Electronic Journal of Mathematical Analysis and Applications
Vol. 6(2) July 2018, pp. 172-184.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

NEW PARAMETRIC HADAMARD TYPE INEQUALITIES WITH APPLICATIONS

M. ADIL KHAN, M. AIZAZ ALI AND TINGSONG DU

Abstract. In this paper, we offer new parametric Hadamard type inequalities for differentiable convex and concave functions. Moreover, some consequent applications to special means of real numbers are obtained.

1. Introduction

A function $\Psi : \mathtt{J} \to \mathbb{R}$ is said to be convex on $\mathtt{J}$ if the following inequality holds:

$$\Psi(t\xi + (1-t)\eta) \leq t\Psi(\xi) + (1-t)\Psi(\eta) \tag{1}$$

where $\xi, \eta \in J$, $t \in [0,1]$. For concavity of Ψ the inequality (1) will revered. The function Ψ is strictly convex if strictly inequality holds in (1). Ψ is strictly concave if inequality (1) is strict in reversed order. Let $\Psi : \mathtt{J} \to \mathbb{R}$ be a convex function define on J, then the following inequality is known as Hermite-Hadamard inequality for convex function.

$$\Psi\left(\frac{\beta_1+\beta_2}{2}\right) \leq \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} \Psi(\xi)d\xi \leq \frac{\Psi(\beta_1)+\Psi(\beta_2)}{2}, \tag{2}$$

where $\beta_1, \beta_2 \in J$, and $\beta_1 < \beta_2$. This inequality is a key to check whether the given function is convex, concave or not. Moreover, it shows that every convex function is integrable. The inequality (2) holds if and only if Ψ is a convex function. It will reverse if Ψ is concave. In the last few decades, a huge class of important inequalities have been designed which is connected with inequality (2), (see [1]-[31]).

Kirmaci [32] proved the following results linked with the left part of (2).

Lemma 1 [[32, Lemma 2.1]] Let $\Psi : J^{\circ} \to \mathbb{R}$ be a differentiable function on J°

2010 *Mathematics Subject Classification.* 26D15, 26A51, 26A42.
Key words and phrases. Hermite-Hadamard inequality, Convex function, Concave function, integral inequalities, means.
Submitted Dec. 18, 2017.

with $\beta_1, \beta_2 \in J^\circ$ and $\beta_1 < \beta_2$. If $\Psi' \in L[\beta_1, \beta_2]$, then the following identity holds:

$$\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta - \Psi\left(\frac{\beta_1+\beta_2}{2}\right) = (\beta_2-\beta_1)\left[\int_0^{\frac{1}{2}} t\Psi'(t\beta_1+(1-t)\beta_2)\,dt + \int_{\frac{1}{2}}^{1}(t-1)\Psi'(t\beta_1+(1-t)\beta_2)\,dt\right],$$

where and in what follows J° denotes the interior of J.

Theorem 1 [[32, Theorem 2.2]] Let $\Psi : J^\circ \to \mathbb{R}$ be a differentiable function on J° with $\beta_1, \beta_2 \in J^\circ$ and $\beta_1 < \beta_2$. If $|\Psi'|$ is convex on the interval J, then the following inequality holds:

$$\left|\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta - \Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right| \le \frac{\beta_2-\beta_1}{8}\big(|\Psi'(\beta_1)|+|\Psi'(\beta_2)|\big).$$

Theorem 2 [[32, Theorem 2.3]] Let $\Psi : J^\circ \to \mathbb{R}$ be a differentiable function on J° with $\beta_1, \beta_2 \in J^\circ$ and $\beta_1 < \beta_2$. If the function $|\Psi'|^{\frac{p}{p-1}}$ is convex on J, for $p > 1$, then we have:

$$\left|\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta - \Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right| \le \frac{\beta_2-\beta_1}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(|\Psi'(\beta_1)|^{\frac{p}{p-1}}+3|\Psi'(\beta_2)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(3|\Psi'(\beta_1)|^{\frac{p}{p-1}}+|\Psi'(\beta_2)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right]$$

The main aim of this paper is to offer certain parametric Hadamard type inequalities for functions whose first derivative in absolute values are concave or convex. As applications, inequalities for some means of real number are obtained.

2. Main Results

To prove our main results, first we need to prove the following lemma.

Lemma 2 Let $\vartheta \in \mathbb{R}$ and $\Psi : J^\circ \to \mathbb{R}$ be a differentiable function on J° with $\beta_1, \beta_2 \in J^\circ$ and $\beta_1 < \beta_2$. If $\Psi' \in L[\beta_1, \beta_2]$, then the following identity holds:

$$\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta - \left[(1-\vartheta)\Psi(\beta_1)+\vartheta\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right]$$
$$= (\beta_2-\beta_1)\left[\int_0^{\frac{1}{2}} t\Psi'(t\beta_1+(1-t)\beta_2)dt + \int_{\frac{1}{2}}^{1}(t-\vartheta)\Psi'(t\beta_1+(1-t)\beta_2)dt\right].$$

Proof. Using integration by parts we have

$$\int_0^{\frac{1}{2}} t\Psi'(t\beta_1+(1-t)\beta_2\, dt + \int_{\frac{1}{2}}^{1} (t-\vartheta)\Psi'(t\beta_1+(1-t)\beta_2)\, dt$$

$$= \frac{\Psi(\frac{\beta_1+\beta_2}{2})}{2(\beta_1-\beta_2)} - \frac{1}{\beta_2-\beta_1}\int_0^{\frac{1}{2}} \Psi(t\beta_1+(1-t)\beta_2 dt + \frac{\Psi(\beta_1)(1-\vartheta)-\Psi(\frac{\beta_1+\beta_2}{2})(\frac{1}{2}-\vartheta)}{\beta_1-\beta_2} - \int_{\frac{1}{2}}^{1} \frac{\Psi(t\beta_1+(1-t)\beta_2)dt}{\beta_1-\beta_2}$$

$$= \frac{\Psi(\frac{\beta_1+\beta_2}{2})}{2(\beta_1-\beta_2)} + \frac{\Psi(\beta_1)(1-\vartheta)-\Psi(\frac{\beta_1+\beta_2}{2})(\frac{1}{2}-\vartheta)}{\beta_1-\beta_2} - \frac{1}{\beta_1-\beta_2}\Big[\int_0^{\frac{1}{2}} \Psi(t\beta_1+(1-t))\beta_2 dt + \int_{\frac{1}{2}}^{1} \Psi(t\beta_1+(1-t)\beta_2)dt\Big]$$

$$= \frac{\Psi(\frac{\beta_1+\beta_2}{2})}{2(\beta_1-\beta_2)} + \frac{\Psi(\beta_1)(1-\vartheta)-\Psi(\frac{\beta_1+\beta_2}{2})(\frac{1}{2}-\vartheta)}{\beta_1-\beta_2} - \frac{1}{\beta_1-\beta_2}\left[\int_0^{1} \Psi(t\beta_1+(1-t))\beta_2 dt\right]$$

