SOLVABILITY OF A COUPLED SYSTEM OF URYSOHN-STIELTJES INTEGRAL EQUATIONS

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Abstract. In this paper, we study the existence of continuous solutions \( x, y \in C(I) \) of the coupled system of Urysohn-Stieltjes integral equations

\[
\begin{align*}
  x(t) &= p_1(t) + \lambda_1 \int_0^1 f_1(t, s, x(s), y(s)) \, ds \, g_1(t, s), \quad t \in I \\
  y(t) &= p_2(t) + \lambda_2 \int_0^1 f_2(t, s, x(s), y(s)) \, ds \, g_2(t, s), \quad t \in I.
\end{align*}
\]

1. Introduction and Preliminaries

The Volterra-Stieltjes integral equations and Urysohn-Stieltjes integral equations have been studied by J. Banaś and some other authors (see [1]-[9] and [14]-[16]). Consider the Urysohn-Stieltjes integral equation

\[
  x(t) = p(t) + \int_0^1 f(t, s, x(s)) \, ds \, g(t, s), \quad t \in [0, 1].
\]

J. Banaś (see [3]) proved the existence of at least one solution \( x \in C(I) \) to the equation (1), where \( g : I \times I \to R \) is nondecreasing in the second argument on \( I \) and the symbol \( ds \) indicates the integration with respect to \( s \).

For the definition, background and properties of the Stieltjes integral we refer to Banaś [1]. However, the coupled system of integral equations have been studied, recently, by some authors (see [11]-[12],[13]).

In this paper, we generalize this result for the coupled system of Urysohn-Stieltjes
The functions $f_i$ of the coupled system of nonlinear integral equations of Urysohn-Stieltjes type (2) are continuous on $I$, $x, y, s, t, u \in R^2$.

\[ T(x,y)(t) = (T_1 x(t), T_2 y(t)) \]

where

\[ T_1 x(t) = p_1(t) + \lambda_1 \int_0^1 f_1(t,s,x(s)) \, d_s g_1(t,s), \quad t \in I \]

\[ T_2 y(t) = p_2(t) + \lambda_2 \int_0^1 f_2(t,s,x(s)) \, d_s g_2(t,s), \quad t \in I \]

in the Banach space $C(I)$.

2. Existence of solutions

In this section we study the existence of continuous solutions $x, y \in C(I)$ for the coupled system of nonlinear integral equations of Urysohn-Stieltjes type (2). Now we formulate assumptions under which coupled system (2) has at least one classical solution in $C(I)$.

\begin{enumerate}
\item $p_i \in C(I), \quad \lambda_i \in R, \quad i = 1, 2.$
\item $f_i : I \times I \times R^2 \to R, \quad (i = 1, 2)$ is continuous on $I$, $\forall x, y \in R^2, \quad t \in I$ such that there exist continuous functions $k_i : I \times I \to I$ and two positive constants $b_i$ such that:

\[ | f_i(t,s,x,y) | \leq k_i(t,s) + b_i \max \{ |x|, |y| \} \]

for $t, s \in I$ and $x, y \in R$.

\item $g_i : I \times I \to R, \quad i = 1, 2$ and for all $t_1, t_2 \in I$ with $t_1 < t_2$, the functions $s \to g_i(t_2, s) - g_i(t_1, s)$ is nondecreasing on $I$.

\item $g_i(0,s) = 0$ for any $s \in I, \quad i = 1, 2$.

\item The functions $t \to g_i(t,t)$ and $t \to g_i(t,0)$ are continuous on $I$, $i = 1, 2$.

\end{enumerate}

Put $\mu = \sup_t | g_i(t,1) | + \sup_t | g_i(t,0) |$ on $I$.

Now, let $X$ be the Banach space of all ordered pairs $(x,y)$, $x, y \in C(I)$ with the norm

\[ \|(x,y)\|_X = \max \{ \|x\|_{C(I)}, \|y\|_{C(I)} \} \]

where

\[ \|x\| = \sup_{t \in I} | x(t) |, \quad \|y\| = \sup_{t \in I} | y(t) | . \]

It is clear that $(X, \|(x,y)\|_X)$ is a Banach space.

