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PERIODICITY IN MULTIPLE DELAY VOLTERRA DIFFERENCE EQUATIONS OF NEUTRAL TYPE

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Abstract. We prove the existence and uniqueness of a periodic solution for the multiple delay difference neutral Volterra equation

$$\begin{aligned}\Delta x(n) &= -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)) + \Delta Q(n, x(n-\tau_1(n)), ..., x(n-\tau_N(n)) \\ &\quad + \sum_{j=1}^{N}\sum_{n-\tau_j(n)}^{n} k_j(n,s)f_j(s,x(s)).\end{aligned}$$

The contraction mapping principle and a Krasnoselskii's fixed point theorem are used in the analysis.

1. Introduction

Research into the qualitative behaviour of solutions of difference equations has received a lot of attention from some mathematicians in recent times. These qualitative properties include stability, and periodicity of solutions of difference equations, see [1], [3], [5], [7], [9], [10] and the references cited therein. In this paper we consider the Volterra difference equation with variable multiple delays

$$\begin{aligned}\Delta x(n) &= -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)) + \Delta Q(n, x(n-\tau_1(n)), ..., x(n-\tau_N(n)) \\ &\quad + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} k_j(n,s)f_j(s,x(s)), \qquad (1)\end{aligned}$$

where $a_j : \mathbb{Z}^+ \to \mathbb{R}$, $k_j : \mathbb{Z}^+ \times \cap \mathbb{Z} \to \mathbb{R}$, $f_j : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$, $Q_j : \mathbb{Z}^+ \times \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R} \to \mathbb{R}$ and $\tau_j : \mathbb{Z}^+ \to \mathbb{Z}^+$, for $j = 1, ..., N$.

Here Δ denotes the forward difference operator. That is, $\Delta x(n) = x(n+1) - x(n)$ for any sequence $\{x(n) : n \in \mathbb{Z}^+\}$. This work is mainly motivated by the work of Raffoul in [5] where he proved the existence and uniqueness of a periodic solution for the equation

$$\Delta x(n) = -a(n)x(n-\tau), \qquad (2)$$

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where τ is a positive constant. In this paper, we obtain sufficient conditions for (1) to have a unique periodic solution..
The rest of the paper is organized as follows. In the next section we state some preliminary results needed in the paper. We state and prove our main result in section 3.

2. Preliminaries

Let T be an integer such that $T \geq 1$. Define $P_T = \{\varphi \in C(\mathbb{Z}, \mathbb{R}) : \varphi(n+T) = \varphi(n)\}$ where $C(\mathbb{Z}, \mathbb{R})$ is the space of all real valued functions. Then $(P_T, ||.||)$ is a Banach space with the maximum norm

$$||\varphi|| = \max_{n \in [0, T-1]} |\varphi(n)|.$$

In this paper we assume that for j=1,...,N,

$$\begin{aligned} &a_j(n+T) = a_j(n),\ \tau_j(n+T) = \tau_j(n), \quad \tau_j(n) \geq \tau_j^* > 0 \\ &k_j(n+T, s+T) = k_j(n,s),\ f_j(n+T, x) = f_j(n,x), \end{aligned} \tag{3}$$

and

$$|f_j(n,x) - f_j(n,y)| \leq \rho_j ||x-y||. \tag{4}$$

Suppose further that

$$Q(n+T, x, x, x, ..., x) = Q(n, x, x, x, ..., x), \tag{5}$$

and

$$|Q(n, x, x, x, ..., x) - Q(n, y, y, y, ..., y)| \leq \sum_{j=1}^{N} L_j ||x-y||. \tag{6}$$

Lemma 1 Let $h_j : \mathbb{Z} \to \mathbb{R}$ be an arbitrary sequence, for $j = 1, ..., N$ and $H(n) = 1 - \sum_{j=1}^{N} h_j(n)$. Suppose that $\prod_{r=n-T}^{n-1} H(r) \neq 1$, for all $n \in \mathbb{Z}$,

$$h_j(n+T) = h_j(n) \text{ for } j = 1, ..., N, \tag{7}$$

and (3) hold. If $x \in P_T$, then x is a solution of equation (1) if and only if

