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GENERATING FUNCTIONS K-FIBONACCI AND K-JACOBSTHAL NUMBERS AT NEGATIVE INDICES

ALI BOUSSAYOUD, SOUHILA BOUGHABA AND MOHAMED KERADA

Abstract. In this paper, we calculate the generating functions by using the concepts of symmetric functions. Although the methods cited in previous works are in principle constructive, we are concerned here only with the question of manipulating combinatorial objects, known as symmetric operators. The proposed generalized symmetric functions can be used to find explicit formulas of the k-Fibonacci and k- Jacobsthal numbers at negative indices and of the Chebychev polynomials of first and second kinds.

1. Introduction and Preliminaries

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field of science and art (e.g., see [12]). Fibonacci numbers F_n are defined by the recurrence relation

$$\begin{cases} F_0 = 1, F_1 = 1 \\ F_{n+1} = F_n + F_{n-1},\ n \geq 1 \end{cases}.$$

There exist a lot of properties about Fibonacci numbers. In particular, there is a beautiful combinatorial identity to Fibonacci numbers [13]

$$Fn =_{i=0}^{\left[\frac{n-1}{2}\right]} \begin{pmatrix} n-i-1 \\ i \end{pmatrix}. \tag{1}$$

From (1), Falcon [14] introduced the incomplete Fibonacci numbers $F_n(s)$. They are defined by

$$F_n(s) =_{j=0}^{s} \begin{pmatrix} n-j-1 \\ j \end{pmatrix},\ 0 \leq s \leq \left[\frac{n-1}{2}\right];\ n = 0, 1, 2, ..,$$

On the other hand, many kinds of generalizations of Fibonacci numbers have been presented in the literature. In particular, a generalization is the k-Fibonacci

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Numbers. For any positive real number k, the k-Fibonacci sequence, say $(F_{n,k})_{n\in\mathbb{N}}$, is defined recurrently by [14]

$$\begin{cases} F_{k,0}=1, F_{k,1}=1 \\ F_{k,n+1}=kF_{k,n}+F_{k,n-1},\ n\geq 1 \end{cases}.$$

In [13], k-Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several papers; see [13, 14].

For any positive real number k, the k-Jacobsthal Numbers, say $(J_{n,k})_{n\in\mathbb{N}}$, is defined recurrently by [11]

$$\begin{cases} J_{k,0}=0, J_{k,1}=1 \\ J_{k,n+1}=kJ_{k,n}+2J_{k,n-1},\ n\geq 1 \end{cases}$$

If $k=1$, we have the classical Jacobsthal numbers appears: $J_0=0,\ J_1=1$ and $J_{n+1}=J_n+2J_{n-1}$ for $n\geq 1$.

In this contribution, we shall define a new useful operator denoted by $\delta_{p_1p_2}$ for which we can formulate, extend and prove new results based on our previous ones [1, 5, 7]. In order to determine generating functions of the product of k-Fibonacci and k-Jacobsthal numbers at negative indices and Chebychev polynomials of first and second kind, we combine between our indicated past techniques and these presented polishing approaches.

In order to render the work self-contained we give the necessary preliminaries tools; we recall some definitions and results.

Definition 1 [7] Let B and P be any two alphabets. We define $S_n(B-P)$ by the following form

$$\frac{\Pi_{p\epsilon P}(1-pt)}{\Pi_{b\epsilon B}(1-bt)}=\sum_{n=0}^{\infty}S_n(B-P)t^n, \tag{2}$$

with the condition $S_n(B-P)=0$ for $n<0$.

