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UNIQUENESS OF MEROMORPHIC FUNCTIONS CONCERNING PRODUCT OF DIFFERENCE POLYNOMIALS

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Abstract. In this paper, we deal with distribution of zeros of certain types of difference polynomial and in addition to this we investigate the uniqueness of product of difference polynomials $f^nP(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $g^nP(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ which are sharing a fixed point z and f, g share ∞ IM. I obtained some results which extends some recent results of Renukadevi S. Dyavnal and Ashwini M. Hattikal[8].

1. Introduction and main results

A meromorphic function $f(z)$ means meromorphic in the whole complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna Theory [3]. As usual, the abbrevation CM stands for counting multiplicities, while IM means ignoring multiplicities. We use $\rho(f)$ to denote the order of $f(z)$ and $N_p(r, \frac{1}{f-a})$ to denote the counting function of the zeros of $f-a$, where an m-fold zero is counted m times if $m \leq p$ and p times if $m > p$.

A meromorphic function a is called small function with respect to f if $T(r, a) = S(r, f)$ and the order, hyper order of meromorphic function f are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{logT(r, f)}{logr}, \ \ \rho_2(f) = \limsup_{r\to\infty} \frac{loglogT(r, f)}{logr}.$$

In 2010, J.F.Xu, F.Lu and H.X.Yi obtained the following result on meromorphic function sharing a fixed point.

Theorem A.[7] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let n, k be two positive integers with $n > 3k + 10$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2}$ where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^nc^2 = -1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.

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Further, Fang and Qiu investigated uniqueness for the same functions as in the Theorem A, when $k = 1$.

Theorem B.[2] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.

In 2012, Cao and Zhang replaced f' with $f^{(k)}$ and obtained the following theorem.

Theorem C.[1] Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, whose zeros are of multiplicities atleast k, where k is a positive integer. Let $n > max\{2k-1, 4+4/k+4\}$ be a positive integer. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and f and g share ∞ IM, then one of the following two conclusions holds.
(1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$
(2) $f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

Recently, X.B.Zhang reduced the lower bond of n and relax the condition on multiplicity of zeros in Theorem C and proved the below result.

Theorem D.[11] Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and n, k two positive integers with $n > k+6$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and f and g share ∞ IM, then one of the following two conclusions holds.
(1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$;
(2) $f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

In 2016, Renukadevi S. Dyavanal and Ashwini M. Hattikal proved the following theorem.

Theorem E.[8] Let f and g be two transcendental meromorphic functions of hyper order $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let k, n, d, λ be positive integers and $n > max\{2d(k+2) + \lambda(k+3) + 7, \lambda_1, \lambda_2\}$. If $F(z) = f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $G(z) = g(z)^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ share z CM an f, g share ∞ IM, then one of the following two conclusions holds.
(1) $F(z) \equiv G(z)$
(2) $\prod_{j=1}^{d} f(z+c_j)^{s_j} = C_1e^{Cz^2}$, $\prod_{j=1}^{d} g(z+c_j)^{s_j} = C_2e^{-Cz^2}$, where C_1, C_2 and C are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.

We define a difference product of meromorphic function $f(z)$ as follows

$$F(z) = f(z)^n P(f) \left[\prod_{j=1}^{d} f(z+c_j)^{s_j} \right]^{(k)} \quad (1)$$

$$F_1(z) = f(z)^n P(f) \prod_{j=1}^{d} f(z+c_j)^{s_j} \quad (2)$$

where $c_j \in \mathbb{C} \setminus \{0\} (j = 1, 2, ..., d)$ are distinct constants. $n, k, d, s_j (j = 1, 2, ..., d)$ are positive integers and $\lambda = \sum_{j=1}^{d} s_j$.

For $j = 1, 2, 3, ...d, \lambda_1 = \sum_{j=1}^{d} \alpha_j s_j$ and $\lambda_2 = \sum_{j=1}^{d} \beta_j s_j$, where $f(z+c_j)$ and $g(z+c_j)$ have zeros with maximum orders α_j and β_j respectively.

In this article, we prove the theorem on product of difference-differential polynomials sharing a fixed point as follows.

Theorem 1. Let f and g be two transcendental meromorphic functions of hyper order $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let k, n, d, λ be positive integers and $n > max\{2d(k+2) + \lambda(k+4) + \Gamma_0 + 8 - m, \lambda_1, \lambda_2\}$. If $F(z)$ and $G(z)$ share z CM and f, g share ∞ IM, then one of the following two conclusions holds.
(1)$F(z) \equiv G(z)$
(2) $\prod_{j=1}^{d} f(z+c_j)^{s_j} = C_1 e^{Cz^2}, \prod_{j=1}^{d} g(z+c_j)^{s_j} = C_2 e^{-Cz^2}$ wher C_1, C_2 and C are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.

Remark.
If $m = 1$ then Theorem 1 reduces to Theorem E.

Theorem 2. Let f and g be two transcendental meromorphic functions. Let k, n, d, λ be positive integers and $n > max\{\frac{(3k+5)d}{2} + \frac{(9+3k)\lambda}{2} + \frac{3}{2}\Gamma_0 + \frac{19}{2} - m, \lambda_1, \lambda_2\}$. If $F(z)$ and $G(z)$ share "$(\alpha(z), 1)$" and f, g share ∞ IM, then $F(z) \equiv G(z)$.