$$\begin{aligned} x(n) \quad = \quad & Q(n, x(n-\tau_1(n)), ..., x(n-\tau_N(n))) + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s) \\ & + \Big[1 - \prod_{r=n-T}^{n-1} H(r)\Big]^{-1} \sum_{s=n-T}^{n-1} \Big[\sum_{j=1}^{N} \{h_j(s-\tau_j(s)) - a_j(s)\} x(s-\tau_j(s)) \\ & - [1-H(s)] Q(s, x(s-\tau_1(s)), ..., x(s-\tau_N(s))) - [1-H(s)] \sum_{j=1}^{N} \sum_{r=s-\tau_j(s)}^{s-1} h_j(r)x(r) \\ & + \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u) f_j(u, x(u))\Big] \prod_{u=s+1}^{n-1} H(u). \end{aligned} \tag{8}$$

Proof. Let $x \in P_T$ be a solution of (1). Rewrite (1) as

$$\begin{aligned}\Delta x(n) &= -\sum_{j=1}^{N} h_j(n)x(n) + \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s)\\ &\quad+\sum_{j=1}^{N}\{h_j(n-\tau_j(n)) - a_j(n)\}x(n-\tau_j(n))\\ &\quad+\Delta Q(n, x(n-\tau_1(n)), ..., x(n-\tau_N(n)))\\ &\quad+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} k_j(n,s)f_j(s,x(s)),\end{aligned}$$

where Δ_n denotes the difference taken with respect to n. The above equation is equivalent to

$$\begin{aligned}x(n+1) &= H(n)x(n) + \Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s)\\ &\quad+\sum_{j=1}^{N}\{h_j(n-\tau_j(n)) - a_j(n)\}x(n-\tau_j(n))\\ &\quad+\Delta Q(n, x(n-\tau_1(n)), ..., x(n-\tau_N(n)))\\ &\quad+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} k_j(n,s)f_j(s,x(s)). \qquad (9)\end{aligned}$$

Rewrite equation (9) as

$$\begin{aligned}\Delta_n\Big[\prod_{u=0}^{n-1} H(u)^{-1}x(u)\Big] &= \Big[\Delta_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s)\\ &\quad+\sum_{j=1}^{N}\{h_j(n-\tau_j(n)) - a_j(n)\}x(n-\tau_j(n))\\ &\quad+\Delta Q(n, x(n-\tau_1(n)), ..., x(n-\tau_N(n))) \qquad (10)\\ &\quad+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} k_j(n,s)f_j(s,x(s))\Big]\prod_{u=0}^{n} H(u)^{-1}.\end{aligned}$$

Summing (10) from $(n-T)$ to $(n-1)$ we obtain

$$\begin{aligned}\sum_{s=n-T}^{n-1}\Delta_s\Big[\prod_{u=0}^{s-1} H(u)^{-1}x(s)\Big] &= \sum_{s=n-T}^{n-1}\Big[\Delta_s \sum_{j=1}^{N} \sum_{r=s-\tau_j(s)}^{s-1} h_j(r)x(r)\\ &\quad+\sum_{j=1}^{N}\{h_j(s-\tau_j(s)) - a_j(s)\}x(s-\tau_j(s))\\ &\quad+Q(s, x(s-\tau_1(s)), ..., x(s-\tau_N(s)))\\ &\quad+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u)f_j(u,x(u))\Big]\prod_{u=0}^{s} H(u)^{-1}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& x(n)\prod_{u=0}^{n-1}H(u)^{-1}-x(n-T)\prod_{u=0}^{n-T-1}H(u)^{-1} \\
&=\sum_{s=n_0}^{n-1}\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r) \\
&\quad+\sum_{j=1}^{N}\{h_j(s-\tau_j(s))-a_j(s)\}x(s-\tau_j(s)) \\
&\quad+\Delta Q(s,x(s-\tau_1(s)),...,x(s-\tau_N(s)) \\
&\quad+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}k_j(s,u)f_j(u,x(u))\Big]\prod_{u=0}^{s}H(u)^{-1}. \qquad (11)
\end{aligned}$$

