and the order, hyper order of meromorphic function $f$ are defined by

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \quad \rho_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$ 

In 2010, J.F.Xu, F.Lu and H.X.Yi obtained the following result on meromorphic function sharing a fixed point.

**Theorem A.** [7] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let $n, k$ be two positive integers with $n \geq 3k + 10.$ If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share $z$ CM, $f$ and $g$ share $\infty$ IM, then either $f(z) = c_1 e^{cz^2}, g(z) = c_2 e^{-cz^2}$ where $c_1, c_2$ and $c$ are three constants satisfying $4n^2(c_1 c_2)^n c^2 = -1,$ or $f \equiv tg$ for a constant $t$ such that $t^n = 1.$
Further, Fang and Qiu investigated uniqueness for the same functions as in the Theorem A, when \( k = 1 \).

**Theorem B.**[2] Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions and let \( n \geq 11 \) be a positive integer. If \( f^n(z)f'(z) \) and \( g^n(z)g'(z) \) share \( z \) CM, then either \( f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2} \), where \( c_1, c_2 \) and \( c \) are three constants satisfying \( 4(c_1c_2)^{n+1}c^2 = -1 \), or \( f(z) \equiv tg(z) \) for a constant \( t \) such that \( t^{n+1} = 1 \).

In 2012, Cao and Zhang replaced \( f' \) with \( f^{(k)} \) and obtained the following theorem.

**Theorem C.**[1] Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions, whose zeros are of multiplicities atleast \( k \), where \( k \) is a positive integer. Let \( n > max\{2k-1, 4+4/k+4\} \) be a positive integer. If \( f^n(z)f^{(k)}(z) \) and \( g^n(z)g^{(k)}(z) \) share \( z \) CM, and \( f \) and \( g \) share \( \infty \) IM, then one of the following two conclusions holds.

1. \( f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z) \)
2. \( f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2} \), where \( c_1, c_2 \) and \( c \) are constants such that \( 4(c_1c_2)^{n+1}c^2 = -1 \).

Recently, X.B. Zhang reduced the lower bond of \( n \) and relax the condition on multiplicity of zeros in Theorem C and proved the below result.

**Theorem D.**[11] Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions and \( n, k \) two positive integers with \( n > k + 6 \). If \( f^n(z)f^{(k)}(z) \) and \( g^n(z)g^{(k)}(z) \) share \( z \) CM, and \( f \) and \( g \) share \( \infty \) IM, then one of the following two conclusions holds.

1. \( f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z) \)
2. \( f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2} \), where \( c_1, c_2 \) and \( c \) are constants such that \( 4(c_1c_2)^{n+1}c^2 = -1 \).

In 2016, Renukadevi S. Dyavanal and Ashwini M. Hattikal proved the following theorem.

**Theorem E.**[8] Let \( f \) and \( g \) be two transcendental meromorphic functions of hyper order \( \rho_2(f) < 1 \) and \( \rho_2(g) < 1 \). Let \( k, n, \lambda \) be positive integers and \( n > max\{2d(k + 2) + \lambda(k + 3) + 7, \lambda_1, \lambda_2\} \). If \( F(z) = f(z)^n \left[ \prod_{j=1}^{d} f(z + c_j)^{s_j} \right]^{(k)} \) and \( G(z) = g(z)^n \left[ \prod_{j=1}^{d} g(z + c_j)^{s_j} \right]^{(k)} \) share \( z \) CM an \( f, g \) share \( \infty \) IM, then one of the following two conclusions holds.

1. \( F(z) \equiv G(z) \)
2. \( \prod_{j=1}^{d} f(z + c_j)^{s_j} = C_1e^{Cz^2}, \prod_{j=1}^{d} g(z + c_j)^{s_j} = C_2e^{-Cz^2} \), where \( C_1, C_2 \) and \( C \) are constants such that \( 4(C_1C_2)^{n+1}C^2 = -1 \).

We define a difference product of meromorphic function \( f(z) \) as follows
\[ F(z) = f(z)^n P(f) \left( \prod_{j=1}^{d} f(z + c_j)^{s_j} \right)^{(k)} \] (1)

\[ F_1(z) = f(z)^n P(f) \prod_{j=1}^{d} f(z + c_j)^{s_j} \] (2)

where \( c_j \in \mathbb{C} \setminus \{0\} \) \((j = 1, 2, ..., d)\) are distinct constants. \( n, k, d, s_j (j = 1, 2, ..., d) \) are positive integers and \( \lambda = \sum_{j=1}^{d} s_j \).

For \( j = 1, 2, 3, ... d, \lambda_1 = \sum_{j=1}^{d} \alpha_j s_j \) and \( \lambda_2 = \sum_{j=1}^{d} \beta_j s_j \), where \( f(z + c_j) \) and \( g(z + c_j) \) have zeros with maximum orders \( \alpha_j \) and \( \beta_j \) respectively.

In this article, we prove the theorem on product of difference-differential polynomials sharing a fixed point as follows.

**Theorem 1.** Let \( f \) and \( g \) be two transcendental meromorphic functions of hyper order \( \rho_2(f) < 1 \) and \( \rho_2(g) < 1 \). Let \( k, n, d, \lambda \) be positive integers and \( n > \max \{2d(k + 2) + \lambda(k + 4) + \Gamma_0 + 8 - m, \lambda_1, \lambda_2 \} \). If \( F(z) \) and \( G(z) \) share \( z \) CM and \( f, g \) share \( \infty \) IM, then one of the following two conclusions holds.

1. \( F(z) \equiv G(z) \)
2. \( \prod_{j=1}^{d} f(z + c_j)^{s_j} = C_1 e^{Cz^2}, \prod_{j=1}^{d} g(z + c_j)^{s_j} = C_2 e^{-Cz^2} \) where \( C_1, C_2 \) and \( C \) are constants such that \( 4(C_1C_2)^{n+1}C^2 = -1 \).

**Remark.**
If \( m = 1 \) then Theorem 1 reduces to Theorem E.

**Theorem 2.** Let \( f \) and \( g \) be two transcendental meromorphic functions. Let \( k, n, d, \lambda \) be positive integers and \( n > \max \{(\frac{3k+5)d}{2} + \frac{(9+3k)\lambda}{2} + \frac{3}{2}\Gamma_0 + \frac{19}{2} - m, \lambda_1, \lambda_2 \} \). If \( F(z) \) and \( G(z) \) share \( "(\alpha(z), 1)" \) and \( f, g \) share \( \infty \) IM, then \( F(z) \equiv G(z) \).