Theorem 3. Let f and g be two transcendental meromorphic functions. Let k, n, d, λ be positive integers and $n > max\{\frac{(5k+6)d}{2} + (7+4k)\lambda + \Gamma_0 + 14 - m, \lambda_1, \lambda_2\}$. If $F(z)$ and $G(z)$ share "$(\alpha(z), 0)$" and f, g share ∞ IM, then $F(z) \equiv G(z)$.

2. **Lemmas**

In this section we present some lemmas needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \quad (3)$$

Lemma 2.1.[9] Let f and g be two non-constant meromorphic functions, $'a'$ be a finite non-zero constant. If f and g share $'a'$ CM and ∞ IM, then one of the following cases holds.
(1)$T(r, f) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + 3\overline{N}(r, f) + S(r, f) + S(r, g)$.
The same inequality holding for $T(r, g)$;

(2)$fg \equiv a^2$;
(3)$f \equiv g$.

Lemma 2.2.[5] Let $f(z)$ be a transcendental meromorphic functions of hyper order $\rho_2(f) < 1$, and let c be a non-zero complex constant Then we have

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f(z)),$$
$$N(r, f(z+c)) = N(r, f(z)) + S(r, f(z)),$$
$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f(z)).$$

Lemma 2.3.[10] Let f be a non-constant meromorphic function, let $P(f) = a_0 + a_1 f + a_2 f^2 + ... + a_n f^n$, where $a_0, a_1, a_2, ...a_n$ are constants and $a_n \neq 0$. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.4.[10] Let f be a non-constant meromorphic function and p, k be positive integers. Then

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, f) + S(r, f), \tag{4}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f), \tag{5}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f), \tag{6}$$

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \tag{7}$$

Lemma 2.5.[3] Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If

$$N(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{f^{(k)}} = S(r, \frac{f'}{f}),$$

then $f(z) = e^{az+b}$, where $a \neq 0, b$ are constants.

Lemma 2.6.[12] If f, g be two nonconstant meromorphic functions such that they share "$(1, 1)$", then

$$2\overline{N}_L(r, 1; f) + 2\overline{N}_L(r, 1; g) + \overline{N}_E^{(2}(r, 1; f) - \overline{N}_{f>2}(r, 1; g) \leq N(r, 1; g) - \overline{N}(r, 1; g).$$

Lemma 2.7.[12] Let f, g share "$(1, 1)$", Then

$$\overline{N}_{f>2}(r, 1; g) \leq \frac{1}{2}\overline{N}(r, 0; f) + \frac{1}{2}\overline{N}(r, \infty; f) - \frac{1}{2}\overline{N}_0(r, 0; f') + S(r, f).$$

Lemma 2.8.[12] Let f, g be two nonconstant meromorphic functions such that they share "$(1, 0)$". Then $\overline{N}_L(r, 1; f) + 2\overline{N}_L(r, 1; g) + \overline{N}_E^{(2}(r, 1; f) - \overline{N}_{f>2}(r, 1; g) - \overline{N}_{g>2}(r, 1; f) \leq N(r, 1; g) - \overline{N}(r, 1; g)$.

Lemma 2.9.[12] Let f, g share "$(1, 0)$". Then $\overline{N}_L(r, 1; f) \leq \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f)$.

Lemma 2.10.[12] Let f, g share "$(1,0)$". Then

$$(i)\overline{N}_{f>1}(r,1;g) \le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - \overline{N}_0(r,0;f') + S(r,f);$$
$$(ii)\overline{N}_{g>1}(r,1;g) \le \overline{N}(r,0;g) + \overline{N}(r,\infty;f) - \overline{N}_0(r,0;f') + S(r,g);$$

Lemma 2.11. Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and $F_1(z)$ be stated as in (2). Then

$$(n+m-\lambda)T(r,f) + S(r,f) \le T(r,F_1(z)) \le (n+m+\lambda)T(r,f) + S(r,f).$$

Proof. Since f is a meromorphic function with $\rho_2(f) < 1$. From Lemma 2.2 and Lemma 2.3 we have

$$\begin{aligned} T(r,F_1(z)) &\le T(r,f(z)^n) + T(r,P(f)) + T\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + S(r,f) \\ &\le (n+m+\lambda)T(r,f) + S(r,f) \end{aligned} \tag{8}$$

On the other hand, from Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} (n+m+\lambda)T(r,f) &= T(r,f^n f^m f^\lambda) + S(r,f) \\ &= m(r,f^n f^m f^\lambda) + N(r,f^n f^m f^\lambda) + S(r,f) \\ &\le m\left(r, \frac{F_1(z)f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + N\left(r, \frac{F_1(z)f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r,f) \\ &\le m(r,F_1(z)) + N(r,F_1(z)) + T\left(r, \frac{f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r,f) \\ &\le T(r,F_1(z)) + 2\lambda T(r,f) + S(r,f) \\ (n+m+\lambda-2\lambda)T(r,f) &\le T(r,F_1(z)) + S(r,f) \end{aligned} \tag{9}$$

$$\Rightarrow (n+m-\lambda)T(r,f) + S(r,f) \le T(r,F_1(z)).$$

Hence we get Lemma 2.11.

3. **Proof of the Theorem**

Proof of the Theorem 1

$$Let\ \ F^* = \frac{F}{z}\ \ and\ \ G^* = \frac{G}{z} \tag{10}$$

From the hypothesis of the Theorem 1, we have F and G share z CM and f, g share ∞ IM. It follows that F^* and G^* share 1 CM and ∞ IM.

By Lemma 2.1, we arrive at 3 cases as follows.