Corollary 1 [3] Taking $B=\{0,0,...,0\}$ in (2) gives

$$\Pi_{p\in P}(1-pt)=\sum_{n=0}^{\infty}S_n(-P)t^n.$$

Equation (2) can be rewritten in the following form

$$\sum_{n=0}^{\infty}S_n(B-P)t^n=\left(\sum_{n=0}^{\infty}S_n(B)t^n\right)\times\left(\sum_{n=0}^{\infty}S_n(-P)t^n\right),$$

where

$$S_n(B-P)=\sum_{j=0}^{n}S_{n-j}(-P)S_j(B).$$

Definition 2 [6] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows

$$\partial_{p_ip_{i+1}}(f)=\frac{f(p_1,\cdots,p_i,p_{i+1},\cdots p_n)-f(p_1,\cdots p_{i-1},p_{i+1},p_i,p_{i+2}\cdots p_n)}{p_i-p_{i+1}}.$$

Definition 3 The symmetrizing operator $\delta^k_{e_1e_2}$ is defined by

$$\delta^k_{p_1p_2}(g(p_1))=\frac{p_1^kg(p_1)-p_2^kg(p_2)}{p_1-p_2}\text{ for all }k\in\mathbb{N}.$$

Proposition 1 [7] Let $P=\{p_1,p_2\}$ an alphabet, we define the operator $\delta^k_{p_1p_2}$ as follows

$$\delta^k_{p_1p_2}g(p_1)=S_{k-1}(p_1+p_2)g(p_1)+p_2^k\partial_{p_1p_2}g\ (p_1),\ \textit{for all } k\in\mathbb{N}.$$

Proposition 2 [1] The relations

$$\begin{aligned} 1)\ F_{k,-n} &= (-1)^{n+1}F_{k,n}, \\ 2)\ J_{k,-n} &= (-1)^{n+1}J_{k,n} \end{aligned}$$

hold for all $n \geq 0$.

2. Theorem and Proof

In our main result, we will combine all these results in a unified way such that they can be considered as a special case of the following Theorem.

Theorem 1 Given two alphabets $P = \{p_1, p_2\}$ and $B = \{b_1, b_2, b_3\}$, we have

$$\sum_{n=0}^{\infty} S_n(B)\partial_{p_1p_2}(p_1^{n+1})t^n = \frac{S_0(-B) - p_1p_2S_2(-B)t^2 - p_1p_2S_3(-B)S_1(P)t^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)}, \quad (3)$$

with $S_0(-B) = 1,\ S_2(-B) = b_1b_2 + b_1b_3 + b_2b_3,\ S_3(-B) = b_1b_2b_3$.

Proof. Let $\sum_{n=0}^{\infty} S_n(B)t^n$ and $\sum_{n=0}^{\infty} S_n(-B)t^n$ be two sequences such that $\sum\limits_{n=0}^{\infty} S_n(B)t^n = \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-B)t^n}$. On one hand, since $g(p_1) = \sum\limits_{n=0}^{\infty} S_n(B)p_1^nt^n$ and $g(p_2) = \sum\limits_{n=0}^{\infty} S_n(B)p_2^nt^n$, we have

$$\begin{aligned} \delta_{p_1p_2}g(p_1) &= \delta_{p_1p_2}\left(\sum_{n=0}^{\infty} S_n(B)p_1^nt^n\right) \\ &= \frac{p_1\sum\limits_{n=0}^{\infty} S_n(B)p_1^nt^n - p_2\sum\limits_{n=0}^{\infty} S_n(B)p_2^nt^n}{p_1 - p_2} \\ &= \sum_{n=0}^{\infty} S_n(B)\left(\frac{p_1^{n+1} - p_2^{n+1}}{p_1 - p_2}\right)t^n \\ &= \sum_{n=0}^{\infty} S_n(B)\partial_{p_1p_2}(p_1^{n+1})t^n \end{aligned}$$

which is the right-hand side of (3). On the other part, since

$$g(p_1) = \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-B)p_1^nt^n},$$

we have

$$\delta_{p_1p_2}g(p_1) = \frac{p_1\prod\limits_{b\in B}(1-bp_2)t - p_2\prod\limits_{b\in B}(1-bp_1t)}{(p_1-p_2)\left(\sum\limits_{n=0}^{\infty}S_n(-B)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-B)p_2^nt^n\right)}.$$

Using the fact that : $\sum\limits_{n=0}^{\infty}S_n(-B)p_1^nt^n = \prod\limits_{b\in B}(1-bp_1t)$, then

$$\begin{aligned}\delta_{p_1p_2}g(p_1) &= \frac{\sum\limits_{n=0}^{\infty}S_n(-B)\frac{p_1p_2^n-p_2p_1^n}{p_1-p_2}t^n}{\left(\sum\limits_{n=0}^{\infty}S_n(-B)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-B)p_2^nt^n\right)}\\ &= \frac{S_0(-B)-p_1p_2S_2(-B)t^2-p_1p_2S_3(-B)S_1(P)\,t^3}{\left(\sum\limits_{n=0}^{\infty}S_n(-B)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty}S_n(-B)p_2^nt^n\right)}.\end{aligned}$$

This completes the proof.

