ASYMPTOTICALLY LACUNARY EQUIVALENT SEQUENCE SPACES DEFINED BY IDEAL CONVERGENCE AND AN ORLICZ FUNCTION

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ABSTRACT. The purpose of this paper is to introduce certain new sequence spaces using ideal convergence, a lacunary sequence $\theta = (k_r)$, a strictly positive sequence $p = (p_k)$, and an Orlicz function and examine some of their properties.

1. Introduction

Let $s, \ell_\infty, c$ denote the spaces of all real sequences, bounded, and convergent sequences, respectively. Any subspace of $s$ is called a sequence space.

Following Freedman et al. [5], we call the sequence $\theta = (k_r)$ lacunary if it is an increasing sequence of integers such that $k_0 = 0, h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_r - 1, k_r]$ and $q_r = k_r / k_{r-1}$. These notations will be used throughout the paper. The sequence space of lacunary strongly convergent sequences $N_\theta$ was defined by Freedman et al. [5], as follows:

$$N_\theta = \{ x = (x_i) \in s : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} |x_i - s| = 0 \text{ for some } s \}.$$ 

Orlicz [8] used the idea of Orlicz function to construct the space $L^M$. An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, nondecreasing and convex with $M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for all values of $u$, if there exists constant $K > 0$, such that $M(2u) \leq KM(u)$ ($u \geq 0$). It is also easy to see that always $K > 2$. The $\Delta_2$-condition is equivalent to the satisfaction of the inequality $M(Lu) \leq KLM(u)$ for all values of $u$ and $L > 1$.

Remark 1. An Orlicz function satisfies the inequality $M(\lambda x) < \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

The following well known inequality will be used throughout the paper;

$$|a_i + b_i|^{p_i} \leq T(|a_i|^{p_i} + |b_i|^{p_i})$$

where $a_i$ and $b_i$ are complex numbers, $T = \max(1, 2^{H-1})$, and $H = \text{supp}_{i} < \infty$.

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Marouf presented definitions for asymptotically equivalent sequences and asymptotically regular matrices in [7]. Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices in [9]. Subsequently, many authors have shown their interest to solve different problems arising in this area (see [1],[3], and [10]).

Kostyrko et al. [6] introduced the notion of I-convergence with the help of an admissible ideal I, which denotes the ideal of subsets of \( \mathbb{N} \), which is a generalization of statistical convergence. Quite recently, Das et al. [4] unified these two approaches to introduce new concepts such as I-statistical convergence and I-lacunary statistical convergence and investigated some of their consequences. For more applications of ideals we refer to [2,6,11] where many important references can be found.

Recently, Karakuş and Bilgin [3] used an Orlicz function to define some notions of asymptotically equivalent sequences and studied some of their connections. This paper extended these concepts by presenting a non-trivial ideal \( I \).

We introduce some new notions, (\( M, p \))-asymptotically equivalent of multiple \( L \), strong (\( M, p \))-asymptotically equivalent of multiple \( L \), and strong (\( M, p \))-asymptotically lacunary equivalent of multiple \( L \) with respect to the ideal \( I \) which is a natural common-trivial ideal \( I \). Lacunary sequence, a strictly positive sequence \( p = (p_k) \), and Orlicz function. In addition to these definitions, we obtain some relevant connections between these notions.

2. Definitions and Notations

Now we recall some definitions of sequence spaces.

**Definition 2.1.** Two nonnegative sequences \([x]\) and \([y]\) are said to be asymptotically equivalent if \( \lim_k \frac{x_k}{y_k} = 1 \) (denoted by \( x \sim y \)).

**Definition 2.2.** Two nonnegative sequences \([x]\) and \([y]\) are said to be asymptotically statistical equivalent of multiple \( L \) provided that for every \( \varepsilon > 0 \),

\[
\lim_{n} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0,
\]

(denoted by \( x \overset{S}{\sim} y \)) and simply asymptotically statistical equivalent, if \( L = 1 \).