Case 1. Suppose that case (1) of Lemma 2.1 holds.

$$T(r,F^*) \le N_2(r,\frac{1}{F^*}) + N_2(r,\frac{1}{G^*}) + 3\overline{N}(r,F^*) + S(r,F^*) + S(r,G^*) \tag{11}$$

We deduce from (11) and obtained the following

$$T(r,F) \le N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + 3\overline{N}(r,F) + S(r,F) + S(r,G) \tag{12}$$

From Lemma 2.2 and Lemma 2.6, we have $S(r,F) = S(r,f)$ and $S(r,G) = S(r,g)$.
From (12), we have

$$\begin{aligned} T(r,F) &= N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + 3\overline{N}(r,F) + S(r,f) + S(r,g) \\ &\le N_2\left(r,\frac{1}{f^n}\right) + N_2\left(r,\frac{1}{P(f)}\right) + N_2\left(r,\frac{1}{(\prod_{j=1}^d f(z+c_j)^{s_j})^{(k)}}\right) + N_2\left(r,\frac{1}{g^n}\right) + N_2\left(r,\frac{1}{P(g)}\right) \\ &+ N_2\left(r,\frac{1}{(\prod_{j=1}^d g(z+c_j)^{s_j})^{(k)}}\right) + 3\overline{N}(r,f^n) + 3\overline{N}(r,P(f)) + 3\overline{N}\left(r,\left(\prod_{j=1}^d f(z+c_j)^{s_j}\right)^{(k)}\right) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{13}$$

Using (5) of Lemma 2.4 in (13) we have

$$\begin{aligned} T(r,F) &\le 2T(r,f) + \Gamma_0 T(r,f) + T\left(r,(\prod_{j=1}^d f(z+c_j)^{s_j})^{(k)}\right) - T\left(r,\prod_{j=1}^d f(z+c_j)^{s_j}\right) \\ &+ N_{k+2}\left(r,\frac{1}{\prod_{j=1}^d f(z+c_j)^{s_j}}\right) + 2T(r,g) + \Gamma_0 T(r,g) + T\left(r,(\prod_{j=1}^d g(z+c_j)^{s_j})^{(k)}\right) \\ &- T\left(r,\prod_{j=1}^d g(z+c_j)^{s_j}\right) + N_{k+2}\left(r,\frac{1}{\prod_{j=1}^d g(z+c_j)^{s_j}}\right) + 6N(r,f) + 3N\left(r,\prod_{j=1}^d f(z+c_j)^{s_j}\right) \\ &+ S(r,f) + S(r,g) \\ &\le (2+\Gamma_0)T(r,f) + T\left(r,(\prod_{j=1}^d f(z+c_j)^{s_j})^{(k)}\right) + T(r,f^n) - T(r,f^n) - T\left(r,\prod_{j=1}^d f(z+c_j)^{s_j}\right) \\ &+ (k+2)dT(r,f) + (2+\Gamma_1)T(r,g) + T\left(r,\prod_{j=1}^d g(z+c_j)^{s_j}\right) + k\overline{N}\left(r,\prod_{j=1}^d g(z+c_j)^{s_j}\right) \\ &- T\left(r,\prod_{j=1}^d g(z+c_j)^{s_j}\right) + (k+2)dT(r,g) + 6T(r,f) + 3\lambda T(r,f) + S(r,f) + S(r,g) \\ T(r,F) &\le (2+\Gamma_0)T(r,f) + T(r,F) - T(r,F_1) + (k+2)dT(r,f) + (2+\Gamma_0)T(r,g) + k\lambda T(r,g) \\ &+ (k+2)dT(r,g) + (6+3\lambda)T(r,f) + S(r,f) + S(r,g) \\ T(r,F_1) &\le (2+\Gamma_0)T(r,f) + (k+2)dT(r,f) + (2+\Gamma_0)T(r,g) + (k+2)dT(r,g) + k\lambda T(r,g) \\ &+ (6+3\lambda)T(r,f) + S(r,f) + S(r,g) \\ &\le (2+\Gamma_0)[T(r,f) + T(r,g)] + (k+2)d[T(r,f) + T(r,g)] + k\lambda T(r,g) + (6+3\lambda)T(r,f) \\ &+ S(r,f) + S(r,g) \end{aligned}$$

From Lemma 2.11, we have

$$\begin{aligned}(n+m-\lambda)T(r,f) \le &((k+2)d+2+\Gamma_0)[T(r,f)+T(r,g)]+k\lambda T(r,g)+(6+3\lambda)T(r,f)\\ &+S(r,f)+S(r,g)\end{aligned} \tag{14}$$

Similarly for $T(r,g)$ we obtain the following

$$\begin{aligned}(n+m-\lambda)T(r,g) \le &((k+2)d+2+\Gamma_0)[T(r,f)+T(r,g)]+k\lambda T(r,f)+(6+3\lambda)T(r,g)\\ &+S(r,f)+S(r,g)\end{aligned} \tag{15}$$

From (14) and (15), we have

$$\begin{aligned}(n+m-\lambda)[T(r,f)+T(r,g)] \le &2((k+2)d+2+\Gamma_0)[T(r,f)+T(r,g)]+(k\lambda+6+3\lambda)\\ &[T(r,f)+T(r,g)]+S(r,f)+S(r,g)\end{aligned}$$

Which is contradiction to $n > 2d(k+2)+\Gamma_0+\lambda(k+4)+8-m$.