3. On the Generating Functions of Some Numbers and Polynomials

We now derive new generating functions of the products of some well-known numbers and polynomials. Indeed, we consider Theorem 1 in order to derive k-Fibonacci and k-Jacobsthal numbers at negative indices and Tchebychev polynomials of first and second kind.

Case 1: Replacing p_2 by $(-p_2)$ and assuming that $p_1p_2 = 1$, $p_1 - p_2 = k$ in Theorem 1, we have the following theorem

Theorem 2 [4] We have the following a generating function of both k-Fibonacci numbers at negative indices and symmetric functions in several variables as

$$\sum_{n=0}^{\infty}S_n(b_1+b_2+b_3)F_{k,-n}t^n = \frac{kS_3(-B)t^3-S_2(-B)t^2-1}{{}_{i=1}^{3}\left(1+kb_it-b_i^2t^2\right)}. \quad (4)$$

Corollary 2 If $k=1$ in the relationship (4) we have [4]

$$\sum_{n=0}^{\infty}S_n(b_1+b_2+b_3)F_{-n}t^n = \frac{b_1b_2b_3t^3-\left(b_1b_2+b_1b_3+b_2b_3\right)t^2-1}{\prod\limits_{i=1}^{3}\left(1+b_it-b_i^2t^2\right)},$$

which representing a new generating function of Fibonacci numbers at negative indices and symmetric functions in several variable.

Setting $b_3 = 0$ and replacing b_2 by $(-b_2)$ in (4), and assuming $b_1 - b_2 = k$; $b_1b_2 = 1$; we deduce the following theorems.

Theorem 3 [4] For $n \in \mathbb{N}$, the generating function of the product of k-Fibonacci numbers and k-Fibonacci numbers at negative indices is given by

$$\sum_{n=0}^{\infty}F_{k,n}F_{k,-n}t^n = \frac{t^2-1}{1+k^2t-2(k^2+1)t^2+k^2t^3+t^4}. \quad (5)$$

Corollary 3 If $k = 1$ in the relationship (5) we have [4]

$$\sum_{n=0}^{\infty} F_n F_{-n} t^n = \frac{t^2 - 1}{1 + t - 4t^2 + t^3 + t^4},$$

which representing a new generating function of the product of Fibonacci numbers and Fibonacci numbers at negative indices.

Theorem 4 [1] For $n \in \mathbb{N}$, the new generating function of the product of k-Fibonacci numbers at negative indices is given by

$$\sum_{n=0}^{\infty} F_{k,-n}^2 t^n = \frac{1 - t^2}{1 - k^2 t - 2(k^2 + 1)t^2 - k^2 t^3 + t^4}.$$

Case 2: Replacing p_2 by $(-p_2)$ and assuming that $p_1 p_2 = 2,\ p_1 - p_2 = k$ in Theorem 1, we have the following theorem

Theorem 5 We have the following a new generating function of both k-Jacobsthal numbers at negative indices and symmetric functions in several variables as

$$\sum_{n\ =0}^{\infty} S_{n-1}(b_1 + b_2 + b_3) J_{k,-n} t^n = \frac{2kS_3(-B)t^4 - 2S_2(-B)t^3 - t}{{}_{i=1}^{3} (1 + kb_i t - 2b_i^2 t^2)}. \tag{6}$$

Corollary 4 If $k = 1$ in the relationship (6) we get

$$\sum_{n\ =0}^{\infty} S_{n-1}(b_1 + b_2 + b_3) J_{-n} t^n = \frac{2S_3(-B)t^4 - 2S_2(-B)t^3 - t}{{}_{i=1}^{3} (1 + b_i t - 2b_i^2 t^2)},$$

which representing a new generating function of the product of Jacobsthal numbers at negative indices and symmetric functions in several variables.