**Theorem 3.** Let \( f \) and \( g \) be two transcendental meromorphic functions. Let \( k, n, d, \lambda \) be positive integers and \( n > \max \{\frac{5k+6)d}{2} + (7+4k)\lambda + \Gamma_0 + 14 - m, \lambda_1, \lambda_2 \} \). If \( F(z) \) and \( G(z) \) share \( "(\alpha(z), 0)" \) and \( f, g \) share \( \infty \) IM, then \( F(z) \equiv G(z) \).

2. **Lemmas**

In this section we present some lemmas needed in the sequel. Let \( F, G \) be two non-constant meromorphic functions. Henceforth we shall denote by \( H \) the following function.

\[ H = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right) \] (3)

**Lemma 2.1.**[9] Let \( f \) and \( g \) be two non-constant meromorphic functions, \( 'a' \) be a finite non-zero constant. If \( f \) and \( g \) share \( 'a' \) CM and \( \infty \) IM, then one of the following cases holds.

1. \( T(r, f) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + 3\overline{N}(r, f) + S(r, f) + S(r, g) \).

The same inequality holding for \( T(r, g) \);
(2) $fg \equiv a^2$;
(3) $f \equiv g$.

**Lemma 2.2.**[5] Let $f(z)$ be a transcendental meromorphic functions of hyper order $\rho_2(f) < 1$, and let $c$ be a non-zero complex constant Then we have

$$T(r, f(z + c)) = T(r, f(z)) + S(r, f(z)), \quad N(r, f(z + c)) = N(r, f(z)) + S(r, f(z)).$$

**Lemma 2.3.**[10] Let $f$ be a non-constant meromorphic function, let $P(f) = a_0 + a_1f + a_2f^2 + \ldots + a_nf^n$, where $a_0, a_1, a_2, \ldots a_n$ are constants and $a_n \neq 0$. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

**Lemma 2.4.**[10] Let $f$ be a non-constant meromorphic function and $p, k$ be positive integers. Then

$$T(r, f^{(k)}) \leq T(r, f) + kN(r, f) + S(r, f), \quad (4)$$

$$N_p \left( r, \frac{1}{f^{(k)}} \right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k} \left( r, \frac{1}{f} \right) + S(r, f), \quad (5)$$

$$N_p \left( r, \frac{1}{f^{(k)}} \right) \leq N_{p+k} \left( r, \frac{1}{f} \right) + kN(r, f) + S(r, f), \quad (6)$$

$$N \left( r, \frac{1}{f^{(k)}} \right) \leq N \left( r, \frac{1}{f} \right) + kN(r, f) + S(r, f). \quad (7)$$

**Lemma 2.5.**[3] Suppose that $f$ is a non-constant meromorphic function, $k \geq 2$ is an integer. If

$$N(r, f) + N(r, \frac{1}{f^{(k)}}) = S(r, \frac{f'}{f}),$$

then $f(z) = e^{az+b}$, where $a \neq 0, b$ are constants.

**Lemma 2.6.**[12] If $f, g$ be two nonconstant meromorphic functions such that they share “$(1, 1)$”, then

$$2N_L(r, 1; f) + 2N_L(r, 1; g) + N^2_E(r, 1; f) - N^2_{f>2}(r, 1; g) \leq N(r, 1; g) - N(r, 1; g).$$

**Lemma 2.7.**[12] Let $f, g$ share “$(1, 1)$”, Then

$$N_{f>2}(r, 1; g) \leq \frac{1}{2}N(r, 0; f) + \frac{1}{2}N(r, \infty; f) - \frac{1}{2}N_0(r, 0; f') + S(r, f).$$

**Lemma 2.8.**[12] Let $f, g$ be two nonconstant meromorphic functions such that they share “$(1, 0)$”. Then $N_L(r, 1; f) + 2N_L(r, 1; g) + N^2_E(r, 1; f) - N^2_{f>2}(r, 1; g) - N_{g>2}(r, 1; f) \leq N(r, 1; g) - N(r, 1; g).$

**Lemma 2.9.**[12] Let $f, g$ share “$(1, 0)$”. Then $N_L(r, 1; f) \leq N(r, 0; f) + N(r, \infty; f) + S(r, f).$
Lemma 2.10.[12] Let \( f, g \) share "(1,0)". Then
\[
(i) \mathcal{N}_{f>1}(r,1;g) \leq \mathcal{N}(r,0;f) + \mathcal{N}(r,\infty;f) - \mathcal{N}_0(r,0;f') + S(r,f);
\]
\[
(ii) \mathcal{N}_{g>1}(r,1;g) \leq \mathcal{N}(r,0;g) + \mathcal{N}(r,\infty;f) - \mathcal{N}_0(r,0;f') + S(r,g);
\]

Lemma 2.11. Let \( f(z) \) be a transcendental meromorphic function of hyper order \( \rho_2(f) < 1 \) and \( F_1(z) \) be stated as in (2). Then
\[
(n + m - \lambda) T(r,f) + S(r,f) \leq T(r,F_1(z)) \leq (n + m + \lambda) T(r,f) + S(r,f).
\]

Proof. Since \( f \) is a meromorphic function with \( \rho_2(f) < 1 \). From Lemma 2.2 and Lemma 2.3 we have
\[
T(r,F_1(z)) \leq T(r,f(z)^n) + T(r,P(f)) + T \left( r, \prod_{j=1}^{d} f(z + c_j)^{s_j} \right) + S(r,f) \tag{8}
\]
\[
\leq (n + m + \lambda) T(r,f) + S(r,f)
\]
On the other hand, from Lemma 2.2 and Lemma 2.3, we have
\[
(n + m + \lambda) T(r,f) = T(r,f^n f^m f^\lambda) + S(r,f)
\]
\[
= m(r, f^n f^m f^\lambda) + N(r, f^n f^m f^\lambda) + S(r,f)
\]
\[
\leq m \left( r, \frac{F_1(z)f^\lambda}{\prod_{j=1}^{d} f(z + c_j)^{s_j}} \right) + N \left( r, \frac{F_1(z)f^\lambda}{\prod_{j=1}^{d} f(z + c_j)^{s_j}} \right) + S(r,f)
\]
\[
\leq m(r,F_1(z)) + N(r,F_1(z)) + T \left( r, \frac{f^\lambda}{\prod_{j=1}^{d} f(z + c_j)^{s_j}} \right) + S(r,f) \tag{9}
\]
\[
\leq T(r,F_1(z)) + 2\lambda T(r,f) + S(r,f)
\]
\[
(n + m + \lambda - 2\lambda) T(r,f) \leq T(r,F_1(z)) + S(r,f)
\]
\[
\Rightarrow (n + m - \lambda) T(r,f) + S(r,f) \leq T(r,F_1(z)).
\]
Hence we get Lemma 2.11.

