OSCILLATION CRITERIA FOR DELAY DYNAMIC EQUATIONS
ON TIME SCALES

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Abstract. The present paper is dedicated to examine the oscillatory behavior of all solutions of first order delay dynamic equation
\[ x(\Delta t) + p(t)x(t(\tau(t))) = 0 \quad \text{for } t \in [t_0, \infty)_T. \tag{*} \]
We obtain a new oscillation criterion for this equation on time scale \( T \). In particular, we show that all solutions of (*\) oscillate under the condition
\[ M > 2m + \frac{2}{\lambda_1} - 1 \]
is satisfied when \( M < 1 \) and \( 0 < m \leq \frac{1}{2} \) such that the numbers \( m \) and \( M \) are defined as
\[ m = \liminf_{t \to \infty} \frac{1}{\tau(t)} \int_{\tau(t)}^t p(s) \Delta s \]
and
\[ M = \limsup_{t \to \infty} \frac{1}{\tau(t)} \int_{\tau(t)}^t p(s) \Delta s \]
where \( \lambda_1 \in [1, e] \) is the unique root of the equation \( \lambda = e^{m \lambda} \).

1. Introduction

In this paper, we study the oscillatory behavior of solutions of the first-order delay dynamic equation
\[ x(\Delta t) + p(t)x(t(\tau(t))) = 0 \quad \text{for } t \in [t_0, \infty)_T, \tag{1.1} \]
where \( T \) is a time scale that is unbounded above with \( t_0 \in T \), \( p \in C_{rd}([t_0, \infty)_T, \mathbb{R}^+) \), \( \tau \in C_{rd}([t_0, \infty)_T, T) \) is nondecreasing on \( T \) and
\[ \tau(t) \leq t, \lim_{t \to \infty} \tau(t) = \infty \quad \text{for } t \in T, \tag{1.2} \]
and \( \sup T = \infty \).

For a reader not familiar to the time scale calculus, it will be helpful to introduce the following introductory information. A time scale, which inherits the standard topology on \( \mathbb{R} \), is a nonempty closed subset of reals. In this paper, a time scale

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will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma := \inf(t, \infty)_{\mathbb{T}}$ while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho := \sup(-\infty, t)_{\mathbb{T}}$, and the graininess function $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined as $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and/or equivalently $\mu(t) = 0$ holds; otherwise it is called right-scattered, and similarly left-dense and left scattered points are defined with respect to the backward jump operator. We also need the set $\mathbb{T}^\kappa$ as follows: If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be $\Delta$-differentiable at the point $t \in \mathbb{T}^\kappa$ provided that there exists $f^\Delta(t)$ such that for every $\varepsilon > 0$ there exists a neighborhood $U$ of $t$ such that

$$||f(\sigma(t) - f(s)) - f^\Delta(t)\sigma(t) - s|| \leq \varepsilon |\sigma(t) - s| \text{ for all } s \in U.$$  

We shall mean the $\Delta$-derivative of a function when we only say derivative if it is not mentioned explicitly. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$, and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$. For $s, t \in \mathbb{T}$ and a function $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, the $\Delta$-integral is defined by

$$\int^t_s f(\eta)\Delta(\eta) = F(t) - F(s)$$

where $F \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ is an anti-derivative of $f$, i.e., $F^\Delta = f$ on $\mathbb{T}^\kappa$. Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$ then $F$ is defined by

$$F(t) = \int^t_{t_0} f(\eta)\Delta(\eta) \text{ for } t \in \mathbb{T}$$

which is an antiderivative of $f$. And, for $t \in \mathbb{T}^\kappa$

$$\int^\sigma_t f(\eta)\Delta(\eta) = \mu(t)f(t).$$

It is obvious that if $f^\Delta \geq 0$, then $f$ is nondecreasing.