Dividing both sides of (11) by $\prod_{u=0}^{n-1}H(u)^{-1}$ we obtain

$$\begin{aligned}
x(n) \quad = \quad & \frac{1}{1-\prod_{r=n-T}^{n-1}H(r)}\sum_{s=n-T}^{n-1}\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r) \\
&+\sum_{j=1}^{N}\{h_j(s-\tau_j(s))-a_j(s)\}x(s-\tau_j(s)) \\
&+\Delta Q(s,x(s-\tau_1(s)),...,x(s-\tau_N(s)) \\
&+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}k_j(s,u)f_j(u,x(u))\Big]\prod_{r=s+1}^{n-1}H(r). \qquad (12)
\end{aligned}$$

Using the summation by parts formula, we obtain

$$\begin{aligned}
& \sum_{s=n-T}^{n-1}\Big[\Delta Q(s,x(s-\tau_1(s)),...,x(s-\tau_N(s))\Big]\prod_{u=s+1}^{n-1}H(u) \\
&= Q(s,x(s-\tau_1(s)),...,x(s-\tau_N(s))\Big[1-\prod_{r=t-T}^{n-1}H(r)\Big] \qquad (13) \\
&\quad -\sum_{s=n-T}^{n-1}Q(s,x(s-\tau_1(s)),...,x(s-\tau_N(s))[1-H(s)]\prod_{u=s+1}^{n-1}H(u),
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{s=n-T}^{n-1}\prod_{u=s+1}^{n-1}H(u)\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r)\Big] \\
&=\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}h_j(s)x(s)\Big[1-\prod_{u=n-T}^{n-1}H(u)\Big] \\
&\quad -\sum_{s=n-T}^{n-1}[1-H(s)]\prod_{u=s+1}^{n-1}H(u)\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r). \qquad (14)
\end{aligned}$$

Substituting (13) and (14) into (12) gives the desired results.

The following result which is found in [8] will be used to establish our main results in the next section.

Theorem 2 [Krasnoselskii] Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, ||.||)$. Suppose that J and A map $\mathbb{M}$ into $\mathbb{B}$ such that

(i) J is compact and continuous,
(ii) A is a contraction mapping,
(iii) $x, y \in \mathbb{M}$, implies $Jx + Ay \in \mathbb{M}$.

Then there exists $z \in \mathbb{M}$ with $z = Jz + Az$.

3. Existence and uniqueness of periodic solutions

In this section we state and prove our main results. We begin by defining the maps $A, J : P_T \to P_T$ by

$$(A\varphi)(n) = Q(n, x(n-\tau_1(n)), ..., x(n-\tau_N(n))) + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s), \qquad (15)$$

and

$$\begin{aligned}(J\varphi)(n) \quad = \quad & \Big[1 - \prod_{r=n-T}^{n-1} H(r)\Big]^{-1} \sum_{s=n-T}^{n-1} \Big[\sum_{j=1}^{N} \{h_j(s-\tau_j(s)) - a_j(s)\} x(s-\tau_j(s)) \\ & -[1-H(s)]Q(s, x(s-\tau_1(s)), ..., x(s-\tau_N(s)) - [1-H(s)] \sum_{j=1}^{N} \sum_{r=s-\tau_j(s)}^{s-1} h_j(r)x(r) \\ & + \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u) f_j(u, x(u))\Big] \prod_{u=s+1}^{n-1} H(u). \qquad (16)\end{aligned}$$

Lemma 3 Suppose (3)-(6) hold. If J is defined by (16) then $J : P_T \to P_T$ is continuous and compact.

Proof. We will first show that $(J\varphi)(n+T) = (J\varphi)(n)$. Let $\varphi \in P_T$. Then using (16) we obtain

$$\begin{aligned}(J\varphi)(n+T) \quad = \quad & \Big[1 - \prod_{r=n}^{n+T-1} H(r)\Big]^{-1} \sum_{s=n}^{n+T-1} \Big[\sum_{j=1}^{N} \{h_j(s-\tau_j(s)) - a_j(s)\} x(s-\tau_j(s)) \\ & -[1-H(s)]Q(s, x(s-\tau_1(s)), ..., x(s-\tau_N(s))) \\ & -[1-H(s)] \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} h_j(u)x(u) \\ & + \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u) f_j(u, x(u))\Big] \prod_{r=s+1}^{n+T-1} H(r)\end{aligned}$$