Case 2. Suppose that $FG \equiv z^2$ holds.

$$i.e., f^nP(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} g^nP(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)} \equiv z^2 \tag{16}$$

Now, (16) can be written as

$$f^nP(f)g^nP(g) \equiv \frac{z^2}{[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}[\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}}$$

By using Lemma 2.2, Lemma 2.3 and (8) of Lemma 2.4, we derive

$$(n+m)[N(r,f)+N(r,g)] \le \lambda[N(r,\frac{1}{f})+N(r,\frac{1}{g})]+kd[N(r,f)+N(r,g)]+S(r,f)+S(r,g) \tag{17}$$

From (16), we can write

$$\frac{1}{f^nP(f)g^nP(g)} = \frac{[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}[\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}}{z^2}$$

Similarly, as (17), we obtain

$$(n+m)[N(r,\frac{1}{f})+N(r,\frac{1}{g})] \le (\lambda+kd)[N(r,f)+N(r,g)]+S(r,f)+S(r,g) \tag{18}$$

From (17) and (18), deduce
$(n+m-(\lambda+2kd))[N(r,f)+N(r,g)]+(n+m-\lambda)[N(r,\frac{1}{f})+N(r,\frac{1}{g})] \le S(r,f)+S(r,g)$
Since $n > 2d(k+2)+\lambda(4+k)+\Gamma_0+8-m$, we have

$$N(r,f)+N(r,g)+N(r,\frac{1}{f})+N(r,\frac{1}{g}) < S(r,f)+S(r,g)$$

Hence, we conclude that f and g have finitely many zeros and poles.

Let z_0 be a pole of f of multiplicity p, then z_0 is pole of f^n of multiplicity np, since f and g share ∞ IM, then z_0 is pole of g of multiplicity q.

If z_0 also zero of $[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}$ and $[\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}$ then we have from (16) that

$$n(p+q) \leq \sum_{j=1}^{d} \alpha_j s_j + \sum_{j=1}^{d} \beta_j s_j - 2k.$$

$\Rightarrow 2n < n(p+q) \leq \sum_{j=1}^{d} \alpha_j s_j + \sum_{j=1}^{d} \beta_j s_j - 2k = \lambda_1 + \lambda_2 - 2k < \lambda_1 + \lambda_2 \leq 2max\{\lambda_1, \lambda_2\}$
$\Rightarrow n < max\{\lambda_1, \lambda_2\}$, which is contradiction to $n > max\{2d(k+2) + \lambda(4+k) + \Gamma_0 + 8 - m, \lambda_1, \lambda_2\}$.
Therefore f has no poles.

Similarly, we can get contradiction for other two cases namely, if z_0 is zero of $[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}$, but not zero of $[\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}$and other way. Therefore f has no poles. Similarly, we get that g also has no poles. By this we conclude that f and g are entire functions and hence $[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}$ and $[\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}$ are entire functions.

Then from (16), we deduce that f and g have no zeros.
Therefore

$$f = e^{\alpha(z)}, g = e^{\beta(z)} \ and$$

$$\prod_{j=1}^{d} f(z+c_j)^{s_j} = \prod_{j=1}^{d} (e^{\alpha(z+c_j)})^{s_j} \ , \ \prod_{j=1}^{d} g(z+c_j)^{s_j} = \prod_{j=1}^{d} (e^{\beta(z+c_j)})^{s_j} \tag{19}$$

where α, β are entire functions with $\rho_2(f) < 1$. Substituting f and g into (16) we get

$$e^{n\alpha(z)} \left[\prod_{j=1}^{d} (e^{\alpha(z+c_j)})^{s_j} \right]^{(k)} e^{n\beta(z)} \left[\prod_{j=1}^{d} (e^{\beta(z+c_j)})^{s_j} \right]^{(k)} \equiv z^2 \tag{20}$$

If $k = 1$, then

$$e^{n\alpha(z)} \left[\prod_{j=1}^{d} (e^{\alpha(z+c_j)})^{s_j} \right]' e^{n\beta(z)} \left[\prod_{j=1}^{d} (e^{\beta(z+c_j)})^{s_j} \right]' \equiv z^2 \tag{21}$$

$$\Rightarrow e^{n(\alpha+\beta)} e^{\sum_{j=1}^{d} (\alpha(z+c_j)+\beta(z+c_j))s_j} \sum_{j=1}^{d} (\alpha'(z+c_j))s_j \sum_{j=1}^{d} (\beta'(z+c_j))s_j \equiv z^2 \tag{22}$$

Since $\alpha(z)$ and $\beta(z)$ are non-constant entire functions, then we have

$$T\left(r, \frac{(\prod_{j=1}^{d} f(z+c_j)^{s_j})'}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) = T\left(r, \frac{(\prod_{j=1}^{d} e^{\alpha(z+c_j)s_j})'}{\prod_{j=1}^{d} e^{\alpha(z+c_j)s_j}}\right) \tag{23}$$

$$T\left(r, \frac{\sum_{j=1}^{d} \alpha'(z+c_j)s_j \prod_{j=1}^{d} e^{\alpha(z+c_j)s_j}}{\prod_{j=1}^{d} e^{\alpha(z+c_j)s_j}}\right) = T\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right) \tag{24}$$

Let

$$\begin{aligned}(n+m)T(r,f) = T(r,f^{n+m}) = T\left(r, \frac{F}{\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}}\right) \leq T(r,F)\\ + T\left(r, \left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right) + S(r,f)\\ \leq T(r,F) + T\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + k\overline{N}\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + S(r,f)\end{aligned}$$

$$(n+m)T(r,f) \leq T(r,F) + (\lambda + kd)T(r,f) + S(r,f)$$

$$(n+m-\lambda-kd)T(r,f) \leq T(r,F) + S(r,f) \tag{25}$$

We obtain from (24) that

$$T(r,f) = O(T(r,F)) \tag{26}$$

as $r \in E$ and $r \to \infty$, where $E \subset (0,+\infty)$ is some subset of finite linear measure. On the other hand, we have

$$\begin{aligned}T(r,F) = T\left(r, f^n P(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}\right)\\ \leq nT(r,f) + mT(r,f) + \lambda T(r,f) + k\overline{N}\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + S(r,f)\\ \leq (n+m+kd+\lambda)T(r,f) + S(r,f)\\ \Rightarrow T(r,F) = O(T(r,f))\end{aligned} \tag{27}$$

as $r \in E$ and $r \to \infty$, where $E \subset (0,+\infty)$ is some subset of finite linear measure.