A function $f \in C_{rd}(\mathbb{T}, \mathbb{C})$ is called regressive if $1 + f\mu \neq 0$ on $\mathbb{T}^\kappa$, and $f \in C_{rd}(\mathbb{T}, \mathbb{C})$ is called positively regressive if $1 + f\mu > 0$ on $\mathbb{T}^\kappa$. The set of regressive functions and the set of positively regressive functions are denoted by $\mathcal{R}(\mathbb{T}, \mathbb{C})$ and $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$ respectively. $\mathcal{R}^-(\mathbb{T}, \mathbb{R})$ is defined similarly. For simplicity, we denote the set of regressive constants by $\mathcal{R}_c(\mathbb{T}, \mathbb{C})$. Similarly, we define the sets $\mathcal{R}^+_c(\mathbb{T}, \mathbb{R})$ and $\mathcal{R}^-_c(\mathbb{T}, \mathbb{R})$.

A function $x : \mathbb{T} \to \mathbb{R}$ is called a solution of the equation (1.1), if $x(t)$ is delta differentiable for $t \in \mathbb{T}^\kappa$ and it satisfies the equation (1.1) for $t \in \mathbb{T}$. We say that a solution $x$ of equation (1.1) has a generalized zero at $t$ if $x(t) = 0$ or $\mu(t) > 0$ and $x(t)x(\sigma(t)) < 0$. Let $\sup \mathbb{T} = \infty$ and then a nontrivial solution $x$ of equation (1.1) is called oscillatory on $[t, \infty)$ if it has arbitrarily large generalized zeros in $[t, \infty)$. 


In recent years, there has been an increasing interest in the oscillation of solutions of some dynamic equations. See [1-27] and the references cited therein. However, few papers ([3,25-27]) deal with only delay dynamic equations even in the case of first order linear equations.

Supposing $T = \mathbb{R}$, then Eq. (1.1) is reduced to the first order delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0.$$  \hspace{1cm} (1.3)

Many authors studied the oscillatory behavior of Eq. (1.3), ([4, 8-11, 13-16, 18-20, 23-24]).

Similarly, in case that $T = \mathbb{N}$, Eq. (1.1) turns into

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, \ldots.$$ \hspace{1cm} (1.4)

Recently, many studies are performed on the oscillation of solutions of Eq. (1.4), [5-7, 21-22].

In 2002, Zhang and Deng [26], studied the oscillatory behavior of solutions of the following delay differential equation on time scales

$$x^\Delta(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad t \in T.$$  

where $p \in C_r(T, \mathbb{R}^+)$, $\tau \in C_r(T, T)$ and $\tau(t) < t$ for $t \in T$, and $\sup T = \infty$. They proved the following result by the help of cylinder transforms.

**Theorem 1.** Define

$$\alpha = \limsup_{t_0 \to \infty} \sup_{\lambda \in E} \{ \lambda \exp_{-\lambda p}(\tau(t), t) \}$$ \hspace{1cm} (1.5)

where

$$\exp_{-\lambda p}(\tau(t), t) = \exp \int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s)) \Delta s,$$

$E = \{ \lambda : \lambda > 0, \ 1 - \lambda p(t) \mu(t) > 0 \}$, and

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & \text{if } h \neq 0 \\ z, & \text{if } h = 0 \end{cases}.$$

If $\alpha < 1$, then all solutions of Eq.(1.1) are oscillatory.

In 2005, Bohner [3] gave the following result by using exponential functions notation for any time scale $T$.

**Theorem 2.** If Eq.(1.1) has an eventually positive solution, then $\alpha$ satisfies the condition $\alpha \geq 1$ defined by (1.5).

Following these studies, Şahiner and Stavroulakis [25] gave the following result for Eq.(1.1).

**Theorem 3.** Assume that there exists a positive constant $L$ such that

$$\liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s > L$$ \hspace{1cm} (1.6)

and

$$\limsup_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s > 1 - \frac{L^2}{4}.$$ \hspace{1cm} (1.7)
Then Eq.(1.1) is oscillatory.

In 2005, the following criterias were given by Zhang et al. [27] for all solutions of Eq.(1.1) to be oscillatory.