**Theorem 1.** Let the assumptions (i)-(v) be satisfied, then the coupled system (2) has at least one classical solution in $X$.

**Proof:** Define the operator $T$ by

\[ T(x,y)(t) = (T_1 x(t), T_2 y(t)) \]

where

\[ T_1 x(t) = p_1(t) + \lambda_1 \int_0^1 f_1(t,s,u(s)) \, d_s g_1(t,s) \]

\[ T_2 y(t) = p_2(t) + \lambda_2 \int_0^1 f_2(t,s,u(s)) \, d_s g_2(t,s) \]

and $u = (x, y)$.

For every $u \in X, \quad t \in I, \quad f_i(t,\cdot,u(\cdot)) \quad (i = 1, 2)$ is continuous on $I$. Observe that
Assumptions (iii) and (iv) imply that the function \( s \to g(t, s) \) is nondecreasing on the interval \( I \), for any fixed \( t \in I \). Indeed, putting \( t_2 = t, t_1 = 0 \) in (iii) and keeping in mind (iv), we obtain the desired conclusion. From this observation, it follows immediately that, for every \( t \in I \), the function \( s \to g(t, s) \) is of bounded variation on \( I \). It follows, \( f_i(t, s, x(s), y(s)) \) are Riemann-Stieltjes integrable on \( I \) with respect to \( s \to g_i(t, s) \). Thus \( T_i \) make sense.

We will prove a few results concerning the continuity and compactness of these operators in the space of continuous functions. We denoted \( K := \max\{k_i(t, s) : t, s \in I, i = 1, 2\} \), and we define the set \( U \) by

\[
U := \{u = (x, y) \in \mathbb{R}^2 : \|(x, y)\|_X \leq r, r = \frac{\|p_i\| + \lambda K \mu}{1 - \lambda b_1 \mu}\}
\]

Also, let us denote

\[
\theta(\epsilon) = \sup\{\|f_1(t_2, s, u) - f_1(t_1, s, u)\|, \|f_2(t_2, s, u) - f_2(t_1, s, u)\| : t_1, t_2 \in I,
\quad |t_2 - t_1| \leq \epsilon, u \in \mathbb{R}^2\}.
\]

The remainder of the proof will be given in four steps.

**Step 1:** The operator \( T \) transforms from \( X \) into \( X \).
For \( u = (x, y) \in U \), for all \( \epsilon > 0, \delta > 0 \) and for each \( t_1, t_2 \in I \), \( t_1 < t_2 \) such that \( |t_2 - t_1| < \delta \), then

\[
|T_1x(t_2) - T_1x(t_1)| \leq |p_1(t_2) - p_1(t_1)| + |\lambda_1 \int_0^1 f_1(t_2, s, x(s), y(s)) \, ds g_1(t_2, s)|
\]

\[
- |\lambda_1 \int_0^1 f_1(t_1, s, x(s), y(s)) \, ds g_1(t_1, s)| \leq |p_1(t_2) - p_1(t_1)|
\]

\[
+ |\lambda_1 \int_0^1 f_1(t_2, s, x(s), y(s)) \, ds g_1(t_2, s) - \lambda_1 \int_0^1 f_1(t_1, s, x(s), y(s)) \, ds g_1(t_1, s)|
\]

\[
+ |\lambda_1 \int_0^1 f_1(t_1, s, x(s), y(s)) \, ds g_1(t_2, s) - \lambda_1 \int_0^1 f_1(t_1, s, x(s), y(s)) \, ds g_1(t_1, s)|
\]

\[
\leq |p_1(t_2) - p_1(t_1)|
\]

\[
+ |\lambda_1 \int_0^1 [f_1(t_1, s, x(s), y(s)) - f_1(t_1, s, x(s), y(s))] \, ds g_1(t_2, s)|
\]

\[
+ |\lambda_1 \int_0^1 f_1(t_1, s, x(s), y(s)) \, ds (g_1(t_2, s) - g_1(t_1, s))|
\]