3. Proof of the Theorem

Proof of the Theorem 1

Let \( F^* = \frac{F}{z} \) and \( G^* = \frac{G}{z} \) \( \tag{10} \)

From the hypothesis of the Theorem 1, we have \( F \) and \( G \) share \( z \) CM and \( f, g \) share \( \infty \) IM. It follows that \( F^* \) and \( G^* \) share 1 CM and \( \infty \) IM.

By Lemma 2.1, we arrive at 3 cases as follows.

Case 1. Suppose that case (1) of Lemma 2.1 holds.
\[
T(r,F^*) \leq N_2(r, \frac{1}{F^*}) + N_2(r, \frac{1}{G^*}) + 3\mathcal{N}(r,F^*) + S(r,F^*) + S(r,G^*) \tag{11}
\]
We deduce from (11) and obtained the following
\[ T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 3N(r, F) + S(r, F) + S(r, G) \quad (12) \]

From Lemma 2.2 and Lemma 2.6, we have \( S(r, F) = S(r, f) \) and \( S(r, G) = S(r, g) \).
From (12), we have
\[
T(r, F) = N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 3N(r, F) + S(r, f) + S(r, g)
\]
\[
\leq N_2 \left( r, \frac{1}{f^n} \right) + N_2 \left( r, \frac{1}{P(f)} \right) + N_2 \left( r, \frac{1}{(\prod_{j=1}^d f(z + c_j)^{s_j})^{(k)}} \right) + N_2 \left( r, \frac{1}{g^n} \right) + N_2 \left( r, \frac{1}{P(g)} \right)
\]
\[
+ N_2 \left( r, \frac{1}{(\prod_{j=1}^d g(z + c_j)^{s_j})^{(k)}} \right) + 3N(r, f^n) + 3N(r, P(f)) + 3N \left( r, \left( \prod_{j=1}^d f(z + c_j)^{s_j} \right)^{(k)} \right)
\]
\[
+ S(r, f) + S(r, g).
\]
Using (5) of Lemma 2.4 in (13) we have
\[
T(r, F) \leq 2T(r, f) + \Gamma_0T(r, f) + T \left( r, \left( \prod_{j=1}^d f(z + c_j)^{s_j} \right)^{(k)} \right) - T \left( r, \left( \prod_{j=1}^d f(z + c_j)^{s_j} \right)^{(k)} \right)
\]
\[
+ N_{k+2} \left( r, \frac{1}{(\prod_{j=1}^d f(z + c_j)^{s_j})^{(k)}} \right) + 2T(r, g) + \Gamma_0T(r, g) + T \left( r, \left( \prod_{j=1}^d g(z + c_j)^{s_j} \right)^{(k)} \right)
\]
\[
- T \left( r, \left( \prod_{j=1}^d g(z + c_j)^{s_j} \right)^{(k)} \right) + N_{k+2} \left( r, \frac{1}{(\prod_{j=1}^d g(z + c_j)^{s_j})^{(k)}} \right) + 6N(r, f) + 3N \left( r, \left( \prod_{j=1}^d f(z + c_j)^{s_j} \right)^{(k)} \right)
\]
\[
+ S(r, f) + S(r, g)
\]
\[
\leq (2 + \Gamma_0)T(r, f) + T \left( r, \left( \prod_{j=1}^d f(z + c_j)^{s_j} \right)^{(k)} \right) + T(r, f^n) - T(r, f^n) - T \left( r, \left( \prod_{j=1}^d f(z + c_j)^{s_j} \right)^{(k)} \right)
\]
\[
+ (k + 2)dT(r, f) + (2 + \Gamma_1)T(r, g) + T \left( r, \left( \prod_{j=1}^d g(z + c_j)^{s_j} \right)^{(k)} \right) + kN \left( r, \left( \prod_{j=1}^d g(z + c_j)^{s_j} \right)^{(k)} \right)
\]
\[
- T \left( r, \left( \prod_{j=1}^d g(z + c_j)^{s_j} \right)^{(k)} \right) + (k + 2)dT(r, g) + 6T(r, f) + 3\lambda T(r, f) + S(r, f) + S(r, g)
\]
\[
T(r, F) \leq (2 + \Gamma_0)T(r, f) + T(r, F) - T(r, F_1) + (k + 2)dT(r, f) + (2 + \Gamma_0)T(r, g) + k\lambda T(r, g)
\]
\[
+ (k + 2)dT(r, g) + (6 + 3\lambda)T(r, f) + S(r, f) + S(r, g)
\]
\[
T(r, F_1) \leq (2 + \Gamma_0)T(r, f) + (k + 2)dT(r, f) + (2 + \Gamma_0)T(r, g) + (k + 2)dT(r, g) + k\lambda T(r, g)
\]
\[
+ (6 + 3\lambda)T(r, f) + S(r, f) + S(r, g)
\]
\[
\leq (2 + \Gamma_0)[T(r, f) + T(r, g)] + (k + 2)d[T(r, f) + T(r, g)] + k\lambda T(r, g) + (6 + 3\lambda)T(r, f)
\]
\[
+ S(r, f) + S(r, g)
\]
From Lemma 2.11, we have
\[ (n + m - \lambda)T(r, f) \leq ((k + 2)d + 2 + \Gamma_0)[T(r, f) + T(r, g)] + k\lambda T(r, g) + (6 + 3\lambda)T(r, f) + S(r, f) + S(r, g) \]  
(14)
Similarly for \( T(r, g) \) we obtain the following
\[ (n + m - \lambda)T(r, g) \leq ((k + 2)d + 2 + \Gamma_0)[T(r, f) + T(r, g)] + k\lambda T(r, f) + (6 + 3\lambda)T(r, g) + S(r, f) + S(r, g) \]  
(15)
From (14) and (15), we have
\[ (n + m - \lambda)[T(r, f) + T(r, g)] \leq 2((k + 2)d + 2 + \Gamma_0)[T(r, f) + T(r, g)] + (k\lambda + 6 + 3\lambda) 
\[ [T(r, f) + T(r, g)] + S(r, f) + S(r, g) \]
Which is contradiction to \( n > 2d(k + 2) + \Gamma_0 + \lambda(k + 4) + 8 - m \).