**Definition 2.3.** Two nonnegative sequences \([x]\) and \([y]\) are said to be strong asymptotically equivalent of multiple \( L \) provided that

\[
\lim_{n} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| = 0,
\]

(denoted by \( x \overset{w}{\sim} y \)) and simply strong asymptotically equivalent, if \( L = 1 \).

**Definition 2.4.** Let \( \theta \) be a lacunary sequence; the two nonnegative sequences \([x]\) and \([y]\) are said to be asymptotically lacunary statistical equivalent of multiple \( L \) provided that for every \( \varepsilon > 0 \),

\[
\lim_{r} \frac{1}{|I_r|} \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} = 0,(denoted by \( x \overset{S}{\sim} y \)) and simply asymptotically lacunary statistical equivalent, if \( L = 1 \).

**Definition 2.5.** Let \( \theta \) be a lacunary sequence; the two nonnegative sequences \([x]\) and \([y]\) are said to be strong asymptotically lacunary equivalent of multiple \( L \) provided that

\[
\lim_{r} \frac{1}{|I_r|} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| = 0 \ (\text{denoted by} \ x \overset{N}{\sim} y \ ) \text{ and simply strong asymptotically lacunary equivalent, if } L = 1.
\]
Definition 2.6. Let $M$ be any Orlicz function; the two nonnegative sequences $[x]$ and $[y]$ are said to be $M$-asymptotically equivalent of multiple $L$ provided that,

$$\lim \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|x_k - L|}{\rho}\right) = 0,$$

for some $\rho > 0$, (denoted by $x \sim_{M} y$) and simply strong $M$-asymptotically equivalent, if $L = 1$.

Definition 2.7. Let $M$ be any Orlicz function; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong $M$-asymptotically equivalent of multiple $L$ provided that,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|x_k - L|}{\rho}\right) = 0,$$

for some $\rho > 0$, (denoted by $x \sim_{wM} y$) and simply strong $M$-asymptotically equivalent, if $L = 1$.

Definition 2.8. Let $M$ be any Orlicz function and $\theta$ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be strong $M$-asymptotically lacunary equivalent of multiple $L$ provided that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} M\left(\frac{|x_k - L|}{\rho}\right) = 0,$$

for some $\rho > 0$, (denoted by $x \sim_{NM} y$) and simply strong $M$-asymptotically lacunary equivalent, if $L = 1$.

For any non-empty set $X$, let $P(X)$ denote the power set of $X$.

Definition 2.9. A family $I \subseteq P(X)$ is said to be an ideal in $X$ if

(i) $\emptyset \in I$;

(ii) $A, B \in I$ imply $A \cup B \in I$ and

(iii) $A \in I, B \subseteq A$ imply $B \in I$.

Definition 2.10. A non-empty family $F \subseteq P(X)$ is said to be a filter in $X$ if

(i) $\emptyset \notin F$;

(ii) $A, B \in F$ imply $A \cap B \in F$ and

(iii) $A \in F, B \supseteq A$ imply $B \in F$.

An ideal $I$ is said to be non-trivial if $I \neq \{\emptyset\}$ and $X \notin I$. A non-trivial ideal $I$ is called admissible if it contains all the singleton sets. Moreover, if $I$ is a non-trivial ideal on $X$, then $F = F(I) = \{X - A : A \in I\}$ is a filter on $X$ and conversely. The filter $F(I)$ is called the filter associated with the ideal $I$.

Definition 2.11. Let $I \subset P(N)$ be a non-trivial ideal in $N$. A sequence $[x]$ in $X$ is said to be $I$-convergent to $\xi$ if for each $\varepsilon > 0$, the set $\{k \in N : |x_k - \xi| \geq \varepsilon\} \in I$.

In this case, we write $I - \lim_{k \to \infty} x_k = \xi$. A sequence $[x]$ in $X$ is said to be $I$-null if $L = 0$. In this case we write $I - \lim_{k \to \infty} x_k = 0$.

Definition 2.12. A sequence $[x]$ of numbers is said to be $I$-statistical convergent or $S(I)$-convergent to $L$, if for every $\varepsilon > 0$ and $\delta > 0$, we have

$$\left\{ n \in N : \frac{1}{n} \sum_{k=1}^{n} |x_k - L| \geq \varepsilon \right\} \in I.$$

In this case, we write $x_k \to L(S(I))$ or $S(I) - \lim_{k \to \infty} x_k = L$.