Thus from (25),(26) and the standard reasoning of removing exceptional set we deduce $\rho(f) = \rho(F)$. Similarly, we have $\rho(g) = \rho(G)$. It follows from (16) that $\rho(F) = \rho(G)$. Hence we get $\rho(f) = \rho(g)$.

We deduce that either both α and β are polynomials or both α and β are transcendental entire functions. Moreover, we have

$$N\left(r, \frac{1}{(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}}\right) \leq N\left(r, \frac{1}{z^2}\right) = O(logr) \tag{28}$$

From (27) and (19), we have

$$N\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + N\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + N\left(r, \frac{1}{(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}}\right) = O(logr).$$

If $k \geq 2$, then it follows from (23),(27) and Lemma 2.5 that $\sum_{j=1}^{d} \alpha'(z+c_j)s_j$ is a polynomial and therefore we have $\alpha(z)$ is a non-constant polynomial.

Similarly, we can deduce that $\beta(z)$ is also a non-constant polynomial. From this, we deduce from (19) that

$$\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)} = e^{\sum_{j=1}^{d} \alpha(z+c_j)s_j} \left[P_{k-1}(\alpha'(z+c_j)) + \left(\sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)^k\right]$$

$$\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)} = e^{\sum_{j=1}^{d} \beta(z+c_j)s_j} \left[Q_{k-1}(\alpha'(z+c_j)) + \left(\sum_{j=1}^{d} \beta'(z+c_j)s_j\right)^k\right]$$

Where P_{k-1} and Q_{k-1} are difference-differential polynomials in $\alpha'(z+c_j)$ with degree at most $k-1$.
Then (20) becomes

$$e^{n(\alpha+\beta)} e^{\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j} \left[\sum_{j=1}^{d} \alpha^{(k)}(z+c_j)s_j + \left(\sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)^k\right]$$

$$\left[\sum_{j=1}^{d} \beta^{(k)}(z+c_j)s_j + \left(\sum_{j=1}^{d} \beta'(z+c_j)s_j\right)^k\right] = z^2 \tag{29}$$

We deduce from (28) that $\alpha(z) + \beta(z) \equiv C$ for a constant C.

If $k = 1$, from (22), we have

$$e^{n(\alpha+\beta)+\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j} \left[\sum_{j=1}^{d} (\alpha'(z+c_j))s_j \sum_{j=1}^{d} (\beta'(z+c_j))s_j\right] \equiv z^2. \tag{30}$$

Next, we let $\alpha + \beta = \gamma$ and suppose that α, β both are transcendental entire functions.

If γ is a constant, then $\alpha' + \beta' = 0$ and $\sum_{j=1}^{d} \alpha'(z+c_j) = -\sum_{j=1}^{d} \beta'(z+c_j)$.
From (29) we have

$$e^{n(\alpha+\beta)+\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j} \left\{-\left[\sum_{j=1}^{d} \alpha'(z+c_j)s_j\right]^2\right\} \equiv z^2$$

$$e^{n\gamma+d\gamma} \left\{\left[-\sum_{j=1}^{d} \alpha'(z+c_j)\right]^2\right\} = z^2 \tag{31}$$

Which implies that α' is a non-constant polynomial of degree 1. This together with $\alpha' + \beta' = 0$ which implies that β' is also non-constant polynomial of degree 1. Which is contradiction to α, β both are transcendental entire functions.

If γ is not a constant, then we have

$\alpha + \beta = \gamma$ and $\sum_{j=1}^{d} \alpha(z + c_j)s_j + \sum_{j=1}^{d} \beta(z + c_j)s_j = \sum_{j=1}^{d} \gamma(z + c_j)s_j$

From (29) we have

$$\left[\sum_{j=1}^{d} \alpha'(z + c_j)s_j\right]\left[\sum_{j=1}^{d} \gamma'(z + c_j)s_j - \sum_{j=1}^{d} \alpha'(z + c_j)s_j\right] e^{n\gamma+\sum_{j=1}^{d} \gamma(z+c_j)s_j} = z^2 \quad (32)$$

Since

$$\begin{aligned} T\left(r, \sum_{j=1}^{d} \gamma'(z + c_j)s_j\right) &= m\left(r, \sum_{j=1}^{d} \gamma'(z + c_j)s_j\right) + N\left(r, \sum_{j=1}^{d} \gamma'(z + c_j)s_j\right) \\ &\leq m\left(r, \frac{(e^{\sum_{j=1}^{d} \gamma'(z+c_j)s_j})'}{e^{\sum_{j=1}^{d} \gamma'(z+c_j)s_j}}\right) + O(1) = S(r, e^{\sum_{j=1}^{d} \gamma(z+c_j)s_j}) \end{aligned} \quad (33)$$