Setting $b_3 = 0$ and replacing b_2 by $(-b_2)$ in (6), and assuming $b_1 - b_2 = k$; $b_1 b_2 = 2$; we deduce the following theorem.

Theorem 6 For $n \in \mathbb{N}$, the new generating function of the product of k-Jacobsthal numbers and k-Jacobsthal numbers at negative indices is given

$$\sum_{n=0}^{\infty} J_{k,n} J_{k,-n} t^n = \frac{4t^3 - t}{1 + k^2 t - 4(k^2 + 1)t^2 + 4kt^3 + 16t^4}. \tag{7}$$

Corollary 5 In the special case $k = 1$ identity (7) gives

$$\sum_{n=0}^{\infty} J_n J_{-n} t^n = \frac{4t^3 - t}{1 + t - 8t^2 + 4t^3 + 16t^4},$$

which representing a new generating function of the product of Jacobsthal numbers and Jacobsthal numbers at negative indices.

Theorem 7 For $n \in \mathbb{N}$, the new generating function of the product of k-Jacobsthal numbers at negative indices is given by

$$\sum_{n=0}^{\infty} J_{k,-n}^2 t^n = \frac{4t^3 - t}{1 - k^2 t - 4(k^2 + 1)t^2 - 4kt^3 + 16t^4}.$$

Case 3: Replacing p_1 by $2p_1$ and p_2 by $(-2p_2)$, and assuming that $4p_1 p_2 = -1$ in Theorem 1, we have the following a new generating function of the product of Chebychev polynomials of second kind and the symmetric functions in several variables, as follows for $y = p_1 - p_2$,

$$\sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)U_n(-y)t^n = \frac{1 - S_2(-B)t^2 - 2yS_3(-B)t^3}{\prod_{i=1}^{3}(1 + 2b_i yt + b_i^2 t^2)}.$$

Theorem 8 The new generating function of the product of Chebychev polynomials of first kind and the symmetric functions in several variables as

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)T_n(-y)t^n = \frac{1 + yS_1(-B)t + S_2(-B)(2y^2 - 1)t^2 + S_3(-B)(4y^3 + y)t^3}{\prod_{i=1}^{3}(1 + 2b_i yt + b_1^2 t^2)}.$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)T_n(-y)t^n &= \sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)(S_n([2p_1] + [-2p_2]) \\
&\quad -yS_{n-1}(([2p_1] + [-2p_2])(-t)^n \\
&= \sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)S_n(([2p_1] + [-2p_2])(-t)^n \\
&\quad -y\sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)S_{n-1}(([2p_1] + [-2p_2])(-t)^n \\
&= \sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)U_n(-y)t^n \\
&\quad -\frac{y}{2(p_1 + p_2)}\sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)([2p_1]^n - [-2p_2]^n)(-t)^n
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)(-t)^n = \frac{1}{\prod_{b\in B}(1 + bt)},$$

Therfore

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)T_n(-y)t^n &= \sum_{n=0}^{\infty} S_n(b_1 + b_2 + b_3)U_n(-y)t^n - \frac{y}{2(p_1 + p_2)}[\frac{1}{\prod_{b\in B}(1 - 2p_1bt)} - \frac{1}{\prod_{b\in B}(1 + 2p_2bt)} \\
&= \frac{1 + yS_1(-B)t + S_2(-B)(2y^2 - 1)t^2 + S_3(-B)(4y^3 + y)t^3}{\prod_{i=1}^{3}(1 + 2b_i yt + b_i^2 t^2)}.
\end{aligned}$$

This completes the proof.