Definition 2.13 Let $I \subset P(N)$ be a non-trivial ideal in $N$. The two non-negative sequences $[x]$ and $[y]$ are said to be strongly asymptotically equivalent of multiple $L$ with respect to the ideal $I$ provided that for each $\varepsilon > 0$

$$\left\{ n \in N : \frac{1}{n} \sum_{k=1}^{n} |x_k - L| \geq \varepsilon \right\} \in I,$$

(denoted by $x \sim_{I(\rho)} y$) and simply strongly asymptotically equivalent with respect to the ideal $I$, if $L = 1$.

Definition 2.14. Let $I \subset P(N)$ be a non-trivial ideal in $N$ and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $[x]$ and $[y]$ are said to
be asymptotically lacunary statistical equivalent of multiple $L$ with respect to the ideal $I$ provided that for each $\varepsilon > 0$ and $\gamma > 0$,

$$\left\{ r \in N; \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in I$$

(denoted by $x \sim_{I(S_h)} y$) and simply asymptotically lacunary statistical equivalent with respect to the ideal $I$, if $L = 1$.

**Definition 2.15.** Let $I \subset P(N)$ be a non-trivial ideal in $N$ and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $[x]$ and $[y]$ are said to be strongly asymptotically lacunary equivalent of multiple $L$ with respect to the ideal $I$ provided that for each $\varepsilon > 0$,

$$\left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \in I$$

(denoted by $x \sim_{I(N_h)} y$) and simply asymptotically lacunary equivalent with respect to the ideal $I$, if $L = 1$.

### 3. Main Results

We now consider our main results. We begin with the following definitions.

**Definition 3.1.** Let $I \subset P(N)$ be a non-trivial ideal in $N$ and $M$ be any Orlicz function. The two non-negative sequences $[x]$ and $[y]$ are said to be $M$-asymptotically equivalent of multiple $L$ with respect to the ideal $I$ provided that for each $\varepsilon > 0$

$$\left\{ k \in N; M\left(\frac{x_k}{y_k} - L\right) / \rho \geq \varepsilon \right\} \in I,$$

for some $\rho > 0$,

(denoted by $x \sim_{I(M)} y$) and simply $M$- asymptotically equivalent with respect to the ideal $I$, if $L = 1$.

**Definition 3.2.** Let $I \subset P(N)$ be a non-trivial ideal in $N$, $M$ be any Orlicz function, and $p = (p_k)$ be a sequence of positive real numbers. Two number sequences $[x]$ and $[y]$ are said to be strongly $(M,p)$-asymptotically equivalent of multiple $L$ with respect to the ideal $I$ provided that for each $\varepsilon > 0$,

$$\left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} \left[ M\left(\frac{x_k}{y_k} - L\right) / \rho \right] p_k \geq \varepsilon \right\} \in I,$$

for some $\rho > 0$,

(denoted by $x \sim_{I(w^{(M,p)})} y$) and simply strongly $(M,p)$- asymptotically equivalent with respect to the ideal $I$, if $L = 1$.

If we take $M(x) = x$ for $x \geq 0$, we write $x \sim_{I(w^p)} y$ instead of $x \sim_{I(w^{(M,p)})} y$ and simply strongly $p$ - asymptotically equivalent with respect to the ideal $I$, if $L = 1$.

If we take $p_k = p$ for all $k \in N$, we write $x \sim_{I(w^p)} y$ instead of $x \sim_{I(w^{(M,p)})} y$. If we take $p = 1$, we write $x \sim_{I(w^1)} y$ instead of $x \sim_{I(w^{(M,p)})} y$. If $L = 1$, we write $x \sim_{I(w^1)} y$ instead of $x \sim_{I(w^{(M,p)})} y$ and simply strongly $M$- asymptotically equivalent with respect to the ideal $I$, if $L = 1$.