By changing of variable we get

$$= \frac{\Psi(\frac{\beta_1+\beta_2}{2})}{2(\beta_1-\beta_2)} + \frac{\Psi(\beta_1)(1-\vartheta)-\Psi(\frac{\beta_1+\beta_2}{2})(\frac{1}{2}-\vartheta)}{\beta_1-\beta_2} - \frac{1}{\beta_1-\beta_2}\int_{\beta_2}^{\beta_1} \frac{\Psi(\eta)d\eta}{\beta_1-\beta_2}$$

$$= \frac{\Psi(\frac{\beta_1+\beta_2}{2})}{2(\beta_1-\beta_2)} + \frac{\Psi(\beta_1)(1-\vartheta)-\Psi(\frac{\beta_1+\beta_2}{2})(\frac{1}{2}-\vartheta)}{\beta_1-\beta_2} + \frac{1}{(\beta_2-\beta_1)^2}\int_{\beta_1}^{\beta_2} \Psi(\eta)d\eta$$

$$= \frac{1}{(\beta_2-\beta_1)^2}\int_{\beta_1}^{\beta_2} \Psi(\eta)d\eta - \frac{1}{\beta_2-\beta_1}\left[\frac{2\Psi(\beta_1)(1-\vartheta)+2\vartheta\Psi(\frac{\beta_1+\beta_2}{2})}{2}\right]$$

$$= \frac{1}{(\beta_2-\beta_1)^2}\int_{\beta_1}^{\beta_2} \Psi(\eta)d\eta - \frac{1}{(\beta_2-\beta_1)}\left[\Psi(\beta_1)(1-\vartheta)+\vartheta\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right]$$

Multiply both side by $(\beta_2-\beta_1)$ we have

$$= \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2} \Psi(\eta)d\eta - \left[(1-\vartheta)\Psi(\beta_1)+\vartheta\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right]$$

which is the required result.

Remark 1 If we put $\vartheta=1$ in Lemma 2, we get Lemma 1.

Lemma 3 Let $\vartheta \in \mathbb{R}$, then

$$\int_{\frac{1}{2}}^{1} |t-\vartheta|dt = \begin{cases} \frac{4\vartheta-3}{8} & \vartheta \geq 1 \\ \frac{8\vartheta^2-12\vartheta+5}{8} & \frac{1}{2} < \vartheta < 1 \\ \frac{3-4\vartheta}{8} & \vartheta \leq \frac{1}{2}, \end{cases}$$

$$\int_{\frac{1}{2}}^{1} t|t-\vartheta|dt = \begin{cases} \frac{9\vartheta-7}{24} & \vartheta \geq 1 \\ \frac{8\vartheta^3-15\vartheta+9}{24} & \frac{1}{2} < \vartheta < 1 \\ \frac{7-9\vartheta}{24} & \vartheta \leq \frac{1}{2}, \end{cases}$$

$$\int_{\frac{1}{2}}^{1} (1-t)|t-\vartheta|dt = \begin{cases} \frac{3\vartheta-2}{24} & \vartheta \geq 1 \\ \frac{-8\vartheta^3+24\vartheta^2-21\vartheta+6}{24} & \frac{1}{2} < \vartheta < 1 \\ \frac{2-3\vartheta}{24} & \vartheta \leq \frac{1}{2}. \end{cases}$$

Theorem 3 Let $\vartheta \in \mathbb{R}$ and $\Psi : J^\circ \to \mathbb{R}$ be a differentiable function on J° with $\beta_1, \beta_2 \in J^\circ$ and $\beta_1 < \beta_2$. If $|\Psi'|$ is convex on the interval J, then the following inequality holds:

$$\left| \frac{1}{\beta_2-\beta_1} \int_{\beta_1}^{\beta_2} \Psi(\eta)d\eta - (1-\vartheta)\Psi(\beta_1) - \vartheta\Psi\big(\frac{\beta_1+\beta_2}{2}\big) \right| \leq \frac{\beta_2-\beta_1}{8} \left(\frac{|\Psi'(\beta_1)| + 2|\Psi'(\beta_2)|}{3} \right)$$

$$+(\beta_2-\beta_1) \begin{cases} |\Psi'(\beta_1)| \left(\frac{9\vartheta-7}{24}\right) + |\Psi'(\beta_2)| \left(\frac{3\vartheta-2}{24}\right), & \text{if } \vartheta \geq 1 \\ |\Psi'(\beta_1)| \left(\frac{8\vartheta^3-15\vartheta+9}{24}\right) + |\Psi'(\beta_2)| \left(\frac{-8\vartheta^3+24\vartheta^2-21\vartheta+6}{24}\right), & \text{if } \frac{1}{2} < \vartheta < 1 \\ |\Psi'(\beta_1)| \left(\frac{7-9\vartheta}{24}\right) + |\Psi'(\beta_2)| \left(\frac{2-3\vartheta}{24}\right), & \text{if } \vartheta \leq \frac{1}{2} \end{cases}$$