**Theorem 4.** Assume that (1.2) and the following inequality holds
\[
\limsup_{t \to \infty} \int_{\tau(t)}^{\sigma(t)} p(s) \Delta s > 1,
\]
then all solutions of Eq.(1.1) are oscillatory.

**Theorem 5.** Assume that (1.2) holds and
\[
m \in [0, \frac{1}{e}],
\]
Furthermore,
\[
\limsup_{t \to \infty} \int_{\tau(t)}^{\sigma(t)} p(s) \Delta s > \frac{1}{\lambda_1} - \frac{1 - m - \sqrt{1 - 2m - m^2}}{2},
\]
where \( \lambda_1 \in [1, e] \) is the unique root of the equation \( \lambda = e^{m\lambda} \), then all solutions of Eq.(1.1) are oscillatory.

This work is inspired by [27], [22] and [14]. In this paper, we use these studies to find a new criteria for all solutions of Eq.(1.1) to be oscillatory. The purpose of the present paper is essentially to extend these results to the dynamic equations on time scale \( T \). Finally, two examples are given for certain cases.

2. Main Results

In this section, we give an oscillatory criteria for all solutions of Eq.(1.1).

Here, we set
\[
m = \liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s) \Delta s.
\]

**Lemma 1** ([27, Lemma 2.3]). Let \( x(t) \) be an eventually positive solution of Eq.(1.1) and \( m \in [0, \frac{1}{e}] \). Then
\[
\liminf_{t \to \infty} \frac{x(\tau(t))}{x(t)} \geq \lambda_1,
\]
where \( \lambda_1 \in [1, e] \) is the unique root of the equation \( \lambda = e^{m\lambda} \).

**Lemma 2.** Let \( x(t) \) be an eventually positive solution of Eq.(1.1) and \( m \in [0, \frac{1}{e}] \). Assume that \( \tau(t) \) is nondecreasing and there exists \( \theta > 0 \) such that
\[
\int_{\tau(u)}^{\tau(t)} p(s) \Delta s \geq \theta \int_{u}^{t} p(s) \Delta s \quad \text{for all } \tau(t) \leq u \leq t.
\]
Then
\[
\liminf_{t \to \infty} \frac{x(\sigma(t))}{x(\tau(t))} \geq \frac{1 - m - \sqrt{(1 - m)^2 - 4A}}{2},
\]
where \( A \) is given by
\[
A = \frac{e^{\lambda_1 \theta m} - \lambda_1 \theta m - 1}{(\lambda_1 \theta)^2}
\]
and \( \lambda_1 \in [1, e] \) is the unique root of the equation \( \lambda = e^{m\lambda} \).
Proof. If \( m = 0 \), then obviously inequality (2.4) holds.

If \( m \neq 0 \), then let \( x(t) \) be eventually positive solution of Eq.(1.1). Define the functions \( \varphi, \underline{p}, \tau \) on \( \mathbb{R} \) as follows

\[
\varphi(t) = \begin{cases} 
    x(t), & t \in \mathbb{T}, \\
    x(s) + (x(\sigma(s)) - x(s)) \frac{t-s}{\sigma(t)-s}, & s < t < \sigma(s), \ s \in \mathbb{T},
\end{cases}
\]

\[
\underline{p}(t) = \begin{cases} 
    p(t), & t \in \mathbb{T}, \\
    p(s), & s < t < \sigma(s), \ s \in \mathbb{T},
\end{cases}
\]

\[
\tau(t) = \begin{cases} 
    \tau(t), & t \in \mathbb{T}, \\
    \tau(s), & s < t < \sigma(s), \ s \in \mathbb{T}.
\end{cases}
\]

Clearly, these functions are well defined under the assumption on \( \mathbb{T} \). It is easy to see that the function \( \varphi \) is continuous, nonincreasing and eventually positive on \( \mathbb{R} \), and the function \( \tau \) is nondecreasing on \( \mathbb{R} \) with \( \lim_{t \to \infty} \tau(t) = \infty, \ t \in \mathbb{R} \). And \( \underline{p}(t) \geq 0 \), for \( t \geq t_0, \ t \in \mathbb{R} \).