**Definition 3.3.** Let $I \subset P(N)$ be a non-trivial ideal in $N$, $M$ be any Orlicz function, $\theta = (k_r)$ be a lacunary sequence, and $p = (p_k)$ be a sequence of positive real numbers. Two number sequences $[x]$ and $[y]$ are said to be $(M,p)$-asymptotically lacunary equivalent of multiple $L$ with respect to the ideal $I$ provided that for each $\varepsilon > 0$, 
and convex, it follows that every \( u \) function which satisfies the \( \Delta^2 \) over \((l)\) with respect to the ideal \( I \), strongly \( M \), \( p \)-asymptotically lacunary equivalent with respect to the ideal \( I \), to the ideal \( I \), and \( y \) yields
\[
\sum_{n=1}^{\infty} M\left( \left| \frac{a_n}{y_k} - L \right| / \rho \right) \geq \varepsilon \in I, \text{ for some } \rho > 0, \text{ (denoted by } x(I_{N_e}^{M(p)})y) \text{ and simply } (M,p) \text{-asymptotically lacunary equivalent with respect to the ideal } I, \text{ if } L = 1. \]

If we take \( p_k = p \) for all \( k \in \mathbb{N} \), we write \( x(I_{N_e}^{M(p)})y \) instead of \( x(I_{N_e}^{M(p)})y \).

Note that, we put \( p = 1 \), we write \( x(I_{N_e}^{M})y \) instead of \( x(I_{N_e}^{M})y \) and simply \( M \)-asymptotically lacunary equivalent with respect to the ideal \( I \), if \( L = 1 \).

Also if we put \( M(x) = x \) for \( x \geq 0 \), we write \( x(I_{N_e}^{M})y \) instead of \( x(I_{N_e}^{M})y \).

Hence \( x(I_{N_e}^{M})y \) is the same as the \( x(I_{N_e}^{M})y \) of Savas and Gumus [11].

We start this section with the following Theorem to show that the relation between strongly \( M \)- asymptotically equivalence and strong asymptotically equivalence with respect to the ideal \( I \).

**Theorem 3.1.** Let \( I \subset P(N) \) be a non-trivial ideal in \( N \), \( M \) be any Orlicz function which satisfies the \( \Delta_2 \)-condition, \( \theta = (k_r) \) be a lacunary sequence, then if \( x(I_{w}^{M})y \) then \( x(I_{w}^{M})y \).

**Proof.** Let \( x(I_{w}^{M})y \) and \( \varepsilon > 0 \). We choose \( 0 < \delta < 1 \) such that \( M(u) < \varepsilon/2 \) for every \( u \) with \( 0 \leq u \leq \delta \). We can write
\[
\frac{1}{n} \sum_{k=1}^{n} M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) = \frac{1}{n} \sum_{k=1}^{n} M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) + \frac{1}{n} \sum_{k=1}^{n} M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right)
\]
where the first summation is over \( \left( \frac{x_k}{y_k} - L \right) / \rho \leq \delta \) and the second summation over \( \left( \frac{x_k}{y_k} - L \right) / \rho \) \( \delta \). Since \( M \) is continuous
\[
\frac{1}{n} \sum_{k=1}^{n} M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) < \varepsilon/2 \text{ and for } \left( \frac{x_k}{y_k} - L \right) / \rho \geq \delta \text{ we use the fact that } \left( \frac{x_k}{y_k} - L \right) / \rho < \left( \frac{x_k}{y_k} - L \right) / \rho \delta + 1 + \left( \frac{x_k}{y_k} - L \right) / \rho \delta. \text{ Since } M \text{ is non-decreasing and convex}, \text{ it follows that } \]
\[
M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) < M\left( 1 + \left( \frac{x_k}{y_k} - L \right) / \rho \delta \right)
\]
\[
< \frac{1}{2} M(2) + \frac{1}{2} M(2) \left( \frac{x_k}{y_k} - L \right) / \rho \delta
\]
Since \( M \) satisfies the \( \Delta_2 \)-condition, therefore
\[
M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) < \frac{1}{2} K\left( \left( \frac{x_k}{y_k} - L \right) / \rho \delta \right) M(2) + \frac{1}{2} K\left( \left( \frac{x_k}{y_k} - L \right) / \rho \delta \right)\]
\[
= K\left( \frac{x_k}{y_k} - L \right) / \delta M(2)
\]
Hence \( \frac{1}{n} \sum_{k=1}^{n} M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \leq \left( KM(2)/\delta \right) \frac{1}{n} \sum_{k=1}^{n} \left( \frac{x_k}{y_k} - L \right) / \rho \right), \text{ which together with } \]
\[
\frac{1}{n} \sum_{k=1}^{n} M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) < \varepsilon
\]
yields
\[
\frac{1}{n} \sum_{k=1}^{n} M\left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \leq \varepsilon/2 + \left( KM(2)/\delta \right) \frac{1}{n} \sum_{k=1}^{n} \left( \frac{x_k}{y_k} - L \right) / \rho \). Thus,
\[ \left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} M \left( \frac{x_k}{y_k} - L \right) / \rho \geq \varepsilon \right\} \subseteq \left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \delta / 2 KM(2) \right\}. \]