Proof. Using the above Lemma 2 we have

$$\begin{aligned} & \left| \frac{1}{\beta_1-\beta_2} \int_{\beta_1}^{\beta_2} \Psi(\eta)d\eta - \Big[(1-\vartheta)\Psi(\beta_1) + \vartheta\Psi\big(\frac{\beta_1+\beta_2}{2}\big)\Big] \right| \\ = & \left| \beta_2-\beta_1 \left[\int_0^{\frac{1}{2}} t\Psi'(t\beta_1+(1-t)\beta_2)\,dt + \int_{\frac{1}{2}}^{1} (t-\vartheta)\Psi'(t\beta_1+(1-t)\beta_2)\,dt \right] \right| \\ \leq & (\beta_2-\beta_1) \left[\int_0^{\frac{1}{2}} |t||\Psi'(t\beta_1+(1-t)\beta_2|\,dt + \int_{\frac{1}{2}}^{1} |t-\vartheta||\Psi'(t\beta_1+(1-t)\beta_2)|dt \right] \\ \leq & (\beta_2-\beta_1) \Bigg[\int_0^{\frac{1}{2}} \big(t^2|\Psi'(\beta_1)| + t(1-t)|\Psi'(\beta_2)|\big)dt \\ + & \int_{\frac{1}{2}}^{1} \big(t|t-\vartheta||\Psi'(\beta_1)| + (1-t)|t-\vartheta||\Psi'(\beta_2)|\big)dt \Bigg] \\ = & (\beta_2-\beta_1) \left[\frac{|\Psi'(\beta_1)|}{24} + \frac{|\Psi'(\beta_2)|}{12} \right] \end{aligned}$$

$$
\begin{aligned}
&+ \quad (\beta_2-\beta_1)\left[\int_{\frac{1}{2}}^{1}\big(t|t-\vartheta||\Psi'(\beta_1)|+(1-t)|t-\vartheta||\Psi'(\beta_2)|\big)dt\right]\\
&= \quad \frac{\beta_2-\beta_1}{8}\left(\frac{|\Psi'(\beta_1)|+2|\Psi'(\beta_2)|}{3}\right)\\
&+ \quad (\beta_2-\beta_1)\begin{cases}|\Psi'(\beta_1)|\left(\frac{9\vartheta-7}{24}\right)+|\Psi'(\beta_2)|\left(\frac{3\vartheta-2}{24}\right), \quad \text{if } \vartheta\geq 1\\ |\Psi'(\beta_1)|\left(\frac{8\vartheta^3-15\vartheta+9}{24}\right)+|\Psi'(\beta_2)|\left(\frac{-8\vartheta^3+24\vartheta^2-21\vartheta+6}{24}\right),\\ \text{if } \frac{1}{2}<\vartheta<1\\ |\Psi'(\beta_1)|\left(\frac{7-9\vartheta}{24}\right)+|\Psi'(\beta_2)|\left(\frac{2-3\vartheta}{24}\right), \text{ if } \vartheta\leq\frac{1}{2}.\end{cases}
\end{aligned}
$$

Which completes the proof.

Remark 2 If we put $\vartheta = 1$ in Theorem 2, we get Theorem 1.

Theorem 4 Let $\vartheta \in \mathbb{R}$ and $\Psi : J^\circ \to \mathbb{R}$ be a differentiable function on J° with $\beta_1, \beta_2 \in J^\circ$ and $\beta_1 < \beta_2$. If the function $|\Psi'|^{\frac{p}{p-1}}$ is convex on J with $p, q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$
\begin{aligned}
&\left|\quad \frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta-\left[(1-\vartheta)\Psi(\beta_1)+\vartheta\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right]\right|\\
&\leq \quad \frac{\beta_2-\beta_1}{4(p+1)^{\frac{1}{p}}}\left(\frac{|\Psi'(\beta_1)|^{\frac{p}{p-1}}+3|\Psi'(\beta_2)|^{\frac{p}{p-1}}}{4}\right)^{\frac{p-1}{p}}\\
&+ \quad \frac{\beta_2-\beta_1}{2^{\frac{p-1}{p}}}\left(\frac{3|\Psi'(\beta_1)|^{\frac{p}{p-1}}+|\Psi'(\beta_2)|^{\frac{p}{p-1}}}{4}\right)^{\frac{p-1}{p}}\begin{cases}\left(\frac{(2\vartheta-1)^{p+1}-(\vartheta-1)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, \quad \text{if } \vartheta\geq 1\\ \left(\frac{(2\vartheta-1)^{p+1}+(1-\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, \quad \text{if } \frac{1}{2}<\vartheta<1\\ \left(\frac{(1-\vartheta)^{p+1}-(1-2\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, \quad \text{if } \vartheta\leq\frac{1}{2}\end{cases}
\end{aligned}
$$

Proof. Using Lemma 2 we have

$$
\begin{aligned}
&\left|\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta-\left[(1-\vartheta)\Psi(\beta_1)+\vartheta\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right]\right|\\
&\leq \quad (\beta_2-\beta_1)\left[\int_0^{\frac{1}{2}}|t||\Psi'(t\beta_1+(1-t)\,\beta_2|\,dt+\int_{\frac{1}{2}}^{1}|t-\vartheta||\Psi'(t\beta_1+(1-t)\,\beta_2)|dt\right]\\
&= \quad (\beta_2-\beta_1)\left[\int_0^{\frac{1}{2}}t|\Psi'(t\beta_1+(1-t)\,\beta_2|dt+\int_{\frac{1}{2}}^{1}|t-\vartheta||\Psi'(t\beta_1+(1-t)\beta_2)|dt\right]\\
&\leq \quad (\beta_2-\beta_1)\bigg[\Big(\int_0^{\frac{1}{2}}t^p dt\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}}|\Psi'(t\beta_1+(1-t)\beta_2)|^q dt\Big)^{\frac{1}{q}}
\end{aligned}
$$

$$+\Big(\int_{\frac{1}{2}}^{1}|t-\vartheta|^p dt\Big)^{\frac{1}{p}}\Big(\int_{\frac{1}{2}}^{1}|\Psi'(t\beta_1+(1-t)\beta_2)|^q dt\Big)^{\frac{1}{q}}\Big]$$

(Using Hölder integral inequality)

$$\leq (\beta_2-\beta_1)\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}[t|\Psi'(\beta_1)|^q+(1-t)|\Psi'(\beta_2)|^q]dt\right)^{\frac{1}{q}}$$

$$+(\beta_2-\beta_1)\left(\int_{\frac{1}{2}}^{1}|t-\vartheta|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1} t|\Psi'(\beta_1)|^q+(1-t)|\Psi'(\beta_2)|^q\right)^{\frac{1}{q}}$$