**Case 2.** Suppose that \( FG \equiv z^2 \) holds.

\[ i.e., f^n P(f) \left[ \prod_{j=1}^{d} f(z + c_j)^{s_j} \right]^{(k)} g^n P(g) \left[ \prod_{j=1}^{d} g(z + c_j)^{s_j} \right]^{(k)} \equiv z^2 \]  
(16)
Now, (16) can be written as
\[ f^n P(f)g^n P(g) \equiv \frac{z^2}{\prod_{j=1}^{d} f(z + c_j)^{s_j} |^{(k)} \prod_{j=1}^{d} g(z + c_j)^{s_j} |^{(k)}} \]
By using Lemma 2.2, Lemma 2.3 and (8) of Lemma 2.4, we derive
\[ (n + m)[N(r, f) + N(r, g)] \leq \lambda[N(r, \frac{1}{f}) + N(r, \frac{1}{g})] + kd[N(r, f) + N(r, g)] + S(r, f) + S(r, g) \]  
(17)
From (16), we can write
\[ \frac{1}{f^n P(f)g^n P(g)} = \frac{\prod_{j=1}^{d} f(z + c_j)^{s_j} |^{(k)} \prod_{j=1}^{d} g(z + c_j)^{s_j} |^{(k)}}{z^2} \]
Similarly, as (17), we obtain
\[ (n + m)[N(r, \frac{1}{f}) + N(r, \frac{1}{g})] \leq (\lambda + kd)[N(r, f) + N(r, g)] + S(r, f) + S(r, g) \]  
(18)
From (17) and (18), deduce
\[ (n + m - (\lambda + 2kd))[N(r, f) + N(r, g)] + (n + m - \lambda)[N(r, \frac{1}{f}) + N(r, \frac{1}{g})] \leq S(r, f) + S(r, g) \]
Since \( n > 2d(k + 2) + \lambda(4 + k) + \Gamma_0 + 8 - m \), we have
\[ N(r, f) + N(r, g) + N(r, \frac{1}{f}) + N(r, \frac{1}{g}) < S(r, f) + S(r, g) \]
Hence, we conclude that \( f \) and \( g \) have finitely many zeros and poles.

Let \( z_0 \) be a pole of \( f \) of multiplicity \( p \), then \( z_0 \) is pole of \( f^n \) of multiplicity \( np \), since \( f \) and \( g \) share \( \infty \) IM, then \( z_0 \) is pole of \( g \) of multiplicity \( q \).
If \( z_0 \) also zero of \( \prod_{j=1}^{d} f(z + c_j)^{s_j} \) and \( \prod_{j=1}^{d} g(z + c_j)^{s_j} \) then we have from (16) that
\[
n(p + q) \leq \sum_{j=1}^{d} \alpha_j s_j + \sum_{j=1}^{d} \beta_j s_j - 2k.
\]
\( \Rightarrow 2n < n(p + q) \leq \sum_{j=1}^{d} \alpha_j s_j + \sum_{j=1}^{d} \beta_j s_j - 2k = \lambda_1 + \lambda_2 - 2k < \lambda_1 + \lambda_2 \leq 2\max\{\lambda_1, \lambda_2\}
\]
\( \Rightarrow n < \max\{\lambda_1, \lambda_2\} \), which is contradiction to \( n > \max\{2d(k + 2) + \lambda(4 + k) + \Gamma_0 + 8 - m, \lambda_1, \lambda_2\} \).
Therefore \( f \) has no poles.

Similarly, we can get contradiction for other two cases namely, if \( z_0 \) is zero of \( \prod_{j=1}^{d} f(z + c_j)^{s_j} \), but not zero of \( \prod_{j=1}^{d} g(z + c_j)^{s_j} \) and other way. Therefore \( f \) has no poles. Similarly, we get that \( g \) also has no poles. By this we conclude that \( f \) and \( g \) are entire functions and hence \( \prod_{j=1}^{d} f(z + c_j)^{s_j} \) and \( \prod_{j=1}^{d} g(z + c_j)^{s_j} \) are entire functions.

Then from (16), we deduce that \( f \) and \( g \) have no zeros.

Therefore
\[
f = e^{\alpha(z)}, g = e^{\beta(z)} \quad \text{and}
\]
\[
\prod_{j=1}^{d} f(z + c_j)^{s_j} = \prod_{j=1}^{d} (e^{\alpha(z + c_j)})^{s_j}, \quad \prod_{j=1}^{d} g(z + c_j)^{s_j} = \prod_{j=1}^{d} (e^{\beta(z + c_j)})^{s_j}
\]
\[
(19)
\]
where \( \alpha, \beta \) are entire functions with \( \rho_2(f) < 1 \). Substituting \( f \) and \( g \) into (16) we get
\[
e^{n\alpha(z)} \left[ \prod_{j=1}^{d} (e^{\alpha(z + c_j)})^{s_j} \right]^{(k)} e^{n\beta(z)} \left[ \prod_{j=1}^{d} (e^{\beta(z + c_j)})^{s_j} \right]^{(k)} \equiv z^2
\]
(20)

If \( k = 1 \), then
\[
e^{n\alpha(z)} \left[ \prod_{j=1}^{d} (e^{\alpha(z + c_j)})^{s_j} \right]' e^{n\beta(z)} \left[ \prod_{j=1}^{d} (e^{\beta(z + c_j)})^{s_j} \right]' \equiv z^2
\]
(21)

\[\Rightarrow e^{n(\alpha + \beta)} e^{\sum_{j=1}^{d}(\alpha + \beta(z + c_j))s_j} \prod_{j=1}^{d} (\alpha'(z + c_j))s_j \prod_{j=1}^{d} (\beta'(z + c_j))s_j \equiv z^2\]
(22)

