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SOLVABILITY OF A COUPLED SYSTEM OF URYSOHN-STIELTJES INTEGRAL EQUATIONS

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Abstract. In this paper, we study the existence of continuous solutions $x, y \in C(I)$ of the coupled system of Urysohn-Stieltjes integral equations

$$x(t) = p_1(t) + \lambda_1 \int_0^1 f_1(t, s, x(s), y(s))\ d_s g_1(t, s),\ \ t \in I$$
$$y(t) = p_2(t) + \lambda_2 \int_0^1 f_2(t, s, x(s), y(s))\ d_s g_2(t, s),\ \ t \in I.$$

1. Introduction and Preliminaries

The Volterra-Stieltjes integral equations and Urysohn-Stieltjes integral equations have been studied by J. Banaś and some other authors (see [1]-[9] and [14]- [16]). Consider the Urysohn-Stieltjes integral equation

$$x(t) = p(t) + \int_0^1 f(t, s, x(s))\ d_s g(t, s),\ \ t \in I = [0, 1]. \tag{1}$$

J. Banaś (see [3]) proved the existence of at least one solution $x \in C(I)$ to the equation (1), where $g : I \times I \to R$ is nondecreasing in the second argument on I and the symbol d_s indicates the integration with respect to s.

For the definition, background and properties of the Stieltjes integral we refer to Banaś [1]. However, the coupled system of integral equations have been studied, recently, by some authors (see [11]-[12],[13]).

In this paper, we generalize this result for the coupled system of Urysohn-Stieltjes

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integral equations

$$x(t) = p_1(t) + \lambda_1 \int_0^1 f_1(t, s, x(s), y(s))\ d_s g_1(t, s),\ t \in I$$
$$y(t) = p_2(t) + \lambda_2 \int_0^1 f_2(t, s, x(s), y(s))\ d_s g_2(t, s),\ t \in I \tag{2}$$

in the Banach space $C(I)$.

2. Existence of solutions

In this section we study the existence of continuous solutions $x,\ y \in C(I)$ for the coupled system of nonlinear integral equations of Urysohn-Stieltjes type (2). Now we formulate assumptions under which coupled system (2) and will be considered. Namely, we shall assume that:

(i) $p_i \in C(I),\ \lambda_i \in R,\ i = 1, 2.$

(ii) $f_i : I \times I \times R^2 \to R,\ (i = 1, 2)$ is continuous on $I,\ \forall x, y \in R^2,\ t \in I$ such that there exist continuous functions $k_i : I \times I \to I$ and two positive constants b_i such that:

$$| f_i(t, s, x, y) | \leq k_i(t, s) + b_i(\max\{| x |, | y |\})$$

for $t, s \in I$ and $x, y \in R$.

(iii) $g_i : I \times I \to R,\ i = 1, 2$ and for all $t_1,\ t_2 \in I$ with $t_1 < t_2$, the functions $s \to g_i(t_2, s) - g_i(t_1, s)$ is nondecreasing on I.

(iv) $g_i(0, s) = 0$ for any $s \in I,\ i = 1, 2.$

(v) The functions $t \to g_i(t, t)$ and $t \to g_i(t, 0)$ are continuous on $I,\ i = 1, 2.$ Put $\mu = \sup | g_i(t, 1) | + \sup | g_i(t, 0) |$ on I.

Now, let X be the Banach space of all ordered pairs $(x, y),\ x, y \in C(I)$ with the norm

$$\|(x, y)\|_X = \max\{\|x\|_{C(I)}, \|y\|_{C(I)}\}$$

where

$$\|x\| = \sup_{t \in I} | x(t) |,\ \| y \| = \sup_{t \in I} | y(t) | .$$

It is clear that $(X, \|(x, y)\|_X)$ is a Banach space.

Theorem 1. Let the assumptions (i)-(v) be satisfied, then the coupled system (2) has at least one classical solution in X.

Proof: Define the operator T by

$$T(x, y)(t) = (T_1 x(t), T_2 y(t))$$

where

$$T_1 x(t) = p_1(t) + \lambda_1 \int_0^1 f_1(t, s, u(s))\ d_s g_1(t, s)$$
$$T_2 y(t) = p_2(t) + \lambda_2 \int_0^1 f_2(t, s, u(s))\ d_s g_2(t, s)$$

and $u = (x, y)$.

For every $u \in X,\ t \in I,\ f_i(t, ., u(.))\ (i = 1, 2)$ is continuous on I. Observe that

Assumptions (iii) and (iv) imply that the function $s \to g(t,s)$ is nondecreasing on the interval I, for any fixed $t \in I$. Indeed, putting $t_2 = t,\ t_1 = 0$ in (iii) and keeping in mind (iv), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \to g(t,s)$ is of bounded variation on I. It follows, $f_i(t,s,x(s),y(s))$ are Riemann-Stieltjes integrable on I with respect to $s \to g_i(t,s)$. Thus T_i make sense.