(By convexity of $|\Psi'|^q$, where $q=\frac{p}{p-1}$)

$$= (\beta_2-\beta_1)\left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\beta_1)|^q+3|\Psi'(\beta_2)|^q}{8}\right)^{\frac{1}{q}}$$

$$+ (\beta_2-\beta_1)\left(\frac{3|\Psi'(\beta_1)|^q+|\Psi'(\beta_2)|^q}{8}\right)^{\frac{1}{q}}\begin{cases}\left(\frac{(2\vartheta-1)^{p+1}-(\vartheta-1)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, & \text{if } \vartheta\geq 1\\ \left(\frac{(2\vartheta-1)^{p+1}+(1-\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, & \text{if } \frac{1}{2}<\vartheta<1\\ \left(\frac{(1-\vartheta)^{p+1}-(1-2\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, & \text{if } \vartheta\leq\frac{1}{2}\end{cases}$$

$$= \frac{\beta_2-\beta_1}{4(p+1)^{\frac{1}{p}}}\left(\frac{|\Psi'(\beta_1)|^{\frac{p}{p-1}}+3|\Psi'(\beta_2)|^{\frac{p}{p-1}}}{4}\right)^{\frac{p-1}{p}}$$

$$+ \frac{\beta_2-\beta_1}{2^{\frac{p-1}{p}}}\left(\frac{3|\Psi'(\beta_1)|^{\frac{p}{p-1}}+|\Psi'(\beta_2)|^{\frac{p}{p-1}}}{4}\right)^{\frac{p-1}{p}}\begin{cases}\left(\frac{(2\vartheta-1)^{p+1}-(\vartheta-1)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, & \text{if } \vartheta\geq 1\\ \left(\frac{(2\vartheta-1)^{p+1}+(1-\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, & \text{if } \frac{1}{2}<\vartheta<1\\ \left(\frac{(1-\vartheta)^{p+1}-(1-2\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, & \text{if } \vartheta\leq\frac{1}{2}\end{cases}$$

This completes the proof.

Remark 3 If we put $\vartheta=1$ in Theorem 2, we get Theorem 1.

Theorem 5 If the function $|\Psi'|^q$ is convex on J, with $q\geq 1$, then the following inequality holds:

$$\left|\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta-\Big[(1-\vartheta)\Psi(\beta_1)+\vartheta\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\Big]\right|$$

$$\leq \frac{\beta_2-\beta_1}{8}\left[\frac{|\Psi'(\beta_1)|^q+2|\Psi'(\beta_2)|^q}{3}\right]^{\frac{1}{q}}$$

$$+ (\beta_2-\beta_1)\begin{cases}(\frac{4\vartheta-3}{8})^{1-\frac{1}{q}}\Big[|\Psi'(\beta_1)|^q\left(\frac{9\vartheta-7}{24}\right)+|\Psi'(\beta_2)|^q\left(\frac{3\vartheta-2}{24}\right)\Big]^{\frac{1}{q}}, & \text{if } \vartheta\geq 1\\ (\frac{8\vartheta^2-12\vartheta+5}{8})^{1-\frac{1}{q}}\Big[|\Psi'(\beta_1)|^q\left(\frac{8\vartheta^3-15\vartheta+9}{24}\right)+|\Psi'(\beta_2)|^q\left(\frac{-8\vartheta^3+24\vartheta^2-21\vartheta+6}{24}\right)\Big]^{\frac{1}{q}}, & \text{if } \frac{1}{2}<\vartheta<1\\ (\frac{3-4\vartheta}{8})^{1-\frac{1}{q}}\Big[|\Psi'(\beta_1)|^q\left(\frac{7-9\vartheta}{24}\right)+|\Psi'(\beta_2)|^q\left(\frac{2-3\vartheta}{24}\right)\Big]^{\frac{1}{q}}, & \text{if } \vartheta\leq\frac{1}{2}\end{cases}$$

Proof. Using Lemma 2 we have:

$$\begin{aligned}
&\left|\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta-\left[(1-\vartheta)\Psi(\beta_1)+\vartheta\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right]\right| \\
\leq\ & (\beta_2-\beta_1)\left[\int_0^{\frac{1}{2}} t|\Psi'(t\beta_1+(1-t)\beta_2)|dt+\int_{\frac{1}{2}}^{1}|t-\vartheta||\Psi'(t\beta_1+(1-t)\beta_2)|dt\right] \\
=\ & (\beta_2-\beta_1)\Big[\int_0^{\frac{1}{2}} t|\Psi'(t\beta_1+(1-t)\beta_2)|dt\Big]+(\beta_2-\beta_1)\Big[\int_{\frac{1}{2}}^{1}|t-\vartheta||\Psi'(t\beta_1+(1-t)\beta_2)|dt\Big] \\
& \text{(By power mean inequality)} \\
\leq\ & (\beta_2-\beta_1)\left(\int_0^{\frac{1}{2}} t.dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t.|\Psi'(t\beta_1+(1-t)\beta_2)|^q dt\right)^{\frac{1}{q}} \\
+\ & (\beta_2-\beta_1)\left(\int_{\frac{1}{2}}^{1}|t-\vartheta|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}|t-\vartheta||\Psi'(t\beta_1+(1-t)\beta_2)|^q dt\right)^{\frac{1}{q}} \\
& \text{(By convexity of } |\Psi'|^q\text{, we have)} \\
\leq\ & (\beta_2-\beta_1)\left(\int_0^{\frac{1}{2}} t.dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^2.|\Psi'(\beta_1)|^q dt+\int_0^{\frac{1}{2}} t(1-t)|\Psi'(\beta_2)|^q dt\right)^{\frac{1}{q}} \\
+\ & (\beta_2-\beta_1)\left(\int_{\frac{1}{2}}^{1}|t-\vartheta|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1} t|t-\vartheta||\Psi'(\beta_1)|^q dt+\int_{\frac{1}{2}}^{1}(1-t)|t-\vartheta||\Psi'(\beta_2)|^q dt\right)^{\frac{1}{q}} \\
=\ & \frac{\beta_2-\beta_1}{8}\left[\frac{|\Psi'(\beta_1)|^q+2|\Psi'(\beta_2)|^q}{3}\right]^{\frac{1}{q}} \\
+\ & (\beta_2-\beta_1)\begin{cases}
(\frac{4\vartheta-3}{8})^{1-\frac{1}{q}}\Big[|\Psi'(\beta_1)|^q\left(\frac{9\vartheta-7}{24}\right)+|\Psi'(\beta_2)|^q\left(\frac{3\vartheta-2}{24}\right)\Big]^{\frac{1}{q}}, & \text{if } \vartheta\geq 1 \\
(\frac{8\vartheta^2-12\vartheta+5}{8})^{1-\frac{1}{q}}\Big[|\Psi'(\beta_1)|^q\left(\frac{8\vartheta^3-15\vartheta+9}{24}\right)+|\Psi'(\beta_2)|^q\left(\frac{-8\vartheta^3+24\vartheta^2-21\vartheta+6}{24}\right)\Big]^{\frac{1}{q}}, & \\
\text{if } \frac{1}{2}<\vartheta<1 & \\
(\frac{3-4\vartheta}{8})^{1-\frac{1}{q}}\Big[|\Psi'(\beta_1)|^q\left(\frac{7-9\vartheta}{24}\right)+|\Psi'(\beta_2)|^q\left(\frac{2-3\vartheta}{24}\right)\Big]^{\frac{1}{q}}, & \text{if } \vartheta\leq\frac{1}{2}
\end{cases}
\end{aligned}$$

Hence proved.