Since \( \alpha(z) \) and \( \beta(z) \) are non-constant entire functions, then we have
\[
T\left( r, \frac{\prod_{j=1}^{d} f(z + c_j)^{s_j}'}{\prod_{j=1}^{d} f(z + c_j)^{s_j}} \right) = T\left( r, \frac{\prod_{j=1}^{d} e^{\alpha(z + c_j)s_j}'}{\prod_{j=1}^{d} e^{\alpha(z + c_j)s_j}} \right)
\]
(23)

\[
T\left( r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \prod_{j=1}^{d} e^{\alpha(z + c_j)s_j} \right) = T\left( r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right)
\]
(24)
Let

\[(n + m)T(r, f) = T(r, f_n^m) = T \left( r, \frac{F}{\prod_{j=1}^{d} f(z + c_j)^{s_j}} \right) \leq T(r, F) \]

\[+ T \left( r, \left( \prod_{j=1}^{d} f(z + c_j)^{s_j} \right)^{(k)} \right) + S(r, f) \]

\[\leq T(r, F) + T \left( r, \prod_{j=1}^{d} f(z + c_j)^{s_j} \right)^{(k)} + kN \left( r, \prod_{j=1}^{d} f(z + c_j)^{s_j} \right) + S(r, f) \]

\[(n + m)T(r, f) \leq T(r, F) + (\lambda + kd)T(r, f) + S(r, f) \]

\[(n + m - \lambda - kd)T(r, f) \leq T(r, F) + S(r, f) \quad (25) \]

We obtain from (24) that

\[T(r, f) = O(T(r, F)) \quad (26)\]

as \(r \in E\) and \(r \to \infty\), where \(E \subset (0, +\infty)\) is some subset of finite linear measure.

On the other hand, we have

\[T(r, F) = T \left( r, f^{n} P(f) \left( \prod_{j=1}^{d} f(z + c_j)^{s_j} \right)^{(k)} \right) \]

\[\leq nT(r, f) + mT(r, f) + \lambda T(r, f) + kN \left( r, \prod_{j=1}^{d} f(z + c_j)^{s_j} \right) + S(r, f) \quad (27)\]

\[\leq (n + m + kd + \lambda)T(r, f) + S(r, f) \]

\[\Rightarrow T(r, F) = O(T(r, f)) \]

as \(r \in E\) and \(r \to \infty\), where \(E \subset (0, +\infty)\) is some subset of finite linear measure.

Thus from (25),(26) and the standard reasoning of removing exceptional set we deduce \(\rho(f) = \rho(F)\). Similarly, we have \(\rho(g) = \rho(G)\). It follows from (16) that \(\rho(F) = \rho(G)\). Hence we get \(\rho(f) = \rho(g)\).

We deduce that either both \(\alpha\) and \(\beta\) are polynomials or both \(\alpha\) and \(\beta\) are transcendental entire functions. Moreover, we have

\[N \left( r, \frac{1}{\prod_{j=1}^{d} f(z + c_j)^{s_j}} \right) \leq N \left( r, \frac{1}{z^2} \right) = O(\log r) \quad (28)\]

From (27) and (19), we have

\[N \left( r, \prod_{j=1}^{d} f(z + c_j)^{s_j} \right) + N \left( r, \frac{1}{\prod_{j=1}^{d} f(z + c_j)^{s_j}} \right) + N \left( r, \frac{1}{\prod_{j=1}^{d} f(z + c_j)^{s_j}} \right) = O(\log r).\]
If \( k \geq 2 \), then it follows from (23), (27) and Lemma 2.5 that \( \sum_{j=1}^{d} \alpha'(z + c_j)s_j \) is a polynomial and therefore we have \( \alpha(z) \) is a non-constant polynomial.

Similarly, we can deduce that \( \beta(z) \) is also a non-constant polynomial. From this, we deduce from (19) that

\[
\left( \prod_{j=1}^{d} f(z + c_j)^{s_j} \right)^{(k)} = e^{\sum_{j=1}^{d} \alpha(z + c_j)s_j} \left[ P_{k-1}(\alpha'(z + c_j)) + \left( \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right)^k \right]
\]

\[
\left( \prod_{j=1}^{d} g(z + c_j)^{s_j} \right)^{(k)} = e^{\sum_{j=1}^{d} \beta(z + c_j)s_j} \left[ Q_{k-1}(\alpha'(z + c_j)) + \left( \sum_{j=1}^{d} \beta'(z + c_j)s_j \right)^k \right]
\]

Where \( P_{k-1} \) and \( Q_{k-1} \) are difference-differential polynomials in \( \alpha'(z + c_j) \) with degree at most \( k - 1 \). Then (20) becomes

\[
e^{n(\alpha + \beta) + \sum_{j=1}^{d} (\alpha(z + c_j) + \beta(z + c_j))s_j} \left[ \sum_{j=1}^{d} \alpha^{(k)}(z + c_j)s_j + \left( \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right)^k \right]
\]

\[
\left[ \sum_{j=1}^{d} \beta^{(k)}(z + c_j)s_j + \left( \sum_{j=1}^{d} \beta'(z + c_j)s_j \right)^k \right] = z^2 \quad \text{(29)}
\]

We deduce from (28) that \( \alpha(z) + \beta(z) \equiv C \) for a constant \( C \).

If \( k = 1 \), from (22), we have

\[
e^{n(\alpha + \beta) + \sum_{j=1}^{d} (\alpha(z + c_j) + \beta(z + c_j))s_j} \left[ \sum_{j=1}^{d} \alpha'(z + c_j)s_j \sum_{j=1}^{d} \beta'(z + c_j)s_j \right] \equiv z^2. \quad \text{(30)}
\]

Next, we let \( \alpha + \beta = \gamma \) and suppose that \( \alpha, \beta \) both are transcendental entire functions.

If \( \gamma \) is a constant, then \( \alpha' + \beta' = 0 \) and \( \sum_{j=1}^{d} \alpha'(z + c_j) = -\sum_{j=1}^{d} \beta'(z + c_j) \).

From (29) we have

\[
e^{n(\alpha + \beta) + \sum_{j=1}^{d} (\alpha(z + c_j) + \beta(z + c_j))s_j} \left\{ - \left[ \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right]^2 \right\} \equiv z^2
\]

\[
e^{n\gamma + d\gamma} \left\{ - \sum_{j=1}^{d} \alpha'(z + c_j) \right\}^2 = z^2 \quad \text{(31)}
\]
Which implies that $\alpha'$ is a non-constant polynomial of degree 1. This together with $\alpha' + \beta' = 0$ which implies that $\beta'$ is also non-constant polynomial of degree 1. Which is contradiction to $\alpha, \beta$ both are transcendental entire functions.