We will prove a few results concerning the continuity and compactness of these operators in the space of continuous functions.
We denoted $K := \max\{k_i(t,s) : t,s \in I,\ i = 1,2\}$, and we define the set U by

$$U := \{u = (x,y) \mid (x,y) \in R^2 : \|(x,y)\|_X \leq r,\ r = \frac{\|p_i\| + \lambda K\mu}{1 - \lambda b_i \mu}\}$$

Also, let us denote

$$\begin{aligned}\theta(\epsilon) = \sup\{\mid f_1(t_2,s,u) - f_1(t_1,s,u) \mid \quad &, \quad \mid f_2(t_2,s,u) - f_2(t_1,s,u) \mid : t_1,t_2 \in I \\ &, \quad \mid t_2 - t_1 \mid \leq \epsilon,\ u \in R^2\}.\end{aligned}$$

The remainder of the proof will be given in four steps.

Step 1: The operator T transforms from X into X.
For $u = (x,y) \in U$, for all $\epsilon > 0,\ \delta > 0$ and for each $t_1, t_2 \in I,\ t_1 < t_2$ such that $\mid t_2 - t_1 \mid < \delta$, then

$$\begin{aligned}
\mid T_1x(t_2) \quad &- \quad T_1x(t_1) \mid \leq \mid p_1(t_2) - p_1(t_1) \mid \\
&+ \quad \mid \lambda_1 \int_0^1 f_1(t_2,s,x(s),y(s))\ d_sg_1(t_2,s) \\
&- \quad \lambda_1 \int_0^1 f_1(t_1,s,x(s),y(s))\ d_sg_1(t_1,s) \mid \leq \mid p_1(t_2) - p_1(t_1) \mid \\
&+ \quad \mid \lambda_1 \int_0^1 f_1(t_2,s,x(s),y(s))\ d_sg_1(t_2,s) - \lambda_1 \int_0^1 f_1(t_1,s,x(s),y(s))\ d_sg_1(t_2,s) \mid \\
&+ \quad \mid \lambda_1 \int_0^1 f_1(t_1,s,x(s),y(s))\ d_sg_1(t_2,s) - \lambda_1 \int_0^1 f_1(t_1,s,x(s),y(s))\ d_sg_1(t_1,s) \mid \\
&\leq \quad \mid p_1(t_2) - p_1(t_1) \mid \\
&+ \quad \mid \lambda_1 \int_0^1 [f_1(t_2,s,x(s),y(s)) - f_1(t_1,s,x(s),y(s))]\ d_sg_1(t_2,s) \mid \\
&+ \quad \mid \lambda_1 \int_0^1 f_1(t_1,s,x(s),y(s))\ d_s(g_1(t_2,s) - g_1(t_1,s)) \mid \\
&\leq \quad \mid p_1(t_2) - p_1(t_1) \mid \\
&+ \quad |\lambda_1| \int_0^1 \mid f_1(t_2,s,x(s),y(s)) - f_1(t_1,s,x(s),y(s)) \mid \ d_s(\bigvee_{z=0}^{s} g_1(t_2,z)) \\
&+ \quad |\lambda_1| \int_0^1 \mid f_1(t_1,s,x(s),y(s)) \mid \ d_s(\bigvee_{z=0}^{s} [g_1(t_2,z) - g_1(t_1,z)])
\end{aligned}$$

$$\begin{aligned}
&\leq && | p_1(t_2) - p_1(t_1) | + \lambda \int_0^1 \theta(\epsilon)\ d_s(\bigvee_{z=0}^{s} g_1(t_2, z)) \\
&+ && \lambda \int_0^1 (k_1(t_1, s) + b_1(\max\{| x(s) |, | y(s) |\}))\ d_s(\bigvee_{z=0}^{s} [g_1(t_2, z) - g_1(t_1, z)]) \\
&\leq && | p_1(t_2) - p_1(t_1) | + \lambda\theta(\epsilon) \int_0^1\ d_s(g_1(t_2, s)) \\
&+ && \lambda(K + rb_1) \int_0^1\ d_s g_1(t_2, s) - g_1(t_1, s) \\
&\leq && | p_1(t_2) - p_1(t_1) | + \lambda\theta(\epsilon)[g(t_2, 1) - g(t_2, 0)] \\
&+ && \lambda(K + rb_1)\{[g_1(t_2, 1) - g_1(t_1, 1)] - [g_1(t_2, 0) - g_1(t_1, 0)]\} \\
&\leq && | p_1(t_2) - p_1(t_1) | + \lambda\theta(\epsilon)[g(t_2, 1) - g(t_2, 0)] \\
&+ && \lambda(K + rb_1)\{[g_1(t_2, 1) - g_1(t_1, 1)] - [g_1(t_2, 0) - g_1(t_1, 0)]\} \\
&\leq && | p_1(t_2) - p_1(t_1) | + \lambda\theta(\epsilon)[g_1(t_2, 1) - g_1(t_2, 0)] \\
&+ && \lambda(K + rb_1)[| g_1(t_2, 1) - g_1(t_1, 1) | + | g_1(t_2, 0) - g_1(t_1, 0) |]
\end{aligned}$$