Remark 4 If we put $\vartheta = 1$ in Theorem 2 we get the following inequality.

$$\left|\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta-\Psi\Big(\frac{\beta_1+\beta_2}{2}\Big)\right|\leq\frac{\beta_2-\beta_1}{8}\Big[\Big(\frac{|\Psi'(\beta_1)|^q+2|\Psi'(\beta_2)|^q}{3}\Big)^{\frac{1}{q}}+\Big(\frac{2|\Psi'(\beta_1)|^q+|\Psi'(\beta_2)|^q}{3}\Big)^{\frac{1}{q}}\Big].$$

Theorem 6 If the function $|\Psi'|^q$ is concave on J, with $q \geq 1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta-\left[(1-\vartheta)\Psi(\beta_1)+\vartheta\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right]\right| \\
\leq\;& \frac{(\beta_2-\beta_1)}{8}\left|\Psi'\left(\frac{\beta_1+2\beta_2}{3}\right)\right| \\
+\;& (\beta_2-\beta_1)\begin{cases}(\frac{4\vartheta-3}{8})\left|\Psi'\Big(\frac{(9\vartheta-7)\beta_1+(3\vartheta-2)\beta_2}{3(4\vartheta-3)}\Big)\right|, & \text{if } \vartheta\geq 1\\ (\frac{8\vartheta^2-12\vartheta+5}{8})\left|\Psi'\Big(\frac{(8\vartheta^3-15\vartheta+9)\beta_1+(-8\vartheta^3+24\vartheta^2-21\vartheta+6)\beta_2}{3(8\vartheta^2-12\vartheta+5)}\Big)\right|, & \\ \text{if } \frac{1}{2}<\vartheta<1 & \\ (\frac{3-4\vartheta}{8})\left|\Psi'\Big(\frac{(7-9\vartheta)\beta_1+(2-3\vartheta)\beta_2}{3(3-4\vartheta)}\Big)\right|, & \text{if } \vartheta\leq\frac{1}{2}\end{cases}
\end{aligned}$$

Proof. By concavity of $|\Psi'|^q$ and power mean inequality we may write:

$$\begin{aligned}
|\Psi'(\lambda x+(1-\lambda)y)|^q &\geq \lambda|\Psi'(x)|^q+(1-\lambda)|\Psi'(y)|^q\\
|\Psi'(\lambda x+(1-\lambda)y)|^q &\geq (\lambda|\Psi'(x)|+(1-\lambda)|\Psi'(y)|)^q\\
|\Psi'(\lambda x+(1-\lambda)y)| &\geq \lambda|\Psi'(x)|+(1-\lambda)|\Psi'(y)|
\end{aligned}$$

So, $|\Psi'|$ is also concave.
Now using Lemma 2, and Jensen's integral inequality we have

$$\begin{aligned}
&\left|\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta-\left[(1-\vartheta)\Psi(\beta_1)+\vartheta\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right]\right|\\
\leq\;&(\beta_2-\beta_1)\left[\int_0^{\frac{1}{2}}t|\Psi'(t\beta_1+(1-t)\beta_2|\,dt+\int_{\frac{1}{2}}^{1}|t-\vartheta||\Psi'(t\beta_1+(1-t)\beta_2)|dt\right]\\
=\;&(\beta_2-\beta_1)\left[\int_0^{\frac{1}{2}}t|\Psi'(t\beta_1+(1-t)\beta_2)|\,dt\right]+(\beta_2-\beta_1)\left[\int_{\frac{1}{2}}^{1}|t-\vartheta||\Psi'(t\beta_1+(1-t)\beta_2|dt\right]\\
\leq\;&(\beta_2-\beta_1)\int_0^{\frac{1}{2}}t.dt\left|\Psi'\left(\frac{\int_0^{\frac{1}{2}}t(t\beta_1+(1-t)\beta_2)\,dt}{\int_0^{\frac{1}{2}}tdt}\right)\right|
\end{aligned}$$

$$+ \quad (\beta_2-\beta_1)\left(\int_{\frac{1}{2}}^{1}|t-\vartheta|\,dt\right)\left|\Psi'\left(\frac{\int_{\frac{1}{2}}^{1}|t-\vartheta|(t\beta_1+(1-t)\,\beta_2)\,dt}{\int_{\frac{1}{2}}^{1}|t-\vartheta|dt}\right)\right|$$

$$= \quad (\beta_2-\beta_1)\int_0^{\frac{1}{2}}t.dt\left|\Psi'\left(\frac{\int_0^{\frac{1}{2}}t^2\beta_1+t(1-t)\,\beta_2\,dt}{\int_0^{\frac{1}{2}}tdt}\right)\right|$$

$$+ \quad (\beta_2-\beta_1)\left(\int_{\frac{1}{2}}^{1}|t-\vartheta|dt\right)\left|\Psi'\left(\frac{\int_{\frac{1}{2}}^{1}(t.|t-\vartheta|\beta_1+(1-t)|t-\vartheta|\,\beta_2)\,dt}{\int_{\frac{1}{2}}^{1}|t-\vartheta|dt}\right)\right|$$

$$\leq \frac{(\beta_2-\beta_1)}{8}\left|\Psi'\left(\frac{\beta_1+2\beta_2}{3}\right)\right|+(\beta_2-\beta_1)\begin{cases}(\frac{4\vartheta-3}{8})\left|\Psi'\left(\frac{(9\vartheta-7)\beta_1+(3\vartheta-2)\beta_2}{3(4\vartheta-3)}\right)\right|, & \text{if } \vartheta\geq 1\\ (\frac{8\vartheta^2-12\vartheta+5}{8})\left|\Psi'\left(\frac{(8\vartheta^3-15\vartheta+9)\beta_1+(-8\vartheta^3+24\vartheta^2-21\vartheta+6)\beta_2}{3(8\vartheta^2-12\vartheta+5)}\right)\right|, & \\ \text{if } \frac{1}{2}<\vartheta<1 & \\ (\frac{3-4\vartheta}{8})\left|\Psi'\left(\frac{(7-9\vartheta)\beta_1+(2-3\vartheta)\beta_2}{3(3-4\vartheta)}\right)\right|, & \text{if } \vartheta\leq\frac{1}{2}.\end{cases}$$

Hence proved.