If $\gamma$ is not a constant, then we have

$$\alpha + \beta = \gamma \text{ and } \sum_{j=1}^{d} \alpha(z + c_j)s_j + \sum_{j=1}^{d} \beta(z + c_j)s_j = \sum_{j=1}^{d} \gamma(z + c_j)s_j$$

From (29) we have

$$\left[ \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right] \left[ \sum_{j=1}^{d} \gamma'(z + c_j)s_j - \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right] e^{n\gamma+\sum_{j=1}^{d} \gamma(z+c_j)s_j} = z^{2} \tag{32}$$

Since

$$T \left( r, \sum_{j=1}^{d} \gamma'(z + c_j)s_j \right) = m \left( r, \sum_{j=1}^{d} \gamma'(z + c_j)s_j \right) + N \left( r, \sum_{j=1}^{d} \gamma'(z + c_j)s_j \right) \leq m \left( r, \frac{\sum_{j=1}^{d} \gamma'(z+c_j)s_j\gamma'}{e^{\sum_{j=1}^{d} \gamma'(z+c_j)s_j}} \right) + O(1) = S \left( r, e^{\sum_{j=1}^{d} \gamma(z+c_j)s_j} \right) \tag{33}$$

And also we have

$$T \left( r, n\gamma' + \sum_{j=1}^{d} \gamma'(z + c_j)s_j \right) = m \left( r, n\gamma' + \sum_{j=1}^{d} \gamma'(z + c_j)s_j \right) + N \left( r, n\gamma' + \sum_{j=1}^{d} \gamma'(z + c_j)s_j \right) \leq m \left( r, \frac{\sum_{j=1}^{d} \gamma'(z+c_j)s_j\gamma'}{e^{\sum_{j=1}^{d} \gamma'(z+c_j)s_j}} \right) + O(1) = S \left( r, e^{n\gamma+\sum_{j=1}^{d} \gamma(z+c_j)s_j} \right) \tag{34}$$

From (31), we have

$$T \left( r, e^{n\gamma+\sum_{j=1}^{d} \gamma(z+c_j)s_j} \right) \leq T \left( r, e^{\sum_{j=1}^{d} \alpha'(z + c_j)s_j \sum_{j=1}^{d} \gamma'(z+c_j)s_j} e^{\sum_{j=1}^{d} \alpha'(z+c_j)s_j} \right) + O(1) = T(r, z^{2}) + T \left( r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \left[ \sum_{j=1}^{d} \gamma'(z+c_j)s_j - \sum_{j=1}^{d} \alpha'(z+c_j)s_j \right] \right) + O(1) \leq 2logr + 2T \left( r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right) + O(1)$$

$$\Rightarrow T \left( r, e^{n\gamma+\sum_{j=1}^{d} \gamma(z+c_j)s_j} \right) \leq O \left( T \left( r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right) \right) \tag{35}$$
Similarly, we have
\[ T \left( r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right) \leq O \left( T \left( r, e^{n\gamma + \sum_{j=1}^{d} \gamma(z+c_j)s_j} \right) \right) \] (36)

Thus, from (32)-(35) we have
\[ T(r, n\gamma' + \sum_{j=1}^{d} \gamma'(z + c_j)s_j) = S \left(r, e^{n\gamma + \sum_{j=1}^{d} \gamma(z+c_j)s_j} \right) = S \left(r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right) \]

By the second fundamental theorem and (31), we have
\[ T \left( r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right) \leq N \left( r, \sum_{j=1}^{d} \frac{1}{\alpha'(z + c_j)s_j} \right) + N \left( r, \sum_{j=1}^{d} \frac{1}{\alpha'(z + c_j)s_j - \sum_{j=1}^{d} \gamma'(z + c_j)s_j} \right) \]
\[ + S \left( r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right) \leq O(\log r) + S \left(r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right) \]

This implies \( \sum_{j=1}^{d} \alpha'(z + c_j)s_j \) is a polynomial, which leads to \( \alpha'(z) \) is a polynomial.

Which contradicts that \( \alpha(z) \) is a transcendental entire function.

Thus \( \alpha \) and \( \beta \) are both polynomials and \( \alpha(z) + \beta(z) \equiv C \) for a constant \( C \).

Hence from (28) and using \( \alpha + \beta = C \) we get
\[ (-1)^k \left( \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right)^{2k} = z^{2k} + P_{2k-1}(\alpha'(z + c_j)s_j) \text{ for } j = 1, 2, ..., d. \]

Where \( P_{2k-1} \) is difference-differential polynomial in \( \alpha'(z + c_j)s_j \) of degree at most \( 2k - 1 \). From (36) we have
\[ 2kT \left( r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j \right) = 2\log r + S \left(r, \alpha'(z + c_j)s_j \right) \]

From (3.28), we can see that \( \sum_{j=1}^{d} \alpha'(z + c_j)s_j \) is a non-constant polynomial of degree 1 and \( k = 1 \).

Which implies,
\[ \sum_{j=1}^{d} \alpha'(z + c_j)s_j = zl_1 \]

Since \( \alpha' + \beta' = 0 \), we get \( \sum_{j=1}^{d} \beta'(z + c_j)s_j = -\sum_{j=1}^{d} \alpha'(z + c_j)s_j \). Which implies \( \sum_{j=1}^{d} \beta'(z + c_j)s_j \) is also a non-constant polynomial of degree 1. Hence we have
\[ \sum_{j=1}^{d} \beta'(z + c_j)s_j = zl_2 \]

Hence, we get
\[ \prod_{j=1}^{d} f(z + c_j)^{s_j} = C_1 e^{C_2z} \]

Similarly, we have
\[ \prod_{j=1}^{d} g(z + c_j)^{s_j} = C_2 e^{-C_2z} \]
where $C_1, C_2$ and $C$ are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.

This proves the conclusion (2) of Theorem 1.

**Case 3.** If $F \equiv G$

$i.e., \ f^n P(f) \left[ \prod_{j=1}^{d} f(z+c_j)^s \right]^{(k)} \equiv g^n P(g) \left[ \prod_{j=1}^{d} g(z+c_j)^s \right]^{(k)}$

This proves the conclusion (1) of Theorem 1.

**Proof of Theorem 2**

Let $F, G$ be given by from the assumption of Theorem 2, we know that $F$ and $G$ share "(1,1)".