- Let $b_3 = 0$, by making the following restrictions: $p_1 - p_2 = k$, $p_1p_2 = 1$, $4b_1b_2 = -1$, and by replacing $(b_1 - b_2)$ by $2(b_1 - b_2)$ in (3.1), we get a new generating function, involving the product of k-Fibonacci numbers at negative indices with Chebychev polynomial of second kind as follows

$$\sum_{n=0}^{\infty} S_n(2b_1+[-2b_2])S_n(p_1+[-p_2])t^n = \frac{1 + t^2}{1 + 2k(b_1 - b_2)t - (4(b_1 - b_2)^2 + (k^2 + 2))t^2 + 2k(b_1 - b_2)t^3 + t^4}.$$

Thus we conclude with the following theorem.

Theorem 9 We have the following a new generating function of the product of k-Fibonacci numbers at negative indicesand Chebychev polynomials of second kind as

$$\sum_{n=0}^{\infty} F_{k,-n} U_n(b_1-b_2)t^n = \frac{1+t^2}{1-2k(b_1-b_2)t-(4(b_1-b_2)^2-(k^2+2))t^2+2k(b_1-b_2)t^3+t^4}. \tag{8}$$

Corollary 6 If $k=1$ in the relationship (8) we get

$$\sum_{n=0}^{\infty} F_{-n} U_n(b_1-b_2)t^n = \frac{1+t^2}{1-2(b_1-b_2)t+(3-4(b_1-b_2)^2)t^2+2(b_1-b_2)t^3+t^4},$$

which represents a new generating function, involving the product of Fibonacci numbers at negative indices with Chebychev polynomial of second kind.

Theorem 10 For $n \in \mathbb{N}$, The new generating function of the product of $k-$Fibonacci numbers at negative indices and Chebychev polynomials of first kind as

$$\sum_{n=0}^{\infty} F_{k,-n} T_n(b_1-b_2)t^n = -\frac{1+k(b_1-b_2)t+(1-2(b_1-b_2)^2)t^2}{1+2k(b_1-b_2)t-(4(b_1-b_2)^2-(k^2+2))t^2-2k(b_1-b_2)t^3+t^4}. \tag{9}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{k,n} T_n(b_1-b_2)(-t)^n &= \sum_{n=0}^{\infty} F_{k,n}(S_n(2b_1+[-2b_2])-(b_1-b_2)S_{n-1}(2b_1+[-2b_2]))(-t)^n \\
&= \sum_{n=0}^{\infty} F_{k,n} S_n(2b_1+[-2b_2])(-t)^n-(b_1-b_2)\sum_{n=0}^{\infty} F_{k,n} S_{n-1}(2b_1+[-2b_2])(-t)^n \\
&= \sum_{n=0}^{\infty} F_{k,n} U_n(b_1-b_2)(-t)^n-\frac{(b_1-b_2)}{2(b_1+b_2)}\sum_{n=0}^{\infty} F_{k,n}((2b_1)^n-(-2b_2)^n)(-t)^n
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty} F_{k,-n} t^n = \frac{1}{t^2-kt+1}, \text{ (see [1])}$$

Therfore

$$\sum_{n=0}^{\infty} F_{k,-n} T_n(b_1-b_2)t^n = -\frac{1+k(b_1-b_2)t+(1-2(b_1-b_2)^2)t^2}{1+2k(b_1-b_2)t-(4(b_1-b_2)^2-(k^2+2))t^2-2k(b_1-b_2)t^3+t^4}.$$

This completes the proof.

Corollary 7 In the special case $k=1$ identity (9) gives

$$\sum_{n=0}^{\infty} F_{-n} T_n(b_1-b_2)t^n = \frac{1+(b_1-b_2)t+(1-2(b_1-b_2)^2)t^2}{1+2(b_1-b_2)t+(3-4(b_1-b_2)^2)t^2-2(b_1-b_2)t^3+t^4},$$

which represents a new generating function, involving the product of Fibonacci numbers at negative indices with Chebychev polynomial of first kind.

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Ali Boussayoud, LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria

E-mail address: aboussayoud@yahoo.fr

Souhila Boughaba, Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria

E-mail address: souhilaboughaba@gmail.com

Mohamed Kerada, LMAM Laboratory and Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria

E-mail address: mkerada@yahoo.fr