Hence

$$\begin{aligned}
| T_1x(t_2) - T_1x(t_1) | &\leq && | p_1(t_2) - p_1(t_1) | + \lambda\theta(\epsilon)[g_1(t_2, 1) - g_1(t_2, 0)] \\
&+ && \lambda(K + rb_1)[| g_1(t_2, 1) - g_1(t_1, 1) | + | g_1(t_2, 0) - g_1(t_1, 0) |]
\end{aligned}$$

Hence, from the continuity of the functions g_1 assumption (v), we deduce that T_1 maps $C(I)$ into $C(I)$.

As done above we can obtain

$$\begin{aligned}
| T_2y(t_2) - T_2y(t_1) | &\leq && | p_2(t_2) - p_2(t_1) | + \lambda\theta(\epsilon)[g_2(t_2, 1) - g_2(t_2, 0)] \\
&+ && \lambda(K + rb_2)[| g_2(t_2, 1) - g_2(t_1, 1) | + | g_2(t_2, 0) - g_2(t_1, 0) |]
\end{aligned}$$

Also, by our assumption (v), we see that T_2 maps $C(I)$ into $C(I)$.

Now, from the definition of the operator T we get

$$\begin{aligned}
Tu(t_2) - Tu(t_1) &= T(x, y)(t_2) - T(x, y)(t_1) \\
&= (T_1x(t_2), T_2y(t_2)) - (T_1x(t_1), T_2y(t_1)) \\
&= (T_1x(t_2) - T_1x(t_1), T_2y(t_2) - T_2y(t_1))
\end{aligned}$$

Therefore, T maps X into X.
Note that the set of values of $Tu(t)$ for all $u \in X$ is an equi-continuous subset of X.

Step 2: The operator T map U into U.

for $(x,y)\in U$, we have

$$\begin{aligned}
\mid T_1x(t)\mid &\le \mid p_1(t)\mid + \mid \lambda_1\int_0^1 f_1(t,s,x(s),y(s))\ d_sg_1(t,s)\mid\\
&\le \mid p_1(t)\mid + |\lambda_1|\int_0^1 \mid f_1(t,s,x(s),y(s))\mid\ \ d_s(\bigvee_{z=0}^{s} g_1(t,z))\\
&\le \|p_1\| + \lambda\int_0^1 (k_1(t,s)+b_1(\max\{\mid x(s)\mid,\mid y(s)\mid\}))\ d_s(\bigvee_{z=0}^{s} g_1(t,z))\\
&\le \|p_1\| + \lambda\int_0^1 (k_1(t,s)+rb_1)\ d_sg_1(t,s))\\
&\le \|p_1\| + \lambda(K+rb_1)\int_0^1\ d_sg_1(t,s)\\
&\le \|p_1\| + \lambda(K+rb_1)(g_1(t,1)-g_1(t,0))\\
&\le \|p_1\| + \lambda(K+rb_1)[\sup_t|g_1(t,1)|+\sup_t|g_1(t,0)|]\\
&\le \|p_1\| + \lambda(K+rb_1)\mu
\end{aligned}$$

Hence

$$\|T_1x\| \ \le\ \|p_1\| + \lambda(K+rb_1)\mu.$$

By a similar way can deduce that

$$\|T_2y\| \ \le\ \|p_2\| + \lambda(K+rb_2)\mu.$$

Therefore,

$$\|Tu\| = \|T(x,y)\| = \|T_1x, T_2y\| = \max\{\|T_1x\|,\|T_2y\|\}\le r.$$

Thus for every $u=(x,y)\in U$, we have $Tu\in U$ and hence $TU\subset U$, (i.e $T:U\to U$). This means that the functions of TU are uniformly bounded on I.

Step 3: The operator T is compact.
It is clear that the set U is nonempty, bounded, closed and convex, then according to Tychonoff's theorem in topological products and Arzela-Ascolli theorem the compactness criteria T is compact.