Let $H$ be defined as in (3) Suppose that $H \neq 0$. Since $F, G$ share "(1,1)", we can get

$$N(r, \infty; H) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 1; F \geq 2) + \overline{N}(r, 0; F \geq 2) + \overline{N}(r, 0; G \geq 2) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f)$$

which

$$N(r, 1; F \geq 1) \leq N(r, 0; H) + S(r, f) \leq N(r, \infty; H) + S(r, f)$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of $F'$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r, 0; G')$ is similarly defined.

By the second fundamental theorem, we see that

$$T(r, F) + T(r, G) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, 1; F) + \overline{N}(r, 1; G)$$

$$- N_0(r, 0; F') - N_0(r, 0; G') + S(r, F) + S(r, G).$$

Using Lemmas 2.6 and 2.7, (37) and (38) we can get

$$\overline{N}(r, 1; F) + \overline{N}(r, 1; G) \leq N(r, 1; F = 1) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_E^2(r, 1; F) + \overline{N}(r, 1; G)$$

$$\leq N(r, 1; F = 1) + N(r, 1; G) - \overline{N}_L(r, 1; F) - \overline{N}_L(r, 1; G) + \overline{N}_{F>2}(r, 1; G)$$

$$\leq \overline{N}(r, 1; F \geq 2) + \overline{N}(r, 1; G \geq 2) + \overline{N}(r, \infty; F) + \overline{N}_{s}(r, 1; F, G) + T(r, G)$$

$$- m(r, 1; G) + O(1) + \frac{1}{2} \overline{N}(r, \infty; F) - \overline{N}_L(r, 1; F) - \overline{N}_L(r, 1; G) + \frac{1}{2} \overline{N}(r, 0, F)$$

$$+ N_0(r, 0; F') + N_0(r, 0; G') + S(r, F) + S(r, G).$$

Combining (39) and (40), we can obtain

$$T(r, F) \leq \frac{7}{2} \overline{N}(r, \infty; F) + N_2(r, 0; F) + N_2(r, 0; G) + \frac{1}{2} \overline{N}(r, 0; F)$$
By the definition of $F, G$ we have

$$
T(r, F) \leq 2T(r, f) + \Gamma_0 T(r, f) + T(r, \prod_{j=1}^{d} f(z + c_j)^{(k)}) - T(r, \prod_{j=1}^{d} f(z + c_j)^{(k)})
+ N_{k+2} \left(r, \frac{1}{\prod_{j=1}^{d} f(z + c_j)^{(k)}}\right) + 2T(r, g) + \Gamma_0 T(r, g) + T(r, \prod_{j=1}^{d} g(z + c_j)^{(k)})

- T(r, \prod_{j=1}^{d} g(z + c_j)^{(k)}) + N_{k+2} \left(r, \frac{1}{\prod_{j=1}^{d} f(z + c_j)^{(k)}}\right) + \frac{1}{2} N(r, \frac{1}{\lfloor n \rceil}) + \frac{1}{2} N(r, \frac{1}{P(f)})

+ \bar{N}(r, \frac{1}{\prod_{j=1}^{d} g(z + c_j)^{(k)}}) + \frac{7}{2} (2N(r, f)) + \frac{7}{2} \left(r, \prod_{j=1}^{d} f(z + c_j)^{(k)}\right) + S(r, f) + S(r, g).

$$

$$
T(r, F) \leq (2 + \Gamma_0) T(r, f) + T(r, \prod_{j=1}^{d} f(z + c_j)^{(k)}) + T(r, f^n) - T(r, f^n) - T(r, \prod_{j=1}^{d} f(z + c_j)^{(k)})
\leq (2 + \Gamma_0) T(r, f) + (k + 2)d T(r, f) + (k + 2)d T(r, f) + (7 + \frac{7}{2} \lambda) T(r, f)

+ \frac{1}{2} T(r, f) + \frac{1}{2} T(r, f) + \frac{1}{2} T(r, \prod_{j=1}^{d} f(z + c_j)^{(k)}) - T(r, \prod_{j=1}^{d} f(z + c_j)^{(k)})
+ N_{1+k} \left(r, \frac{1}{\prod_{j=1}^{d} f(z + c_j)^{(k)}}\right) + S(r, f) + S(r, g).

$$

$$
T(r, F) \leq (2 + \Gamma_0) T(r, f) + T(r, f) - T(r, F_1) + (k + 2)d T(r, f) + (2 + \Gamma_0) T(r, g) + (k \lambda + k + 2)d T(r, g)
\leq (2 + \Gamma_0) T(r, f) + (k + 2)d T(r, f) + (2 + \Gamma_0) T(r, g) + (k \lambda + (k + 2)d) T(r, f)
\leq (2 + \Gamma_0) T(r, f) + (k + 2)d T(r, f) + (2 + \Gamma_0) T(r, g) + (k \lambda + (k + 2)d) T(r, f)
\leq (2 + \Gamma_0) T(r, f) + (1 + \Gamma_0) T(r, f) + \frac{1}{2} (1 + \Gamma_0) T(r, f) + \frac{1}{2} (k \lambda + (k + 1)d) T(r, f)
+ S(r, f) + S(r, g)

$$

From Lemma 2.11, we have

$$
(n + m - \lambda) T(r, f) \leq (2 + \Gamma_0 + (k + 2)d) [T(r, f) + T(r, g)] + k \lambda T(r, g)
+ \frac{15}{2} + \frac{7}{2} \lambda + \frac{\Gamma_0}{2} + \frac{k \lambda}{2} + \frac{(1 + k)d}{2} T(r, f) + S(r, f) + S(r, g). \tag{42}
$$

Similarly for $T(r, g)$ we obtain the following

$$
(n + m - \lambda) T(r, g) \leq (2 + \Gamma_0 + (k + 2)d) [T(r, f) + T(r, g)] + k \lambda T(r, f)
+ \frac{15}{2} + \frac{7}{2} \lambda + \frac{\Gamma_0}{2} + \frac{k \lambda}{2} + \frac{(1 + k)d}{2} T(r, g) + S(r, f) + S(r, g). \tag{43}
$$