And also we have

$$\begin{aligned} T\left(r, n\gamma' + \sum_{j=1}^{d} \gamma'(z + c_j)s_j\right) &= m\left(r, n\gamma' + \sum_{j=1}^{d} \gamma'(z + c_j)s_j\right) + N\left(r, n\gamma' + \sum_{j=1}^{d} \gamma'(z + c_j)s_j\right) \\ &\leq m\left(r, \frac{(e^{\sum_{j=1}^{d} \gamma'(z+c_j)s_j})'}{e^{\sum_{j=1}^{d} \gamma'(z+c_j)s_j}}\right) + O(1) = S\left(r, e^{n\gamma+\sum_{j=1}^{d} \gamma(z+c_j)s_j}\right) \end{aligned} \quad (34)$$

From (31), we have

$$\begin{aligned} T\left(r, e^{n\gamma+\sum_{j=1}^{d} \gamma(z+c_j)s_j}\right) &\leq T\left(r, \frac{z^2}{\sum_{j=1}^{d} \alpha'(z + c_j)s_j[\sum_{j=1}^{d} \gamma'(z + c_j)s_j - \sum_{j=1}^{d} \alpha'(z + c_j)s_j]}\right) + O(1) \\ &= T(r, z^2) + T\left(r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j\left[\sum_{j=1}^{d} \gamma'(z + c_j)s_j - \sum_{j=1}^{d} \alpha'(z + c_j)s_j\right]\right) + O(1) \\ &\leq 2logr + 2T\left(r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j\right) + O(1) \end{aligned}$$

$$\Rightarrow T\left(r, e^{n\gamma+\sum_{j=1}^{d} \gamma(z+c_j)s_j}\right) \leq O\left(T\left(r, \sum_{j=1}^{d} \alpha'(z + c_j)s_j\right)\right) \quad (35)$$

Similarly, we have

$$T\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right) \leq O\left(T\left(r, e^{n\gamma + \sum_{j=1}^{d} \gamma(z+c_j)s_j}\right)\right) \tag{36}$$

Thus, from (32)-(35) we have

$$T(r, n\gamma' + \sum_{j=1}^{d} \gamma'(z+c_j)s_j) = S\left(r, e^{n\gamma + \sum_{j=1}^{d} \gamma(z+c_j)s_j}\right) = S\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)$$

By the second fundamental theorem and (31), we have

$$T\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right) \leq \overline{N}\left(r, \frac{1}{\sum_{j=1}^{d} \alpha'(z+c_j)s_j}\right) + \overline{N}\left(r, \frac{1}{\sum_{j=1}^{d} \alpha'(z+c_j)s_j - \sum_{j=1}^{d} \gamma'(z+c_j)s_j}\right)$$
$$+ S\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right) \leq O(logr) + S\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)$$

This implies $\sum_{j=1}^{d} \alpha'(z+c_j)s_j$ is a polynomial, which leads to $\alpha'(z)$ is a polynomial. Which contradicts that $\alpha(z)$ is a transcendental entire function.
Thus α and β are both polynomials and $\alpha(z) + \beta(z) \equiv C$ for a constant C.
Hence from (28) and using $\alpha + \beta = C$ we get
$(-1)^k \left(\sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)^{2k} = z^2 + P_{2k-1}(\alpha'(z+c_j)s_j)$ for $j = 1, 2, ..., d$.
Where P_{2k-1} is difference-differential polynomial in $\alpha'(z+c_j)s_j$ of degree at most $2k-1$. From (36) we have

$$2kT\left(r, \sum_{j=1}^{d} \alpha'(z+c_j)s_j\right) = 2logr + S\left(r, \alpha'(z+c_j)s_j\right)$$

From (3.28), we can see that $\sum_{j=1}^{d} \alpha'(z+c_j)s_j$ is a non-constant polynomial of degree 1 and $k = 1$.
Which implies,

$$\sum_{j=1}^{d} \alpha'(z+c_j)s_j = zl_1$$

Since $\alpha' + \beta' = 0$, we get $\sum_{j=1}^{d} \beta'(z+c_j)s_j = -\sum_{j=1}^{d} \alpha'(z+c_j)s_j$. Which implies $\sum_{j=1}^{d} \beta'(z+c_j)s_j$ is also a non-constant polynomial of degree 1. Hence we have

$$\sum_{j=1}^{d} \beta'(z+c_j)s_j = zl_2$$

Hence, we get

$$\prod_{j=1}^{d} f(z+c_j)^{s_j} = C_1 e^{Cz^2}$$

Similarly, we have

$$\prod_{j=1}^{d} g(z+c_j)^{s_j} = C_2 e^{-Cz^2}$$

where C_1, C_2 and C are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.

This proves the conclusion (2) of Theorem 1.

Case 3. If $F \equiv G$
i.e, $f^nP(f)\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} \equiv g^nP(g)\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$

This proves the conclusion (1) of Theorem 1.