Step 4: The operator T is continuous.
Firstly, we prove that T_1 is continuous. Let $\epsilon^*>0$, the continuity of f_i yields $\exists\ \delta=\delta(\epsilon^*)$ such that $|f_i(t,s,x,y)-f_i(t,s,u,y)|<\epsilon^*$ whenever $\|x-u\|\le\delta$, thus if $\|x-u\|\le\delta$, we arrive at:

$$\begin{aligned}
\mid T_1x(t)-T_1u(t)\mid &\le \mid \lambda_1\int_0^1 f_1(t,s,x(s),y(s))\ d_sg_1(t,s)\\
&- \lambda_1\int_0^1 f_1(t,s,u(s),y(s))\ d_sg_1(t,s)\mid
\end{aligned}$$

$$
\begin{aligned}
&\leq \ |\lambda_1| \int_0^1 \mid f_1(t,s,x(s),y(s)) - f_1(t,s,u(s),y(s)) \mid \ d_s(\bigvee_{z=0}^{s} g_1(t,z)) \\
&\leq \ \epsilon^* \lambda \int_0^1 \ d_s(\bigvee_{z=0}^{s} g_1(t,z)) \\
&\leq \ \epsilon^* \lambda \int_0^1 \ d_s g_1(t,s) \\
&\leq \ \epsilon^* \lambda \ [g_1(t,1) - g_1(t,0)] \\
&\leq \ \epsilon^* \lambda \ [\mid g_1(t,1) \mid + \mid g_1(t,0) \mid] \\
&\leq \ \epsilon^* \lambda \ [\sup_{t \in I} \mid g_1(t,1) \mid + \sup_{t \in I} \mid g_1(t,0) \mid] \leq \epsilon
\end{aligned}
$$

where $\epsilon := \epsilon^* \lambda \mu$.
Therefore,

$$\mid T_1 x(t) - T_1 u(t) \mid \leq \epsilon.$$

This means that the operator T_1 is continuous.
By a similar way as done above we can prove that for any $y, \ v \in C[0,T]$ and $\| \ y - v \ \| < \delta$, we have

$$\mid T_2 y(t) - T_2 v(t) \mid \leq \epsilon.$$

Hence T_2 is continuous operator.
The operators $T_i \ (i = 1, 2)$ is continuous operator it imply that T is continuous operator.
Since all conditions of Schauder fixed point theorem are satisfied, then T has at least one fixed point $u = (x, y) \in U$, which completes the proof. ■

In what follows, we provide some examples illustrating the above obtained results.

Example : Consider the functions $g_i : I \times I \to R$ defined by the formula

$$
\begin{aligned}
g_1(t,s) &= \begin{cases} t \ln \frac{t+s}{t}, & \text{for } t \in (0,1], \ s \in I, \\ 0, & \text{for } t = 0, \ s \in I. \end{cases} \\
g_2(t,s) &= t(t+s-1), \ t \in I.
\end{aligned}
$$

It can be easily seen that the functions $g_1(t,s)$ and $g_2(t,s)$ satisfies assumptions (iii)-(v) given in Theorem 1, and $g_1(t,s)$ is function of bounded variation but it is not continuous on I. In this case, the coupled system of Urysohn-Stieltjes integral equations (2) has the form

$$
\begin{aligned}
x(t) &= p_1(t) + \lambda_1 \int_0^1 \frac{t}{t+s} f_1(t,s,x(s),y(s)) \ ds, \ t \in I \\
y(t) &= p_2(t) + \lambda_2 \int_0^t t f_1(t,s,x(s),y(s)) \ ds, \ t \in I.
\end{aligned} \tag{3}
$$

Also, consider the functions $f_i : I \times I \times R^2 \to R$ defined by the formula

$$f_1(t,s,x,y) = t + s + x + y,$$
$$f_2(t,s,x,y) = t + s + x^2 - y^2.$$

Now, it can be easily seen that the functions f_1 and f_2 satisfies assumptions (ii) given in Theorem 1:

$$\begin{aligned} \mid f_1(t,s,x,y) \mid &\leq \mid t+s+x+y \mid \\ &\leq \mid t+s \mid + \mid x \mid + \mid y \mid \\ &\leq 2T + 2\max\{\mid x \mid, \mid y \mid\} \end{aligned}$$

And

$$\begin{aligned} \mid f_2(t,s,x,y) \mid &\leq \mid t+s+x^2-y^2 \mid \\ &\leq \mid t+s \mid + \mid x^2-y^2 \mid \\ &\leq 2T + \mid (x-y)(x+y) \mid \\ &\leq 2T + 2\max\{\mid x \mid, \mid y \mid\} \end{aligned}$$

Hence, $k_i(t,s) = 2T$, and $b_i = 2$
Therefore, the functions f_i satisfies the assumption

$$\mid f_i(t,s,x,y) \mid \leq k_i(t,s) + b_i(\max\{\mid x \mid, \mid y \mid\}).$$

Therefore, the coupled system (3) has at least one solution $x, y \in C[0,1]$.

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