From the proof of Lemma 2.4 in [27] we know that \( \varphi \) is a solution of the following differential equation

\[
\varphi'_+(t) + \underline{p}(t)\varphi(\tau(t)) = 0, \quad t \geq t_0, \ t \in \mathbb{R}, \quad (2.6)
\]

where \( \varphi'_+(t) \) means the right derivative of \( \varphi \) at \( t \).

On the other hand, from (2.3), we get

\[
\frac{\tau(t)}{\tau(u)} \int_{\tau(u)}^{\tau(t)} \underline{p}(s) \Delta s \geq \theta \int_{\tau(u)}^{\tau(t)} \underline{p}(s) \Delta s \quad \text{for all} \quad \tau(t) \leq u \leq t.
\]

Therefore, from Lemma 2 in [14] we have

\[
\varphi(t) \geq \frac{1}{2} \left[ 1 - m - \sqrt{(1 - m)^2 - 4A} \right] \varphi(\tau(t)).
\]

for \( t \in \mathbb{R} \).

If \( s \leq t < \sigma(s), \ s \in \mathbb{T} \), then we have \( \varphi(\tau(t)) = \varphi(\tau(s)) = x(\tau(s)) \). So, we get

\[
\varphi(t) \geq \frac{1}{2} \left[ 1 - m - \sqrt{(1 - m)^2 - 4A} \right] x(\tau(s)).
\]

Let \( t \to \sigma(s) - 0 \) and from the continuity of \( \varphi \),

\[
\varphi(\sigma(s)) \geq \frac{1}{2} \left[ 1 - m - \sqrt{(1 - m)^2 - 4A} \right] x(\tau(s)).
\]

It should be noted that \( \lim_{t \to \sigma(s) - 0} \varphi(t) = \varphi(\sigma(s)) = x(\sigma(s)) \). Thus, we prove that for all \( s \leq t < \sigma(s), \ s \in \mathbb{T} \),

\[
x(\sigma(s)) \geq \frac{1}{2} \left[ 1 - m - \sqrt{(1 - m)^2 - 4A} \right] x(\tau(s)).
\]

Finally, we obtain (2.4). \( \square \)

**Theorem 6.** Consider the Eq.(1.1) and let \( M < 1, \ m \in [0, \frac{1}{e}] \). Assume that (1.2) holds and there exists \( \theta > 0 \) such that (2.3) holds. If \( \tau(t) \) is nondecreasing and

\[
M > \frac{1 + \ln \lambda_1}{\lambda_1} - \frac{1 - m - \sqrt{(1 - m)^2 - 4A}}{2}, \quad (2.7)
\]

where \( \lambda_1 \in [1, e] \) is the unique root of the equation \( \lambda = e^{k\lambda} \) and \( A \) is given by (2.5), then all solutions of Eq.(1.1) oscillate.
Proof. If \( m = 0 \), then the inequality (2.7) reduces to (1.8). Thus with the help of Theorem 1.4, we get the conclusion.

Let \( 0 < m \leq \frac{1}{\varepsilon} \). Assume for the sake of contradiction that \( x \) is an eventually positive solution of Eq.(1.1). Then there exists \( t_0 \leq t_1 \in \mathbb{T} \) such that \( x(t) > 0 \) for \( t > t_1 \). We define \( \mathcal{I}, \mathcal{P}, \mathcal{P} \) as in Lemma 2.2, then \( \mathcal{I} \) satisfies the delay differential equation (2.6). From Lemma 2.1 and Lemma 2.2, it follows that

\[
\liminf_{t \to \infty} \frac{x(t \tau)}{x(t)} \geq \lambda_1, \quad \liminf_{t \to \infty} \frac{x(\tau(t))}{x(t \tau)} \geq \frac{1-m-\sqrt{(1-m)^2-4A}}{2} : = \beta.
\]