Remark 5 If we put $\vartheta = 1$ in Theorem 2 we get the following inequality.

$$\left|\frac{1}{\beta_2-\beta_1}\int_{\beta_1}^{\beta_2}\Psi(\eta)d\eta-\Psi\left(\frac{\beta_1+\beta_2}{2}\right)\right|\leq\frac{(\beta_2-\beta_1)}{8}\left[\left|\Psi'\left(\frac{\beta_1+2\beta_2}{3}\right)\right|+\left|\Psi'\left(\frac{2\beta_1+\beta_2}{3}\right)\right|\right].$$

3. APPLICATIONS TO SPECIAL MEANS

Here we consider some particular means for two positive real numbers μ_1, μ_2. Therefore we recall the following definitions:

(1) The arithmetic mean:

$$\mathtt{A} \quad = \quad \mathtt{A}(\mu_1,\mu_2) := \frac{\mu_1+\mu_2}{2},\ \mu_1,\mu_2\ \in\mathbb{R}^+.$$

(2) The logarithmic mean:

$$L \quad = \quad \mathtt{L}(\mu_1,\mu_2) := \frac{\mu_2-\mu_1}{\ln\mu_2-\ln\mu_1},\ \mu_1\neq\mu_2,\ \mu_1,\mu_2\ \in\mathbb{R}^+.$$

(3)The harmonic mean:

$$H \quad = \quad H(\mu_1,\mu_2) = \frac{2}{\frac{1}{\mu_1}+\frac{1}{\mu_2}}$$

(4) The generalized logarithmic mean:

$$\mathtt{L}_m = \mathtt{L}_m(\mu_1,\mu_2) := \left[\frac{\mu_2^{m+1}-\mu_1^{m+1}}{(\mu_2-\mu_1)(m+1)}\right]^{\frac{1}{m}}, \quad \mu_1 \neq \mu_2,\ m \neq 0,-1,\ m \in \mathbb{R}.$$

(5) The weighted arithmetic mean:

$$\mathtt{A}(\mu_1,\mu_2;w_1,w_2) = \frac{w_1\mu_1+w_2\mu_2}{w_1+w_2},$$

where $\mu_1,\mu_2,w_1,w_2 \in \mathbb{R}^+$.

Proposition 1 If $\beta_1,\beta_2 \in \mathbb{R}^+$ and $m \geq 2$, then the following inequality holds:

$$|L_m^m(\beta_1,\beta_2)-(1-\vartheta)\beta_1^m-\vartheta A^m(\beta_1,\beta_2)| \leq \frac{m(\beta_2-\beta_1)}{8}A\Big(|\beta_1|^{m-1},|\beta_2|^{m-1};1,2\Big)$$

$$+m(\beta_2-\beta_1)\begin{cases} |\beta_1|^{m-1}\left(\frac{9\vartheta-7}{24}\right)+|\beta_2|^{m-1}\left(\frac{3\vartheta-2}{24}\right), & \text{if } \vartheta \geq 1\\ |\beta_1|^{m-1}\left(\frac{8\vartheta^3-15\vartheta+9}{24}\right)+|\beta_2|^{m-1}\left(\frac{-8\vartheta^3+24\vartheta^2-21\vartheta+6}{24}\right), & \text{if } \frac{1}{2}<\vartheta<1\\ |\beta_1|^{m-1}\left(\frac{7-9\vartheta}{24}\right)+|\beta_2|^{m-1}\left(\frac{2-3\vartheta}{24}\right), & \text{if } \vartheta \leq \frac{1}{2}\end{cases}$$

Proof. The proof directly follows from Theorem 2 applied for $\Psi(\eta)=\eta^m$, $\eta \in \mathbb{R}^+$.

Proposition 2 Let $\beta_1,\beta_2 \in \mathbb{R}^+$ and $\beta_1<\beta_2$ with $m \geq 2$, and $p>1$, then the following inequality holds:

$$\begin{aligned}\Big|L_m^m(\beta_1,\beta_2) &- (1-\vartheta)\beta_1^m-\vartheta A^m(\beta_1,\beta_2)\Big|\\ &\leq \frac{m(\beta_2-\beta_1)}{4(p+1)^{\frac{1}{p}}}\left[A\left(|\beta_1|^{\frac{(m-1)p}{p-1}},|\beta_2|^{\frac{(m-1)p}{p-1}};1,3\right)\right]^{\frac{p-1}{p}}\\ &+ \frac{m(\beta_2-\beta_1)}{2^{\frac{p-1}{p}}}\left[A\left(|\beta_1|^{\frac{(m-1)p}{p-1}},|\beta_2|^{\frac{(m-1)p}{p-1}};3,1\right)\right]^{\frac{p-1}{p}}\begin{cases}\left(\frac{(2\vartheta-1)^{p+1}-(\vartheta-1)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, \\ \text{if } \vartheta \geq 1\\ \left(\frac{(2\vartheta-1)^{p+1}+(1-\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\\ \text{if } \frac{1}{2}<\vartheta<1\\ \left(\frac{(1-\vartheta)^{p+1}-(1-2\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}\\ \text{if } \vartheta \leq \frac{1}{2}\end{cases}\end{aligned}$$

Proof. The proof directly follows from Theorem 2 applied for $\Psi(\eta)=\eta^m$, $\eta \in \mathbb{R}^+$.