\[
\leq |p_1(t_2) - p_1(t_1)|
\]

\[
+ |\lambda_1 \int_0^1 |f_1(t_2, s, x(s), y(s)) - f_1(t_1, s, x(s), y(s))| \, ds (\sqrt{z_0^s g_1(t_2, z)})
\]

\[
+ |\lambda_1 \int_0^1 |f_1(t_1, s, x(s), y(s))| \, ds (\sqrt{z_0^s [g_1(t_2, z) - g_1(t_1, z)]})
\]
As done above we can obtain

\[
T(u(t_2)) - T(u(t_1)) = T(x,y)(t_2) - T(x,y)(t_1)
= (T_1x(t_2), T_2y(t_2)) - (T_1x(t_1), T_2y(t_1))
= (T_1x(t_2) - T_1x(t_1), T_2y(t_2) - T_2y(t_1))
\]

Therefore, \( T \) maps \( X \) into \( X \).

Note that the set of values of \( Tu(t) \) for all \( u \in X \) is an equi-continuous subset of \( X \).

**Step 2:** The operator \( T \) map \( U \) into \( U \).
for \((x, y) \in U\), we have

\[
|T_1 x(t)| \leq |p_1(t)| + |\lambda_1 \int_0^1 f_1(t, s, x(s), y(s)) \, ds g_1(t, s)| \\
\leq |p_1(t)| + |\lambda_1 \int_0^1 f_1(t, s, x(s), y(s)) \, |ds| \left( \sup_{z=0}^t g_1(t, z) \right) \\
\leq \|p_1\| + \lambda \int_0^1 (k_1(t, s) + b_1(\max\{|x(s)|, |y(s)|\})) \, ds \left( \sup_{z=0}^t g_1(t, z) \right) \\
\leq \|p_1\| + \lambda \int_0^1 (k_1(t, s) + rb_1) \, ds g_1(t, s) \\
\leq \|p_1\| + \lambda (K + rb_1) \int_0^1 \, ds g_1(t, s) \\
\leq \|p_1\| + \lambda (K + rb_1)(g_1(t, 1) - g_1(t, 0)) \\
\leq \|p_1\| + \lambda (K + rb_1)\sup_t |g_1(t, 1)| + \sup_t |g_1(t, 0)| \\
\leq \|p_1\| + \lambda (K + rb_1)\mu
\]

Hence

\[
\|T_1 x\| \leq \|p_1\| + \lambda (K + rb_1)\mu.
\]

By a similar way can deduce that

\[
\|T_2 y\| \leq \|p_2\| + \lambda (K + rb_2)\mu.
\]

Therefore,

\[
\|Tu\| = \|T(x, y)\| = \|T_1 x, T_2 y\| = \max\{|\|T_1 x\|, \|T_2 y\|\} \leq r.
\]

Thus for every \(u = (x, y) \in U\), we have \(Tu \in U\) and hence \(TU \subset U\), (i.e \(T : U \rightarrow U\)). This means that the functions of \(TU\) are uniformly bounded on \(I\).

**Step 3:** The operator \(T\) is compact.

It is clear that the set \(U\) is nonempty, bounded, closed and convex, then according to Tychonoff’s theorem in topological products and Arzela-Ascoli theorem the compactness criteria \(T\) is compact.

**Step 4:** The operator \(T\) is continuous.

Firstly, we prove that \(T_1\) is continuous. Let \(\epsilon^* > 0\), the continuity of \(f_i\) yields \(\exists \delta = \delta(\epsilon^*)\) such that \(|f_i(t, s, x, y) - f_i(t, s, u, y)| < \epsilon^*\) whenever \(\|x - u\| \leq \delta\), thus if \(\|x - u\| \leq \delta\), we arrive at:

\[
|T_1 x(t) - T_1 u(t)| \leq |\lambda_1 \int_0^1 f_1(t, s, x(s), y(s)) \, ds g_1(t, s)| \\
- |\lambda_1 \int_0^1 f_1(t, s, u(s), y(s)) \, ds g_1(t, s)|
\]
\[ \begin{align*}
&\leq |\lambda_1| \int_0^1 \Big| f_1(t,s,x(s),y(s)) - f_1(t,s,u(s),y(s)) \Big| \, ds \left( \sqrt{ \int_0^1 g_1(t,z) \, dz } \right) \\
&\leq \epsilon^* \lambda \int_0^1 ds \left( \sqrt{ \int_0^1 g_1(t,z) \, dz } \right) \\
&\leq \epsilon^* \lambda \int_0^1 ds g_1(t,s) \\
&\leq \epsilon^* \lambda \left[ g_1(t,1) - g_1(t,0) \right] \\
&\leq \epsilon^* \lambda \left[ \sup_{t \in I} |g_1(t,1)| + \sup_{t \in I} |g_1(t,0)| \right] \\
&\leq \epsilon^* \lambda \left[ \sup_{t \in I} |g_1(t,1)| + \sup_{t \in I} |g_1(t,0)| \right] \leq \epsilon
\end{align*} \]

where \( \epsilon := \epsilon^* \lambda \mu \).

Therefore,
\[ |T_1 x(t) - T_1 u(t)| \leq \epsilon. \]

This means that the operator \( T_1 \) is continuous.

By a similar way as done above we can prove that for any \( y, v \in C[0,T] \) and \( \| y - v \| < \delta \), we have
\[ |T_2 y(t) - T_2 v(t)| \leq \epsilon. \]

Hence \( T_2 \) is continuous operator.

The operators \( T_i \) \((i = 1, 2)\) is continuous operator it imply that \( T \) is continuous operator.

Since all conditions of Schauder fixed point theorem are satisfied, then \( T \) has at least one fixed point \( u = (x, y) \in U \), which completes the proof. \( \blacksquare \)

In what follows, we provide some examples illustrating the above obtained results.

**Example**: Consider the functions \( g_i : I \times I \rightarrow R \) defined by the formula
\[ g_1(t,s) = \begin{cases} 
  t \ln \frac{t+s}{s}, & \text{for } t \in (0,1], \ s \in I, \\
  0, & \text{for } t = 0, \ s \in I. 
\end{cases} \]
\[ g_2(t,s) = t(t+s-1), \ t \in I. \]

It can be easily seen that the functions \( g_1(t,s) \) and \( g_2(t,s) \) satisfies assumptions (iii)-(v) given in Theorem 1, and \( g_1(t,s) \) is function of bounded variation but it is not continuous on \( I \). In this case, the coupled system of Urysohn-Stieltjes integral equations (2) has the form
\[ x(t) = p_1(t) + \lambda_1 \int_0^t \frac{t}{t+s} f_1(t,s,x(s),y(s)) \, ds, \ t \in I \]
\[ y(t) = p_2(t) + \lambda_2 \int_0^t t f_1(t,s,x(s),y(s)) \, ds, \ t \in I. \]

(3)

Also, consider the functions \( f_i : I \times I \times R^2 \rightarrow R \) defined by the formula
\[ f_1(t,s,x,y) = t + s + x + y, \]
\[ f_2(t,s,x,y) = t + s + x^2 - y^2. \]
Now, it can be easily seen that the functions $f_1$ and $f_2$ satisfies assumptions (ii) given in Theorem 1:

$$|f_1(t, s, x, y)| \leq |t + s + x + y|$$
$$\leq |t + s| + |x| + |y|$$
$$\leq 2T + 2 \max\{|x|, |y|\}$$

And

$$|f_2(t, s, x, y)| \leq |t + s + x^2 - y^2|$$
$$\leq |t + s| + |x^2 - y^2|$$
$$\leq 2T + |(x - y)(x + y)|$$
$$\leq 2T + 2 \max\{|x|, |y|\}$$

Hence, $k_i(t, s) = 2T$, and $b_i = 2$
Therefore, the functions $f_i$ satisfies the assumption

$$|f_i(t, s, x, y)| \leq k_i(t, s) + b_i(\max\{|x|, |y|\}).$$

Therefore, the coupled system (3) has at least one solution $x, y \in C[0,1]$.

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References

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