Hence, for \( \forall \varepsilon > 0 \) such that \( \varepsilon < \min \left\{ \lambda_1, \frac{1-m-\sqrt{(1-m)^2-4A}}{2} \right\} \), we have

\[
x(t \tau) \geq \lambda_1 - \varepsilon, \quad \frac{x(\tau(t))}{x(t \tau)} \geq \beta - \varepsilon, \quad \text{for} \quad t > t_2 \geq t_1, \quad t \in \mathbb{T}.
\]

By the definitions of \( \mathcal{I}, \mathcal{P}, \mathcal{P} \) in Lemma 2.2, we also have

\[
\mathcal{I}(\tau(t)) \geq \lambda_1 - \varepsilon, \quad \mathcal{I}(\tau(t)) \geq \beta - \varepsilon, \quad \text{for} \quad t > t_2, \quad t \in \mathbb{R}.
\]

Hence, for a fixed \( t > t_2 \), \( t \in \mathbb{R} \), there exists \( t^* \in (\tau(t), t) \), \( t^* \in \mathbb{R} \) such that

\[
\mathcal{I}(\tau(t)) = \mathcal{I}(t^*).
\]

Integrating Eq.(2.6) from \( t^* \) to \( \sigma(t) \) and using the monotonicity of \( \mathcal{I} \) and \( \mathcal{P} \), we have

\[
0 = \mathcal{I}(\sigma(t)) - \mathcal{I}(t^*) + \int_{t^*}^{\sigma(t)} \mathcal{I}(\tau(s)) \mathcal{P}(s) ds
\]

\[
= \mathcal{I}(\sigma(t)) - \mathcal{I}(t^*) + \int_{t^*}^{\sigma(t)} \mathcal{I}(\tau(s)) \mathcal{P}(s) ds + \int_{t}^{\sigma(t)} \mathcal{I}(\tau(s)) \mathcal{P}(s) ds
\]

\[
\geq \mathcal{I}(\sigma(t)) - \mathcal{I}(t^*) + \mathcal{I}(\tau(t)) \int_{t^*}^{\sigma(t)} \mathcal{P}(s) ds
\]

and then,

\[
\int_{t^*}^{\sigma(t)} \mathcal{P}(s) ds \leq \frac{\mathcal{I}(t^*) - \mathcal{I}(\tau(t))}{\mathcal{I}(\tau(t))} < \frac{1}{\lambda_1 - \varepsilon} - (\beta - \varepsilon).
\]  \( \quad (2.8) \)

Dividing Eq.(2.6) by \( \mathcal{I}(t) \) and integrating it from \( \tau(t) \) to \( t^* \), we have

\[
\int_{\tau(t)}^{t^*} \frac{\mathcal{I}(s)}{\mathcal{I}(t)} ds = - \int_{\tau(t)}^{t^*} \mathcal{P}(s) \frac{\mathcal{I}(\tau(t))}{\mathcal{I}(s)} ds \leq - (\lambda_1 - \varepsilon) \int_{\tau(t)}^{t^*} \mathcal{P}(s) ds
\]

and then

\[
\int_{\tau(t)}^{t^*} \mathcal{P}(s) ds \leq - \frac{1}{\lambda_1 - \varepsilon} \int_{\tau(t)}^{t^*} \frac{\mathcal{I}(s)}{\mathcal{I}(t)} ds = \frac{\ln (\lambda_1 - \varepsilon)}{\lambda_1 - \varepsilon}.
\]  \( \quad (2.9) \)
On the other hand, from [27] we get
\[ \int_{t_1}^{t_2} p(s)ds = \int_{t_1}^{t_2} p(s)\Delta s, \quad \forall t_1 \leq t_2, \quad t_1, t_2 \in T. \]