Proposition 3 Let $\beta_1,\beta_2 \in \mathbb{R}^+$ and $\beta_1<\beta_2$ with $m \geq 2$, $q \geq 1$, then the following inequality holds:

$$\left|L_m^m(\beta_1,\beta_2)-(1-\vartheta)\beta_1^m-\vartheta A^m(\beta_1,\beta_2)\right| \leq \frac{m(\beta_2-\beta_1)}{8}A\Big(|\beta_1|^{(m-1)q},|\beta_2|^{(m-1)q};1,2\Big)$$

$$+ \quad (\beta_2-\beta_1)\begin{cases}(\frac{4\vartheta-3}{8})^{1-\frac{1}{q}}\Big[|\beta_1|^{(m-1)q}\left(\frac{9\vartheta-7}{24}\right)+|\beta_2|^{(m-1)q}\left(\frac{3\vartheta-2}{24}\right)\Big]^{\frac{1}{q}}, \quad \text{if } \vartheta \geq 1\\ (\frac{8\vartheta^2-12\vartheta+5}{8})^{1-\frac{1}{q}}\Big[|\beta_1|^{(m-1)q}\left(\frac{8\vartheta^3-15\vartheta+9}{24}\right)+|\beta_2|^{(m-1)q}\left(\frac{-8\vartheta^3+24\vartheta^2-21\vartheta+6}{24}\right)\Big]^{\frac{1}{q}},\\ \text{if } \frac{1}{2}<\vartheta<1\\ (\frac{3-4\vartheta}{8})^{1-\frac{1}{q}}\Big[|\beta_1|^{(m-1)q}\left(\frac{7-9\vartheta}{24}\right)+|\beta_2|^{(m-1)q}\left(\frac{2-3\vartheta}{24}\right)\Big]^{\frac{1}{q}}, \quad \text{if } \vartheta \leq \frac{1}{2}\end{cases}$$

Proof. The proof directly follows from Theorem 2 applied for $\Psi(\eta)=\eta^m$, $\eta\in\mathbb{R}^+$.
Proposition 4 Let $\beta_1,\beta_2\in\mathbb{R}^+$, and $\beta_1<\beta_2$, then the following inequality holds:

$$|L^{-1}(\beta_1,\beta_2)-(\frac{1-\vartheta}{\beta_1})-\vartheta H(\beta_1,\beta_2)|\leq\frac{\beta_2-\beta_1}{8}A\big(|\beta_1|^{-2},|\beta_2|^{-2};1,2\big)$$

$$+(\beta_2-\beta_1)\begin{cases}|\beta_1|^{-2}\left(\frac{9\vartheta-7}{24}\right)+|\beta_2|^{-2}\left(\frac{3\vartheta-2}{24}\right), & \text{if } \vartheta\geq 1\\ |\beta_1|^{-2}\left(\frac{8\vartheta^3-15\vartheta+9}{24}\right)+|\beta_2|^{-2}\left(\frac{-8\vartheta^3+24\vartheta^2-21\vartheta+6}{24}\right), & \text{if } \frac{1}{2}<\vartheta<1\\ |\beta_1|^{-2}\left(\frac{7-9\vartheta}{24}\right)+|\beta_2|^{-2}\left(\frac{2-3\vartheta}{24}\right), & \text{if } \vartheta\leq\frac{1}{2}\end{cases}$$

Proof. The proof directly follows from Theorem 2 applied for $\Psi(\eta)=\frac{1}{\eta},\eta\in\mathbb{R}^+$.
Proposition 5 Suppose $\beta_1,\beta_2\in\mathbb{R}^+$ with $0<\beta_1<\beta_2$, then for $p>1$, the following inequality holds:

$$\left|L^{-1}(\beta_1,\beta_2)-(\frac{1-\vartheta}{\beta_1})-\vartheta H(\beta_1,\beta_2)\right|\leq\frac{\beta_2-\beta_1}{4(p+1)^{\frac{1}{p}}}\left[A\left(|\beta_1|^{\frac{-2p}{p-1}},|\beta_2|^{\frac{-2p}{p-1}};1,3\right)\right]^{\frac{p-1}{p}}$$

$$+\quad\frac{\beta_2-\beta_1}{2^{\frac{p-1}{p}}}\left[A\left(|\beta_1|^{\frac{-2p}{p-1}},|\beta_2|^{\frac{-2p}{p-1}};3,1\right)\right]^{\frac{p-1}{p}}\begin{cases}\left(\frac{(2\vartheta-1)^{p+1}-(\vartheta-1)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, & \text{if } \vartheta\geq 1\\ \left(\frac{(2\vartheta-1)^{p+1}+(1-\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, & \text{if } \frac{1}{2}<\vartheta<1\\ \left(\frac{(1-\vartheta)^{p+1}-(1-2\vartheta)^{p+1}}{2^{p+1}(p+1)}\right)^{\frac{1}{p}}, & \text{if } \vartheta\leq\frac{1}{2}\end{cases}$$

Proof. The proof directly follows from Theorem 2 applied for $\Psi(\eta)=\frac{1}{\eta}$, $\eta\in\mathbb{R}^+$.

Proposition 6 Suppose $\beta_1,\beta_2\in\mathbb{R}^+$ with $0<\beta_1<\beta_2$, then for $q\geq 1$, the following inequality holds:

$$|L^{-1}(\beta_1,\beta_2)-(\frac{1-\vartheta}{\beta_1})-\vartheta H(\beta_1,\beta_2)|\leq\frac{\beta_2-\beta_1}{8}\left[A\left(|\beta_1|^{-2q},|\beta_2|^{-2q};1,2\right)\right]^{\frac{1}{q}}$$

$$+\quad(\beta_2-\beta_1)\begin{cases}(\frac{4\vartheta-3}{8})^{1-\frac{1}{q}}\left[|\beta_1|^{-2q}\left(\frac{9\vartheta-7}{24}\right)+|\beta_2|^{-2q}\left(\frac{3\vartheta-2}{24}\right)\right]^{\frac{1}{q}}, & \text{if } \vartheta\geq 1\\ (\frac{8\vartheta^2-12\vartheta+5}{8})^{1-\frac{1}{q}}\left[|\beta_1|^{-2q}\left(\frac{8\vartheta^3-15\vartheta+9}{24}\right)+|\beta_2|^{-2q}\left(\frac{-8\vartheta^3+24\vartheta^2-21+6}{24}\right)\right]^{\frac{1}{q}}, & \\ \text{if } \frac{1}{2}<\vartheta<1 & \\ (\frac{3-4\vartheta}{8})^{1-\frac{1}{q}}\left[|\beta_1|^{-2q}\left(\frac{7-9\vartheta}{24}\right)+|\beta_2|^{-2q}\left(\frac{2-3\vartheta}{24}\right)\right]^{\frac{1}{q}}, & \text{if } \vartheta\leq\frac{1}{2}\end{cases}$$

Proof The proof directly follows from Theorem 2 applied for $\Psi(\eta)=\frac{1}{\eta},\ \eta\in\mathbb{R}^+$.