From (42) and (43), we have

$$
(n + m - \lambda) [T(r, f) + T(r, g)] \leq (2 + \Gamma_0 + (k + 2)d) [T(r, f) + T(r, g)] + [k \lambda + \frac{15}{2} + \frac{7}{2} \lambda + \frac{k \lambda}{2} \tag{44}
+ \frac{\Gamma_0}{2} + \frac{(1 + k)d}{2} ] [T(r, f) + T(r, g)] + S(r, f) + S(r, g)
$$

which is contradiction to \( n > \frac{19}{2} + \frac{19 + 3k}{2} + \frac{3k + 5}{2} - m. \)

**Proof of Theorem 3**

Let \( F, G \) be given by (1), from the assumption of Theorem 3, we know that \( F \) and \( G \) share “(1, 0)”. Let \( H \) be defined as in (3) Suppose that \( H \neq 0 \). Since \( F, G \) share “(1, 0)”, we can get

\[
N(r, \infty; H) \leq \overline{N}(r, \infty; F) + \overline{N}(1, F | \geq 2) + \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, f)
\]  

(45)

and

\[
N_E^1(r, 1; F) \leq N_E^1(r, 1; G) + S(r, f)
\]

\[
N_E^2(r, 1; F) \leq N_E^2(r, 1; G) + S(r, f)
\]

(46)

Using Lemmas 2.8-2.10 and (45) and (46), we get

\[
\overline{N}(r, 1; F) + \overline{N}(r, 1; G) \leq \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_E^2(r, 1; F) + \overline{N}(r, 1; G)
\]

\[
\leq N_E^1(r, 1; F) + N(r, 1; G) - N_L(r, 1; G) + N_{F > 1}(r, 1; G) + N_{G > 1}(r, 1; G)
\]

\[
\leq \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2) + \overline{N}(r, \infty; F) + \overline{N}_G(r, 1; F, G) + T(r, G)
\]

\[- m(r, 1; G) + O(1) - N_L(r, 1; G) + \overline{N}_{F > 1}(r, 1; G) + \overline{N}_{G > 1}(r, 1; G) + N_0(r, 0; F')
\]

\[+ N_0(r, 0; G') + S(r, F) + S(r, G).
\]

(47)

Combining (39) and (47) and by Lemma we can obtain

\[
T(r, F) \leq 6\overline{N}(r, \infty; F) + N(r, 0; F) + 2\overline{N}(r, 0; F) + 2\overline{N}(r, 0; G) + S(r, f)
\]

\[\leq N_2(r, 0; F) + 2\overline{N}(r, 0; F) + 2\overline{N}(2, 0; G) + 6\overline{N}(r, \infty; F)
\]

\[
T(r, F) \leq 2T(r, f) + \Gamma_0 T(r, f) + T(r, \prod_{j=1}^d f(z + c_j)^{x_j})\]

\[+ N_{k+2} \left( \frac{1}{\prod_{j=1}^d f(z + c_j)^{x_j}} \right) + 2[T(r, \prod_{j=1}^d f(z + c_j)^{x_j}) + k\overline{N}(r, \prod_{j=1}^d f(z + c_j)^{x_j})
\]

\[- T(r, \prod_{j=1}^d f(z + c_j)^{x_j}) + (1 + k)dT(r, f)] + 2[T(r, \prod_{j=1}^d g(z + c_j)^{x_j}) + k\overline{N}(r, \prod_{j=1}^d g(z + c_j)^{x_j})
\]

\[- T(r, \prod_{j=1}^d g(z + c_j)^{x_j}) + (1 + k)dT(r, g)] + 6[N(r, f) + N(r, f) + N(r, \prod_{j=1}^d f(z + c_j)^{x_j})] + S(r, f)
\]
\[\leq (2 + \Gamma_0)T(r, f) + T(r, \prod_{j=1}^{d} f(z + c_j)^{r_{i,j}}(k_i) + T(r, f_n) - T(r, f_n) - T(r, \prod_{j=1}^{d} f(z + c_j)^{r_{i,j}}) + (k + 2)dT(r, f) + 2T(r, \prod_{j=1}^{d} f(z + c_j)^{r_{i,j}}) + kN(r, \prod_{j=1}^{d} f(z + c_j)^{r_{i,j}}) - T(r, \prod_{j=1}^{d} g(z + c_j)^{r_{i,j}}) + (1 + k)dT(r, f) + 2[T(r, \prod_{j=1}^{d} g(z + c_j)^{r_{i,j}}) + kN(r, \prod_{j=1}^{d} g(z + c_j)^{r_{i,j}}) - T(r, \prod_{j=1}^{d} g(z + c_j)^{r_{i,j}}) + (1 + k)dT(r, g)] + 12T(r, f) + 6\lambda T(r, f) + S(r, f)\]

\[T(r, F) \leq (2 + \Gamma_0)T(r, f) + T(r, F) - T(r, F_1) + (k + 2)dT(r, f) + (2k\lambda + 2(1 + k)d)T(r, f) + (2k\lambda + 2(1 + k)d)T(r, g) + (12 + 6\lambda)T(r, f) + S(r, f)\]

\[T(r, F_1) \leq (2 + \Gamma_0)T(r, f) + (k + 2)dT(r, f) + 2(k\lambda + (1 + k)d)[T(r, f) + T(r, g)] + (12 + 6\lambda)T(r, f) + S(r, f)\]

From Lemma 2.11 we have

\[(n + m - \lambda)T(r, f) \leq (2 + \Gamma_0 + (k + 2)d)T(r, f) + 2(k\lambda + (1 + k)d)[T(r, f) + T(r, g)] + (12 + 6\lambda)T(r, f) + S(r, f)\]  \hspace{1cm} (49)

Similarly for \(T(r, g)\) we obtain the following

\[(n + m - \lambda)T(r, g) \leq (2 + \Gamma_0 + (k + 2)d)T(r, g) + 2(k\lambda + (1 + k)d)[T(r, f) + T(r, g)] + (12 + 6\lambda)T(r, g) + S(r, g)\]  \hspace{1cm} (50)

from (49) and (50), we have

\[(n + m - \lambda)[T(r, f) + T(r, g)] \leq (2 + \Gamma_0 + (k + 2)d)[T(r, f) + T(r, g)] + 4(k\lambda + (1 + k)d)[T(r, f) + T(r, g)] + (12 + 6\lambda)[T(r, f) + T(r, g)] + S(r, f) + S(r, g)\]  \hspace{1cm} (51)

which is contradiction to \(n > d(5k + 6) + \Gamma_0 + \lambda(4k + 7) + 14 - m\).

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