Since \( x \overset{I}{\sim} y \) it follows the later set, and hence, the first set in above expression belongs to \( I \). This proves that \( x \overset{I(w^M)}{\sim} y \)

**Theorem 3.2.** Let \( M_1, M_2 \) be Orlicz functions that satisfy the \( \Delta_2 \)-condition.

Then

(i) if \( x \overset{I(M_1)}{\sim} y \) then \( x \overset{I(M_1 \circ M_2)}{\sim} y \),

(ii) if \( x \overset{I(M_1 \circ M_2)}{\sim} y \) then \( x \overset{I(M_1 + M_2)}{\sim} y \)

**Proof.** (i) Let \( x \overset{I}{\sim} y \). Then there exists \( \rho > 0 \) such that

\[ \left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} M_2 \left( \frac{x_k}{y_k} - L \right) / \rho \geq \varepsilon \right\} \in I \]

Let \( \varepsilon > 0 \) and choose \( 0 < \delta < 1 \) such that \( M_1(u) < \varepsilon / 2 \) for every \( u \) with \( 0 \leq u \leq \delta \). Write \( A_k = M_2 \left( \frac{x_k}{y_k} - L \right) / \rho \) By the Remark, we have, for \( A_k \leq \delta \)

\[ M_1(M_2) \left( \frac{x_k}{y_k} - L \right) / \rho \delta \]

\[ M_1(M_2) \left( \frac{x_k}{y_k} - L \right) / \rho \delta \]

For \( A_k > \delta \), we have \( A_k < A_k / \delta < 1 + A_k / \delta \). Since \( M \) is non-decreasing and convex, it follows that

\[ M_1(A_k) \leq M_1(1 + A_k / \delta) \]

Since \( M \) satisfies the \( \Delta_2 \)-condition, therefore

\[ M_1(A_k) < \frac{1}{2} K(A_k / \delta) M_1(1) + \frac{1}{2} K(A_k / \delta) \]

\[ = K(A_k / \delta) M_1(1) \]

Hence \( M_1(M_2) \left( \frac{x_k}{y_k} - L \right) / \rho \delta \)

\[ \left\{ n \in N; M_1(M_2) \left( \frac{x_k}{y_k} - L \right) / \rho \delta \geq \varepsilon \right\} \]

\[ \subseteq \left\{ n \in N; M_1(M_2) \left( \frac{x_k}{y_k} - L \right) / \rho \delta \geq \varepsilon / 2 \max \left( 1, K \delta^{-1} M_1(1) \right) \right\} \]

we have \( \left\{ n \in N; M_1(M_2) \left( \frac{x_k}{y_k} - L \right) / \rho \delta \geq \varepsilon \right\} \in I \). Hence \( x \overset{I(M_1 \circ M_2)}{\sim} y \).