References

[1] Y. M. Chu, M. Adil Khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, J. Inequal. Appl., Article ID 93, 12 pages, 2017.
[2] Y. M. Chu, M. Adil Khan, T. U. Khan, T. Ali, Generalizations of Hermite-Hadamard type inequalities for MT-convex functions, J. Nonlinear Sci. Appl., 9, 4305–4316, 2016.
[3] M. Adil Khan, T. Ali and S. S. Dragomir, Hermite-Hadamard type inequalities for conformable fractional integrals, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math., DOI 10.1007/s13398-017-0408-5, 2017.
[4] M. Adil Khan, Y. Khurshid, T. Ali and N. Rehman, Inequalities for three times differentiable functions, Punjab Univ. J. Math., 2, 35–48, 2016.
[5] M. Adil Khan, Y. M. Chu, T. U. Khan, J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math., 15, 2017, 1414-1430.

[6] M.A. Latif, Dragomir, S. S., New inequalities of Hermite-Hadamard inequality type for function whose derivatives in absolute value are convex with applications, Acta. Univ. M. Belli Ser. Math., 21, 27-42, 2013.

[7] I. İşcan, Hermite-Hadamard Inequalities for pre-invex function via fractional integrals and related fractional inequalities, Amer. J. Math. Anal., 1, 33-38, 2013 .

[8] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, RGMIA Monographs, 2000.

[9] M. A. Latif, On Hermite-Hadamard type inequalities for n-times differentiable pre-invex functions with applications, Stud. Univ. Babes-Bolyai Math., 58, 325-343, 2013.

[10] L. Ciurdaria, A note concerning sevral Hermite-Hadamard inequalities for different types of convex functions, Int. J. Math. Anal., 6, 1623-1639, 2012.

[11] A. Barani, S. Barani, Hermite-Hadamard type inequalities for functions when a power of the absolute value of the first derivative is P-convex, Bull. Aust. Math. Soc., 86, 126-134, 2012.

[12] T. S. Du, J. G. Liao and Y. J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex function, J. Nonlinear Sci. Appl., 9, 3112-3126, 2016.

[13] T. S. Du, Y. J. Li and Z. Q. Yang, A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions, Appl. Math. Comput., 293, 358-369, 2017.

[14] M. Adil Khan, Y.M. Chu, A. Kashuri, R. Liko, G. Ali,, New Hermite-Hadamard inequalities for conformable fractional integrals, J. Funct. Spaces, (to appear).

[15] Y. M. Li, B. Y. Long and Y. M. Chu, Sharp bounds for the Neuman-Sandor mean in terms of generalized logarithmic mean, J. Math. Inequal., 4, 567-577, 2012.

[16] Y. M. Chu, S. W. Hou and W. F. Xia, Optimal convex combinations bounds of centroidal and harmonic means for logarithmic and inentric means, Bull. Iranian Math. Soc., 39(2), 259-269, 2013.

[17] T. H. Zhao, Y. M. Chu, Y. L. Jiang, Y. M. Li, Best possible bounds for Neuman-Sandor mean by the identric, quadratic and contraharmonic means, Abstr. Appl. Anal., Article ID 348326, 12 pages, 2013.

[18] M. K. Wang, Z. K. Wang, Y. M. Chu, An optimal double inequality between geometric and identric means, Appl. Math. Lett., 25(3), 471-475, 2012.

[19] Y. M. Chu, M. K. Wang and Z. K. Wang, A sharp double inequality between harmonic and identric means, Abstr. Appl. Anal., Article ID 657935, 7 pages, 2011.

[20] M. K. Wang, Y. M. Chu and Y. F. Qiu, Some comparsion inequalities for generalized Muirhead and identric means, J. Inequal. Appl., Article ID 295620, 10 pages, 2010.

[21] W. F. Xia, Y. M. Chu and G. D. Wang, The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means, Abstr. Appl. Anal., Article ID 604804, 9 pages, 2010.

[22] Y. M. Chu and W. F. Xia, Two optimal double inequalites beween power mean and logarithmic mean, Comput. Math. Appl., 60(1), 83-90, 2010.

[23] Y. M. Chu, M. Shi and Y. F. Jiang, Optimal inequalities for the power, harmonic and logarithmic means, Bull. Iranian Math. Soc., 38(3), 597-606, 2012.

[24] Y. M. Chu and W. F Xia, Inequalities for generalized logarithmic means, J. Inequal. Appl., Article ID 763252, 7 pages, 2009.

[25] B. Y. Long and Y. M. Chu, Optimal inequalities for generalized logarithmic, arithmetic, and geometric means, J. Inequal. Appl., Article ID 806825, 10 pages, 2010.

[26] Z. H. Yang, Y. M. Chu and Y. Q. Song, Sharp bounds for Toader-Qi mean in terms of logarithmic and identric means, Math. Inequal. Appl., 19(2), 721-730, 2016.

[27] Y. M. Chu, G. D. Wang and X. H. Zhang, Schur convexity and Hadamard's inequality, Math. Inequal. Appl., 13(4), 725-731, 2010.

[28] Y. M. Zhang, Y. M. Chu and X. H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, J. Inequal. Appl., Article ID 507560, 11 pages, 2010.

[29] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized (s, m, φ)-preinvex functions, Aust. J. Math. Anal. Appl., 13(1), Article 16, 1-11, 2016.

[30] A. Kashuri and R. Liko, Generalizations of Hermite- Hadamard and Ostrowski type inequalities for MTm-preinvex functions, Proyecciones, 36(1), 45-80, 2017.

[31] A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for generalized $(r; s, m, \varphi)$-preinvex functions, Eur. J. Pure Appl. Math., 10(3), 495-505, 2017.

[32] U. S. Kirmacai, Inequalities for differentiable mappings and applications to special means of real numbers and to mid point formula, Appl. Math. Comput., 147, 137-146, 2004.

M. Adil Khan, M. Aizaz Ali
Department of Mathematics, University of Peshawar, Peshawar, Pakistan
E-mail address: adilswati@gmail.com, malikaizazali@gmail.com

Tingsong Du
Department of Mathematics, College of Science, China Three Gorges University Yichang 443002, Hubei, P. R. China

E-mail address: tingsongdu@ctgu.edu.cn