Let $v = s - T$, then

$$\begin{aligned}
(J\varphi)(n+T) \quad = \quad & \Big[1 - \prod_{r=n}^{n+T-1} H(r)\Big]^{-1} \sum_{v=n-T}^{n-1} \Big[\sum_{j=1}^{N} \{h_j(v+T-\tau_j(v+T)) \\
& -a_j(v+T)\} x(v+T-\tau_j(v+T)) \\
& -[1-H(v+T)]Q(v+T, x(v+T-\tau_1(v+T)), ..., x(v+T-\tau_N(v+T)) \\
& -[1-H(v+T)] \sum_{j=1}^{N} \sum_{u=v+T-\tau_j(v+T)}^{v+T-1} h_j(u)x(u) \\
& + \sum_{j=1}^{N} \sum_{u=v+T-\tau_j(v+T)}^{v+T-1} k_j(v+T,u) f_j(u,x(u))\Big] \prod_{r=v+T+1}^{n+T-1} H(r)
\end{aligned}$$

$$\begin{aligned}
= \quad & \Big[1 - \prod_{r=n}^{n+T-1} H(r)\Big]^{-1} \sum_{v=n-T}^{n-1} \Big[\sum_{j=1}^{N} \{h_j(v-\tau_j(v)) - a_j(v)\} x(v-\tau_j(v)) \\
& -[1-H(v)]Q(v, x(v-\tau_1(v)), ..., x(v-\tau_N(v)) \\
& -[1-H(v)] \sum_{j=1}^{N} \sum_{u=v-\tau_j(v)}^{v-1} h_j(u)x(u) \\
& + \sum_{j=1}^{N} \sum_{u=v-\tau_j(v)}^{v-1} k_j(v,u) f_j(u,x(u))\Big] \prod_{i=v+1}^{n-1} H(i)
\end{aligned}$$

Now let $i = r - T$, then

$$\begin{aligned}
(J\varphi)(n+T) \quad = \quad & \Big[1 - \prod_{i=n-T}^{n-1} H(i)\Big]^{-1} \sum_{v=n-T}^{n-1} \Big[\sum_{j=1}^{N} \{h_j(v-\tau_j(v)) - a_j(v)\} x(v-\tau_j(v)) \\
& -[1-H(v)]Q(v, x(v-\tau_1(v)), ..., x(v-\tau_N(v)) \\
& -[1-H(v)] \sum_{j=1}^{N} \sum_{u=v-\tau_j(v)}^{v-1} h_j(u)x(u) \\
& + \sum_{j=1}^{N} \sum_{u=v-\tau_j(v)}^{v-1} k_j(v,u) f_j(u,x(u))\Big] \prod_{i=v+1}^{n-1} H(i) = (J\varphi)(n).
\end{aligned}$$

We next show that J is continuous. Let $\varphi, \psi \in P_T$ with $||\varphi|| \le \gamma$ and $||\psi|| \le \gamma$. Let

$$\begin{aligned}
& \beta_1 = \max_{n\in[0,T-1]} \Big|\frac{1}{1-\prod_{r=n-T}^{n-1} H(r)}\Big|, \;\; \beta_2 = \max\Big\{\max_{n\in[n-T,n]} |h_1(n)|, \; ..., \max_{n\in[n-T,n]} |h_N(n)|\Big\} \\
& \beta_3 = \max\Big\{\max_{n\in[n-T,n]} |k_1(n,u)|, \; ..., \max_{n\in[n-T,n]} |k_N(n,u)|\Big\}, \; \beta_4 = \max_{n\in[0,n-1]} \Big|\prod_{i=n-T}^{n-1} H(i)\Big| \\
& \beta_5 = \max\Big\{\max_{n\in[n-T,n]} |a_1(n)|, \; ..., \max_{n\in[n-T,n]} |a_N(n)|\Big\}, \; L = \max\{L_j, \; j = 1, ..., N\}, \\
& \tau = \max\Big\{\max_{n\in[n-T,n]} |\tau_1(n)|, \; ..., \max_{n\in[n-T,n]} |\tau_N(n)|\Big\}, \; \rho = \max\{\rho_j, \; j = 1, ..., N\}. \qquad (17)
\end{aligned}$$

Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M}$ such that $||\varphi - \psi|| < \delta$, where $M = T\beta_1\beta_4[N(\beta_2 + \beta_5) + N\beta_2 L + N^2\beta_2^2\tau + N\tau\beta_3\rho]$. Using (16) we obtain

$$\begin{aligned}
|(J\varphi)(n) - (J\psi)(n)| &\leq \beta_1 \sum_{s=n-T}^{n-1} \Big[N(\beta_2 + \beta_5)||\varphi - \psi|| \\
&\quad + N\beta_2 L||\varphi - \psi|| + N^2\beta_2^2\tau||\varphi - \psi|| \\
&\quad + N\tau\beta_3\rho||\varphi - \psi||\Big]\beta_4 \\
&= T\beta_1\beta_4\Big[N(\beta_2 + \beta_5) + N\beta_2 L + N^2\beta_2^2\tau \\
&\quad + N\tau\beta_3\rho\Big]||\varphi - \psi|| \\
&< T\beta_1\beta_4\Big[N(\beta_2 + \beta_5) + N\beta_2 L + N^2\beta_2^2\tau + N\tau\beta_3\rho\Big]\delta \\
&= \epsilon.
\end{aligned}$$

Thus,

$$||(J\varphi) - (J\psi)|| \leq \epsilon.$$

This proves that J is continuous.

We next show that J maps bounded subsets into compact sets. Let μ be given such that $S = \{\varphi \in P_T \ : \ ||\varphi|| \leq \mu\}$ and $F = \{(J\varphi)(n) \ : \ \varphi \in S\}$ then S is a subset of R^T which is closed and bounded thus compact. Since J is continuous in φ it maps compact sets into compact sets. Then $F = J(S)$ is compact. This completes the proof.

Lemma 4 Suppose that (6) hold. If A is given by (15) and

$$N(L + \tau\beta_2) \leq \alpha_1 < 1 \tag{18}$$

then A is a contraction.

Proof. Let A be defined by (15). Let L, β_2 and τ be given by (17). Then for $\varphi, \psi \in P_T$ we have

$$\begin{aligned}
|(A\varphi)(n) - (A\psi)(n)|| &\leq \sum_{j=1}^{N} L_j||\varphi - \psi|| + \sum_{j=1}^{N}\Big|\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\Big|||\varphi - \psi|| \\
&\leq N(L + \tau\beta_2)||\varphi - \psi|| \\
&\leq \alpha_1||\varphi - \psi||.
\end{aligned}$$

Hence, A is a contraction.

Theorem 5 Let $v_1 = \max\{f_j(n, 0), \ j = 1, ..., N\}$ and $v_2 = ||Q(n, 0, 0, ..., 0)||$. Let $\beta_1, \beta_2, \beta_3, \beta_4$, and β_5 be given by (17). Suppose (3)-(6) and (18) hold. Suppose further that there is a positive constant ν such that all solutions of (1), $x(t) \in P_T$ satisfy $|x(t)| \leq \nu$, the inequality

$$\begin{aligned}
&\Big\{T\beta_1\beta_4\big(N(\beta_2 + \beta_5) + N\beta_2 L + N^2\beta_2^2\tau + N\tau\beta_3\rho\big) \\
&+ NL + N\tau\beta_2\Big\}\nu + T\beta_1\beta_4(N\beta_2 v_1 + N\tau\beta_3 v_2) + v_1 \leq \nu
\end{aligned} \tag{19}$$

holds. Then equation (1) has a T-periodic solution.

Proof. Define $\mathbb{M} = \{\varphi \in P_T \ : \ ||\varphi|| \leq \nu\}$. Then Lemma 3 implies $J : M \to P_T$ and is continuous and compact. Moreover, it follows from Lemma 4 that the mapping A is a contraction and it is clear that $A : M \to P_T$. We finally show that if $\varphi, \psi \in \mathbb{M}$, we have $||J\varphi + A\psi|| \leq G$. To this end, let $\varphi, \psi \in \mathbb{M}$, then from (15)-(16) and the fact that $|Q(n, x, x, ..., x)| \leq \sum_{j=1}^{N} L_j||x|| + v_2$ and $|f_j(n, x)| \leq \rho_j||x|| + ||f_j(n, 0)||$ we obtain