Proof of Theorem 2
Let F, G be given by from the assumption of Theorem 2, we know that F and G share "$(1,1)$".
Let H be defined as in (3) Suppose that $H \not\equiv 0$. Since F,G share "$(1,1)$", we can get

$$\begin{aligned} N(r,\infty;H) \le \overline{N}(r,\infty;F) + \overline{N}(r,1;F\ge 2) + \overline{N}(r,0;F\ge 2) + \overline{N}(r,0;G\mid\ge 2) + \overline{N}_0(r,0;F') \\ + \overline{N}_0(r,0;G') + S(r,f) \end{aligned} \tag{37}$$

and

$$N(r,1;F\mid=1) \le N(r,0;H) + S(r,f) \le N(r,\infty;H) + S(r,f) \tag{38}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.
By the second fundamental theorem, we see that

$$\begin{aligned} T(r,F) + T(r,G) \le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;F) + \overline{N}(r,1;G) \\ - N_0(r,0;F') - N_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{39}$$

Using Lemmas 2.6 and 2.7, (37) and (38) we can get

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1;G) &\le N(r,1;F\mid=1) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}(r,1;G) \\ &\le N(r,1;F\mid=1) + N(r,1;G) - \overline{N}_L(r,1;F) - \overline{N}_L(r,1;G) + \overline{N}_{F>2}(r,1;G) \\ &\le \overline{N}(r,1;F\mid\ge 2) + \overline{N}(r,1;G\mid\ge 2) + \overline{N}(r,\infty;F) + \overline{N}_*(r,1;F,G) + T(r,G) \\ &- m(r,1;G) + O(1) + \frac{1}{2}\overline{N}(r,\infty;F) - \overline{N}_L(r,1;F) - \overline{N}_L(r,1;G) + \frac{1}{2}\overline{N}(r,0,F) \\ &+ N_0(r,0;F') + N_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{40}$$

Combining (39) and (40), we can obtain

$$T(r,F) \le \frac{7}{2}\overline{N}(r,\infty;F) + N_2(r,0;F) + N_2(r,0;G) + \frac{1}{2}\overline{N}(r,0;F) \tag{41}$$

By the definition of F, G we have

$$\begin{aligned}
T(r,F) \leq\ & 2T(r,f) + \Gamma_0 T(r,f) + T(r,(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}) - T(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}) \\
& + N_{k+2}\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + 2T(r,g) + \Gamma_0 T(r,g) + T(r,\prod_{j=1}^{d} g(z+c_j)^{s_j})^{(k)} \\
& - T(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}) + N_{k+2}\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + \frac{1}{2}\overline{N}(r,\frac{1}{f^n}) + \frac{1}{2}\overline{N}(r,\frac{1}{P(f)}) \\
& + \overline{N}(r,\frac{1}{\prod_{j=1}^{d} g(z+c_j)^{s_j})^{(k)}}) + \frac{7}{2}(2N(r,f)) + \frac{7}{2}N\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + S(r,f) + S(r,g).
\end{aligned}$$

$$\begin{aligned}
T(r,F) \leq\ & (2+\Gamma_0)T(r,f) + T(r,(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}) + T(r,f^n) - T(r,f^n) - T(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}) \\
& + (k+2)dT(r,f) + (2+\Gamma_0)T(r,g) + k\lambda T(r,g) + (k+2)dT(r,g) + (7+\frac{7}{2}\lambda)T(r,f) \\
& + \frac{1}{2}T(r,f) + \frac{\Gamma_0}{2}T(r,f) + \frac{1}{2}[T(r,(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}) - T(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}) \\
& + N_{1+k}\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}}\right)] + S(r,f) + S(r,g).
\end{aligned}$$

$$\begin{aligned}
T(r,F) \leq\ & (2+\Gamma_0)T(r,f) + T(r,F) - T(r,F_1) + (k+2)dT(r,f) + (2+\Gamma_0)T(r,g) + (k\lambda+k+2)dT(r,g) \\
& + (7+\frac{7}{2}\lambda)T(r,f) + \frac{1}{2}(1+\Gamma_0)T(r,f) + \frac{1}{2}(k\lambda+(1+k)d)T(r,f) + S(r,f) + S(r,g) \\
\leq\ & (2+\Gamma_0)T(r,f) + (k+2)dT(r,f) + (2+\Gamma_0)T(r,g) + (k\lambda+(k+2)d)T(r,g) \\
& + (7+\frac{7}{2}\lambda)T(r,f) + \frac{1}{2}(1+\Gamma_0)T(r,f) + \frac{1}{2}(1+\Gamma_0)T(r,f) + \frac{1}{2}(k\lambda+(1+k)d)T(r,f) \\
& + S(r,f) + S(r,g)
\end{aligned}$$

From Lemma 2.11, we have

$$\begin{aligned}
(n+m-\lambda)T(r,f) \leq\ & (2+\Gamma_0+(k+2)d)[T(r,f)+T(r,g)] + k\lambda T(r,g) \\
& + \left[\frac{15}{2} + \frac{7}{2}\lambda + \frac{\Gamma_0}{2} + \frac{k\lambda}{2} + \frac{(1+k)d}{2}\right] T(r,f) + S(r,f) + S(r,g)
\end{aligned} \tag{42}$$

Similarly for $T(r,g)$ we obtain the following

$$\begin{aligned}
(n+m-\lambda)T(r,g) \leq\ & (2+\Gamma_0+(k+2)d)[T(r,f)+T(r,g)] + k\lambda T(r,f) \\
& + \left[\frac{15}{2} + \frac{7}{2}\lambda + \frac{\Gamma_0}{2} + \frac{k\lambda}{2} + \frac{(1+k)d}{2}\right] T(r,g) + S(r,f) + S(r,g)
\end{aligned} \tag{43}$$

From (42) and (43), we have

$$\begin{aligned}
(n+m-\lambda)[T(r,f)+T(r,g)] \leq\ & (2+\Gamma_0+(k+2)d)[T(r,f)+T(r,g)] + [k\lambda + \frac{15}{2} + \frac{7}{2}\lambda + \frac{k}{2}\lambda \\
& + \frac{\Gamma_0}{2} + \frac{(1+k)d}{2}][T(r,f)+T(r,g)] + S(r,f) + S(r,g)
\end{aligned} \tag{44}$$

which is contradiction to $n > \frac{19}{2} + \frac{(9+3k)\lambda}{2} + \frac{3}{2}\Gamma_0 + \frac{(3k+5)d}{2} - m$.