(ii) Let \( x \overset{I(M_1 \circ M_2)}{\sim} y \). Then there exists \( \rho > 0 \) such that

\[ \left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} M_1 \left( \frac{x_k}{y_k} - L \right) / \rho \geq \varepsilon \right\} \in I \text{ and} \]

\[ \left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} M_2 \left( \frac{x_k}{y_k} - L \right) / \rho \geq \varepsilon \right\} \in I. \]

The rest of the proof follows from the following equality

\[ (M_1 + M_2) \left( \frac{x_k}{y_k} - L \right) / \rho = M_1 \left( \frac{x_k}{y_k} - L \right) / \rho + M_2 \left( \frac{x_k}{y_k} - L \right) / \rho \]

The next theorem shows the relationship between the strongly \( M \)-asymptotically equivalence and the \( M \)-asymptotically lacunary equivalence with respect to the ideal \( I \).

**Theorem 3.3.** Let \( I \subset P(N) \) be a non-trivial ideal in \( N \), \( M \) be an Orlicz function, and \( \theta = (k_r) \) be a lacunary sequence, then

(i) if \( \lim \sup_r q_r < \infty \) then \( x \overset{I(N_{k_r})}{\sim} y \) implies \( x \overset{I(w_{k_r})}{\sim} y \),
(ii) if \( \lim \inf q_r > 1 \) then \( x^{I_{w_M}} \) implies \( x^{I_{N^M}} \).

(iii) if \( 1 < \lim \inf q_r \leq \lim \sup q_r < \infty \), then \( x^{I_{w_M}} \) if and only if \( x^{I_{N^M}} \).

**Proof.** Part (i): If \( \lim \sup q_r < \infty \) then there exists \( K > 0 \) such that \( q_r < K \) for every \( r \). Now suppose that \( x^{I_{N^M}} \) and \( \varepsilon > 0 \). Let

\[
A = \left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho) \leq \varepsilon \right\}, \text{for some } \rho > 0.
\]

Hence, for all \( j \in A \) and for some \( \rho > 0 \), we have \( H_j = \frac{1}{h_j} \sum_{k \in I_j} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho) < \varepsilon \). Let \( n \) be any integer with \( k_r \geq n > k_{r-1} \). Now write

\[
\frac{1}{n} \sum_{k=1}^{n} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho) \leq \frac{1}{k_{r-1}} \sum_{k \in I_m} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho)
\]

\[
= \frac{1}{k_{r-1}} \sum_{m=1}^{r} \sum_{k \in I_m} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho)
\]

\[
= \frac{1}{k_{r-1}} \sum_{m=1}^{r} \left( \frac{k_m - k_{m-1}}{h_m} \right) \sum_{k \in I_m} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho)
\]

\[
= \frac{1}{k_{r-1}} \sum_{m=1}^{r} (k_m - k_{m-1}) \sup_{j \in A} H_j
\]

\[
= k_r \sup_{j \in A} H_j
\]

\[
< K \varepsilon = \varepsilon'
\]

It follows that for any \( \varepsilon' > 0 \), \( \left\{ n \in N; \frac{1}{n} \sum_{k=1}^{n} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho) < \varepsilon' \right\} \in F(I) \)

which yields that \( x^{I_{w_M}} \). Because for any set \( A \in F(I), \cup \{ n : k_{r-1} < n < k_r, r \in A \} \in F(I) \).

Part (ii): Let \( x^{I_{w_M}} \) and \( \lim \inf q_r > 1 \). There exists \( \delta > 0 \) such that

\[
q_r = (k_r/k_{r-1}) \geq 1 + \delta \text{ for all } r \geq 1.
\]

We have, for sufficiently large \( r \), that \( (k_r/h_r) \leq \frac{1 + \delta}{\delta} \) and \( (k_{r-1}/h_r) \leq \frac{1}{\delta} \). Let \( \varepsilon > 0 \) and define the set

\[
A = \left\{ k_r \in N; \frac{1}{k_r} \sum_{k=1}^{k_r} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho) < \varepsilon \right\}, \text{for some } \rho > 0.
\]