$$\begin{aligned}
& |(J\varphi)(n) + (A\psi)(n)| \\
= & \Big|\Big[1 - \prod_{r=n-T}^{n-1} H(r)\Big]^{-1} \sum_{s=n-T}^{n-1} \Big[\sum_{j=1}^{N}\{h_j(s - \tau_j(s)) - a_j(s)\}\varphi(s - \tau_j(s)) \\
& -[1 - H(s)]Q(s, \varphi(s - \tau_1(s)), ..., \varphi(s - \tau_N(s)) - [1 - H(s)] \sum_{j=1}^{N} \sum_{r=s-\tau_j(s)}^{s-1} h_j(r)\varphi(r) \\
& + \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} k_j(s, u) f_j(u, \varphi(u))\Big] \prod_{u=s+1}^{n-1} H(u) \\
& +Q(n, \psi(n - \tau_1(n)), ..., \psi(n - \tau_N(n)) + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\psi(s)\Big| \\
\leq & \beta_1 \sum_{s=n-T}^{n-1} \Big[N(\beta_2 + \beta_5)||\varphi|| \\
& +N\beta_2 L||\varphi|| + N\beta_2 v_1 + N^2\beta_2^2\tau||\varphi|| \\
& + N\tau\beta_3\rho||\varphi|| + N\tau\beta_3 v_2\Big]\beta_4 \\
& + NL||\psi|| + v_1 + N\tau\beta_2||\psi|| \\
\leq & T\beta_1\beta_4\Big[\big(N(\beta_2 + \beta_5) + N\beta_2 L + N^2\beta_2^2\tau + N\tau\beta_3\rho\big)\nu \\
& \qquad +(N\beta_2 v_1 + N\tau\beta_3 v_2)\Big] + NL\nu + v_1 + N\tau\beta_2\nu \\
& = \Big\{T\beta_1\beta_4\big(N(\beta_2 + \beta_5) + N\beta_2 L + N^2\beta_2^2\tau + N\tau\beta_3\rho\big) \\
& \qquad +NL + N\tau\beta_2\Big\}\nu + T\beta_1\beta_4(N\beta_2 v_1 + N\tau\beta_3 v_2) + v_1 \\
& \leq \nu.
\end{aligned}$$

Thus,

$$||(J\varphi) + (A\psi)|| \leq \nu.$$

Therefore, all the conditions of the Krasnoselskii's theorem are satisfied on the set $\mathbb{M}$. Thus, equation (1) has a T-periodic solution.

Theorem 6 Suppose (3)-(6) and (18) hold. Let $\beta_1, \beta_2, \beta_3, \beta_4$, and β_5 be given by (17). If

$$\begin{aligned}
& N(L + \tau\beta_2) + T\beta_1\beta_4[N(\beta_2 + \beta_5) \\
& + N\beta_2 L + N^2\beta_2^2\tau + N\tau\beta_3\rho] \leq \alpha_2 < 1,
\end{aligned} \tag{20}$$

then equation (1) has a unique T-periodic solution.

Proof. Define a mapping $H : P_T \to P_T$ by $(H\varphi)(n) = (A\varphi)(n) + (J\varphi)(n)$. Then for $\varphi, \psi \in P_T$ we have

$$
\begin{aligned}
&|(H\varphi)(n) - (H\psi)(n)| \\
&= \Big|\Big((A\varphi)(n) + (J\varphi)(n)\Big) - \Big((A\psi)(n) + (J\psi)(n)\Big)\Big| \\
&= \Big|\Big((A\varphi)(n) - (A\psi)(n)\Big) - \Big((J\varphi)(n) - (J\psi)(n)\Big)\Big| \\
&\leq N(L + \tau\beta_2)||\varphi - \psi|| + T\beta_1\beta_4\Big[N(\beta_2 + \beta_5) + N\beta_2 L \\
&\quad + N^2\beta_2^2\tau + N\tau\beta_3\rho\Big]||\varphi - \psi|| \\
&= \Big\{N(L + \tau\beta_2) + T\beta_1\beta_4\Big[N(\beta_2 + \beta_5) + N\beta_2 L \\
&\quad + N^2\beta_2^2\tau + N\tau\beta_3\rho\Big]\Big\}||\varphi - \psi|| \\
&\leq \alpha_2||\varphi - \psi||.
\end{aligned}
$$

By the contraction mapping principle, (1) has a unique T-periodic solution.

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