Proof of Theorem 3

Let F, G be given by (1), from the assumption of Theorem 3, we know that F and G share "$(1,0)$".
Let H be defined as in (3) Suppose that $H \not\equiv 0$. Since F, G share "$(1,0)$", we can get

$$\begin{aligned} N(r,\infty;H) \le \overline{N}(r,\infty;F) + \overline{N}(1,F \mid\ge 2) + \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) \\ + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) \end{aligned} \tag{45}$$

and

$$\begin{aligned} N_E^{1)}(r,1;F) &\le N_E^{1)}(r,1;G) + S(r,f) \\ N_E^{(2}(r,1;F) &\le N_E^{(2}(r,1;G) + S(r,f) \\ N_E^{1)}(r,1;F) &\le N(r,\infty;H) + S(r,f) \end{aligned} \tag{46}$$

Using Lemmas 2.8-2.10 and (45) and (46), we get

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1;G) &\le \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) + \overline{N}(r,1;G) \\ &\le N_E^{1)}(r,1;F) + N(r,1;G) - \overline{N}_L(r,1;G) + \overline{N}_{F>1}(r,1;G) + \overline{N}_{G>1}(r,1;G) \\ &\le \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,0;G \mid\ge 2) + \overline{N}(r,\infty;F) + \overline{N}_*(r,1;F,G) + T(r,G) \\ &\quad - m(r,1;G) + O(1) - \overline{N}_L(r,1;G) + \overline{N}_{F>1}(r,1;G) + \overline{N}_{G>1}(r,1;G) + N_0(r,0;F') \\ &\quad + N_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{47}$$

Combining (39) and (47) and by Lemma we can obtain

$$\begin{aligned} T(r,F) &\le 6\overline{N}(r,\infty;F) + N_2(r,0;F) + 2\overline{N}(r,0;F) + 2\overline{N}_{(2}(r,0;G) + S(r,f) \\ &\le N_2(r,0;F) + 2\overline{N}(r,0;F) + 2\overline{N}_{(2}(r,0;G) + 6\overline{N}(r,\infty;F) \end{aligned}$$

$$\begin{aligned} T(r,F) \le{}& 2T(r,f) + \Gamma_0 T(r,f) + T(r,(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}) - T(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}) \\ &+ N_{k+2}\left(r,\frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + 2[T(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}) + k\overline{N}(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}) \\ &- T(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}) + (1+k)dT(r,f)] + 2[T(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}) + k\overline{N}(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}) \\ &- T(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}) + (1+k)dT(r,g)] + 6[N(r,f) + N(r,f) + N(r,\prod_{j=1}^{d} f(z+c_j)^{s_j})] + S(r,f) \end{aligned}$$

$$\leq (2+\Gamma_0)T(r,f) + T(r,(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}) + T(r,f^n) - T(r,f^n) - T(r,\prod_{j=1}^{d} f(z+c_j)^{s_j})$$

$$+ (k+2)dT(r,f) + 2[T(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}) + k\overline{N}(r,\prod_{j=1}^{d} f(z+c_j)^{s_j})$$

$$- T(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}) + (1+k)dT(r,f)] + 2[T(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}) + k\overline{N}(r,\prod_{j=1}^{d} g(z+c_j)^{s_j})$$

$$- T(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}) + (1+k)dT(r,g)] + 12T(r,f) + 6\lambda T(r,f) + S(r,f)$$

$$\begin{aligned} T(r,F) &\leq (2+\Gamma_0)T(r,f) + T(r,F) - T(r,F_1) + (k+2)dT(r,f) + (2k\lambda + 2(1+k)d)T(r,f) \\ &+ (2k\lambda + 2(1+k)d)T(r,g) + (12+6\lambda)T(r,f) + S(r,f) \\ T(r,F_1) &\leq (2+\Gamma_0)T(r,f) + (k+2)dT(r,f) + 2(k\lambda + (1+k)d)[T(r,f) + T(r,g)] \\ &+ (12+6\lambda)T(r,f) + S(r,f) \end{aligned} \tag{48}$$

From Lemma 2.11 we have

$$\begin{aligned} (n+m-\lambda)T(r,f) &\leq (2+\Gamma_0 + (k+2)d)T(r,f) + 2(k\lambda + (1+k)d)[T(r,f) + T(r,g)] \\ &+ (12+6\lambda)T(r,f) + S(r,f) \end{aligned} \tag{49}$$

Similarly for $T(r,g)$ we obtain the following

$$\begin{aligned} (n+m-\lambda)T(r,g) &\leq (2+\Gamma_0 + (k+2)d)T(r,g) + 2(k\lambda + (1+k)d)[T(r,f) + T(r,g)] \\ &+ (12+6\lambda)T(r,g) + S(r,g) \end{aligned} \tag{50}$$

from (49) and (50), we have

$$\begin{aligned} (n+m-\lambda)[T(r,f) + T(r,g)] &\leq (2+\Gamma_0 + (k+2)d)[T(r,f) + T(r,g)] + 4(k\lambda + (1+k)d) \\ &[T(r,f) + T(r,g)] + (12+6\lambda)[T(r,f) + T(r,g)] + S(r,f) + S(r,g) \end{aligned} \tag{51}$$

which is contradiction to $n > d(5k+6) + \Gamma_0 + \lambda(4k+7) + 14 - m$.

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