We have \( A \in F(I) \), which is the filter of the ideal \( I \). For each \( k_r \in A \), we have, for some \( \rho > 0 \),

\[
\frac{1}{h_r} \sum_{k \in I_r} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho) = \frac{1}{h_r} \sum_{k=1}^{k_r} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho) - \frac{1}{h_r} \sum_{k=1}^{k_{r-1}} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho)
\]

\[
= \frac{k_r}{h_r} \sum_{k=1}^{k_r} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho) - \frac{k_{r-1}}{h_r} \sum_{k=1}^{k_{r-1}} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho)
\]

\[
\leq \frac{k_r}{h_r} \sum_{k=1}^{k_r} M(\left\lfloor \frac{x_k}{y_k} - L \right\rfloor / \rho)
\]

\[
< \left( \frac{1 + \delta}{\delta} \right) \varepsilon = \varepsilon'
\]

It follows that for any \( \varepsilon' > 0 \),
\[
\left\{ r \in N : \frac{1}{n_r} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho \right\} \in F(I) \text{ which yields that } x^{I(N^M_{\sim})} y.
\]

Part (iii): This immediately follows from (i) and (ii).
Now we give relation between asymptotically lacunary statistical equivalence and $M$-asymptotically lacunary equivalence with respect to the ideal $I$.

**Theorem 3.4.** Let $I \subset P(N)$ be a non-trivial ideal in $N$, $M$ be an Orlicz function, and $\theta = (k_r)$ be a lacunary sequence, then

(i) if $x^{I(N^M_{\sim})} y$ then $x^{I(S_\theta)} y$,

(ii) if $M$ is bounded then $x^{I(N^M_{\sim})} y \iff x^{I(S_\theta)} y$.

**Proof.** Part (i): Take $\varepsilon > 0$ and let \( \sum_{r=1}^{\infty} \) denote the sum over $k \in I_r$ for some $\rho > 0$, with $\left| \frac{x_k}{y_k} - L \right| / \rho \geq \varepsilon$. Then

\[
\begin{aligned}
\frac{1}{n_r} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho &\geq \frac{1}{n_r} \sum_{r=1}^{\infty} M\left( \frac{x_k}{y_k} - L \right) / \rho \\
&\geq \frac{1}{n_r} \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| / \rho \geq \varepsilon \right\} \\
&\geq \left\{ r \in N : \frac{1}{n_r} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho \geq \gamma \right\} \\
&\subseteq \left\{ r \in N : \frac{1}{n_r} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho \geq \gamma \right\} \in I. But then, by definition of an ideal, later set belongs to $I$, and therefore $x^{I(S_\theta)} y$.
\end{aligned}
\]

Part (ii): Suppose that $M$ is bounded and $x^{I(S_\theta)} y$. Since $M$ is bounded, there exists an integer $T$ such that $|M(x)| \leq T$ for all $x \geq 0$. We see that

\[
\frac{1}{n_r} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho \leq T \frac{1}{n_r} \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| / \rho \geq \varepsilon \right\} + M(\varepsilon)
\]

so we have

\[
\left\{ r \in N : \frac{1}{n_r} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho \geq \varepsilon \right\} \\
\subseteq \left\{ r \in N : \frac{1}{n_r} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho \geq \varepsilon \right\} \geq \frac{\varepsilon - M(\varepsilon)}{T} \in I. Therefore we have $x^{I(N^M_{\sim})} y$.
\]

Let $p_k = p$ for all $k$, $t_k = t$ for all $k$ and $0 < p \leq t$. Then it follows following Theorem.

**Theorem 3.5.** Let $I \subset P(N)$ be a non-trivial ideal in $N$, $M$ be an Orlicz function, and $\theta = (k_r)$ be a lacunary sequence, then

$x^{I(N^M_{\sim})} y$ implies $x^{I(N^M_{\sim})} y$.

**Proof.** Let $x^{I(N^M_{\sim})} y$. It follows from Holder’s inequality

\[
\frac{1}{n_r} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho \leq \left( \frac{1}{n_r} \sum_{k \in I_r} M\left( \frac{x_k}{y_k} - L \right) / \rho \right)^{p/t}
\]
and \( \{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[ M\left( \frac{x_k}{y_k} - L \right) \right]^p \geq \varepsilon \} \)
\[ \leq \left\{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[ M\left( \frac{x_k}{y_k} - L \right) \right]^t \geq \varepsilon^{t/p} \right\} \in I. \]

Thus we have \( x \in \mathcal{I}(N_{\infty}^{M, p}) \).