Combining inequalities (2.8) and (2.9), we have
\[ \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s = \int_{\tau(t)}^{\sigma(t)} p(s)ds \leq \frac{1 + \ln (\lambda_1 - \varepsilon)}{\lambda_1 - \varepsilon} - (\beta - \varepsilon). \]

Letting \( t \to \infty \) and \( \varepsilon \to 0 \), we have
\[ \limsup_{t \to \infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \leq \frac{1 + \ln \lambda_1}{\lambda_1} - \frac{1 - m - \sqrt{(1 - m)^2 - 4A}}{2}, \]

which contradicts to (2.7). Thus, the proof is completed. \( \square \)

**Remark 1.** Observe that when \( \theta = 1 \), then
\[ A = \left( \frac{e^{\lambda_1 m} - \lambda_1 m - 1}{(\lambda_1)^2} \right) \]
and (2.7) reduces to
\[ M > 2m + \frac{2}{\lambda_1} - 1. \]

Now, we give two examples in cases \( T = \mathbb{R} \) and \( T = \mathbb{N} \).

**Example 1.** For \( T = \mathbb{R} \), consider the delay differential equation
\[ x'(t) + \frac{1}{e} x(t - \sin^2 \sqrt{t} - 1) = 0, \quad (2.10) \]
where \( p = \frac{1}{e}, \quad a = 1 \) and \( pa = \frac{1}{e} \). Then
\[ m = \liminf_{t \to \infty} \int_{\tau(t)}^{t} \frac{1}{e}ds = \liminf_{t \to \infty} \frac{1}{e} (\sin^2 \sqrt{t} + 1) = \frac{1}{e} \]
and
\[ M = \limsup_{t \to \infty} \int_{\tau(t)}^{t} \frac{1}{e}ds = \limsup_{t \to \infty} \frac{1}{e} (\sin^2 \sqrt{t} + 1) = \frac{1}{e} + \frac{1}{e} = \frac{2}{e}. \]

Thus, according to Theorem 2.3, all solutions of Eq.(2.10) oscillate.

**Example 2.** For \( T = \mathbb{N} \), consider the following delay difference equation
\[ \Delta x(n) + p(n)x(n - 5) = 0, \quad n = 0, 1, \ldots, \quad (2.11) \]
where
\[ p(6n) = \frac{1}{5e}, \quad p(6n + 1) = \ldots = p(6n + 4) = \frac{1}{5e}, \]
\[ p(6n + 5) = \frac{1}{5e} + 0.113, \quad n = 0, 1, \ldots. \]
Then
\[ m = \liminf_{n \to \infty} \sum_{s=n-5}^{n-1} p(s) = \frac{1}{e} \cong 0.3678 \quad \text{and} \quad \lambda_1 = e \]
and
\[ M = \limsup_{n \to \infty} \sum_{s=n-5}^{n} p(s) = \frac{6}{5e} + 0.113 \cong 0.55446 < 1 \]
hold. Since
\[ M = 0.55446 > 2m + \frac{2}{\lambda_1} - 1 \cong 0.47135, \]
all solutions of Eq. (2.11) oscillate by Theorem 2.3.

In [22], the authors gave some incorrect results. Finally, we give a correction to the [22].

**Correction.**

i) In Lemma 2.1 [22], the condition (2.1) was given by
\[ p(\tau(n)) \Delta (\tau(n)) \geq \theta p(n). \]
This condition should be changed as follows
\[ \sum_{j=\tau(u)}^{\tau(n)-1} p(j) \geq \theta \sum_{j=u}^{n-1} p(j) \quad \text{for all} \quad \tau(n) \leq u \leq n. \]

ii) In the proof of Lemma 2.1 [22], the \( \sigma(t) \) is defined as follows
\[ \sigma(t) = \tau(n) + (\Delta \tau(n))(t-n) \quad \text{for} \quad n \leq t < n + 1, \quad n = 0, 1, \ldots. \]
This definition should be changed as follows
\[ \sigma(t) = \tau(n) \quad \text{for} \quad n \leq t < n + 1, \quad n = 0, 1, \ldots. \]

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