We now consider that \((p_k)\) and \((t_k)\) are not constant sequences.

**Theorem 3.6.** Let \( I \subset P(N) \) be a non-trivial ideal in \( N \), \( M \) be an Orlicz function, \( \theta = (k_r) \) be a lacunary sequence, \( 0 < p_k \leq t_k \) for all \( k \) and \((t_k/p_k)\) be bounded, then \( x \in \mathcal{I}(N_{\infty}^{M, (p_k, t_k)}) \) implies \( x \in \mathcal{I}(N_{\infty}^{M, (p_k)}) \).

**Proof.** Let \( x \in \mathcal{I}(N_{\infty}^{M, (p_k, t_k)}) \) and \( \lambda_k = (p_k / t_k) \), so that

\[ 0 < \lambda \leq \lambda_k \leq 1; \text{we define the sequences} (u_k) \text{and} (v_k) \text{as follows: For} z_k \geq 1; \text{let} u_k = z_k \text{and} v_k = 0 \text{and for} z_k < 1; \text{let} v_k = z_k \text{and} u_k = 0. \text{Then we have} z_k = u_k + v_k; z_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}. \text{Now it follows that} u_k^{\lambda_k} \leq u_k \leq z_k \text{and} v_k^{\lambda_k} \leq v_k. \text{Therefore}
\]

\[ \frac{1}{h_r} \sum_{k \in I_r} z_k^{\lambda_k} = \frac{1}{h_r} \sum_{k \in I_r} (u_k^{\lambda_k} + v_k^{\lambda_k}) \leq \frac{1}{h_r} \sum_{k \in I_r} z_k + \frac{1}{h_r} \sum_{k \in I_r} v_k^{\lambda_k} \]

Now for each \( r \):

\[ \frac{1}{h_r} \sum_{k \in I_r} v_k^{\lambda_k} = \sum_{k \in I_r} \left( \frac{1}{h_r} v_k \right)^{\lambda_k} \left( \frac{1}{h_r} \right)^{1-\lambda} \leq \left( \sum_{k \in I_r} \left( \frac{1}{h_r} v_k \right)^{\lambda_k} \right)^{1/\lambda_k} \left( \sum_{k \in I_r} \left( \frac{1}{h_r} \right)^{1-\lambda} \right)^{1/(1-\lambda)} < \left( \frac{1}{h_r} \sum_{k \in I_r} v_k \right)^{\lambda_k} \text{and so}
\]

\[ \frac{1}{h_r} \sum_{k \in I_r} \left[ M\left( \frac{x_k}{y_k} - L \right) / \rho \right]^{p_k} \geq \varepsilon \; \text{then}
\]

\[ \left\{ \frac{1}{h_r} \sum_{k \in I_r} \left[ M\left( \frac{x_k}{y_k} - L \right) / \rho \right]^{t_k} \geq \varepsilon, \; z_k \geq 1 \right\}\]

\[ \left\{ \frac{1}{h_r} \sum_{k \in I_r} \left[ M\left( \frac{x_k}{y_k} - L \right) / \rho \right]^{t_k} \geq \varepsilon, \; z_k < 1 \right\}
\]

Hence \( \{ r \in N; \frac{1}{h_r} \sum_{k \in I_r} \left[ M\left( \frac{x_k}{y_k} - L \right) / \rho \right]^{p_k} \geq \varepsilon \} \)
\[ r \in \mathbb{N}; \frac{1}{h_r} \sum_{k \in I_r} \left[ M \left( \left| \frac{x_k}{y_k} - L \right| / \rho \right) \right] t_k \geq \min \left\{ \varepsilon, \left( \frac{1}{2} \right)^{1/\lambda} \right\} \in I. \]

Thus we have \( x \sim y. \)

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REFERENCES


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