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NUMERICAL SOLUTIONS OF BENJAMIN-BONA-MAHONY-BURGERS EQUATION VIA NONSTANDARD FINITE DIFFERENCE SCHEME

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Abstract. The aim of this paper is to construct an unconditionally stable numerical scheme for the Benjamin-Bona-Mahony-Burgers (BBMB) equation. Kind of nonstandard finite difference discretization is used to achieve this goal. Stability analysis of this scheme is studied using John von Neumann technique, moreover, the accuracy of the proposed scheme is proved. The convergent of the proposed scheme is secured depending on Lax equivalence theorem. Numerical results with comparisons are given to confirm the reliability of the proposed method for BBMB equation.

1. Introduction

BBMB equation has been proposed in [1] as a model to study the propagation of unidirectional long waves of small-amplitudes in water, which is an alternative to the Korteweg-de Vries equation. This equation is used in many branches of science and engineering, for more details on both the mathematical theory and the physical significance of this model we refer to [2] and [3], and the references therein.

The BBMB model has been tackled and inspected by many authors. Manickam *et al.* used a spline collocation method for approximating the solutions of BBMB in [4]. Crank-Nicolson-type finite difference method is used in [5] to solve numerically BBMB by Omrani *et al.*. Kadri *et al.* used finite element methods to study this problem in [6]. Also, Fardi *et al.* in [7] used homotopy analysis method to give analytic solution of BBMB. The tanh method with the aid of symbolic computational system is employed to investigate exact solutions of BBMB equation in [8] and [9]. Recently, Zarebnia *et al.* in [10] used cubic B-spline collocation method to approximate the solutions of BBMB, and more recently, Zarebnia and Parvaz in [11] used quadratic B-spline collocation method to solve this problem.

On the other hand, the nonstandard finite difference method (NSFDM) is proposed by Mickens ([12]-[16]) for improving special discretizations of some terms in the differential equations, such that depending on the denominator function and

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the specific discretization this method be more accurate and more stable than standard method ([17], [18]), in addition this method can be easy to formulate [19]. The positive applications of the NSFDM can be found in the fields of physics, chemistry, engineering ([20], [21], [22] and [23]). Especially, the most attractive applications are in mathematical biology and ecology ([24], [25]) such that the merit of the NSFDM has been shown prominently.

The main goal of this manuscript is to constract a novel nonstandard finite difference scheme (NSFDS) for solving numerically the following BBMB ([10], [11]):

$$u_t - u_{xxt} - \alpha u_{xx} + (\beta + u)u_x = 0, \quad x \in [a, b], \quad t \in [0, T], \tag{1}$$

subject to the following Dirichlet boundary conditions:

$$u(a, t) = 0, \; u(b, t) = 0, \tag{2}$$

and initial condition:

$$u(x, 0) = f(x), \quad x \in [a, b], \tag{3}$$

where α and β are positive constants.

Also, In order to highlight the accuracy of the proposed algorithm, some numerical examples and comparisons are introduced.

The remainder of the paper is organized as follows. In the next section, we recall the preliminaries of the NSFDM. In Section 3, we develop a NSFDS of the BBMB equation. Section 4 is devoted to study the stability analysis of the proposed scheme and to study the truncating error of this scheme. In Section 5, some numerical results are reported to show the efficiency and the accuracy of the suggested algorithms. Finally, a conclusion is given in Section 6.

2. The Nonstandard Finite Difference Method

The technique of the NSFDM was firstly proposed by Mickens ([12]-[16]). It is method to construct a numerical discrete scheme for ordinary differential equations (ODEs) or partial differential equations (PDEs). The NSFDS is able to maintain the properties of the exact solution of the original ODEs or PDEs with the following rules [15]:

1. The orders of the discrete derivatives should be equal to the orders of the corresponding derivatives of the differential equations.
2. Denominator functions for the discrete derivatives must, in general, be expressed in terms of more complicated functions of the step sizes than those conventionally used.
3. Nonlinear terms must be approximated in a nonlocal way.
4. Special conditions that hold for the solutions of the differential equations should also be special discrete for the finite difference scheme.
5. The scheme should not have solutions that do not correspond to solutions of the differential equations.

When we want to approximate $\frac{dy}{dt}$ using Euler method we use $\frac{y(t+h) - y(t)}{\phi(h)}$ instead of $\frac{y(t+h) - y(t)}{h}$, where $\phi(h)$ is a continuous function of step size h, and the function $\phi(h)$ satisfies the following conditions:

$$\phi(h) = h + O(h^2), \quad 0 < \phi(h) < 1, \quad h \longrightarrow 0.$$

In addition to this replacement, if there are nonlinear terms in the differential equation, these are replaced by non-local approximation like for example

$$yx \longrightarrow \begin{cases} y_n x_{n+1}, \\ y_{n+1} x_n. \end{cases}$$

3. Construction of the NSFDS for the BBMB equation

In this section we construct an implict and unconditionally stable NSFDS to obtain numerical solutions of the BBMB equation (1).

Let M, N be natural numbers and the coordinates of the mesh points are:

$$x_n = nh, \quad n = 0,\ 1,\ 2,\ ...,\ N, \qquad t_m = m\triangle t, \quad m = 0,\ 1,\ 2,\ ...,\ M,$$

where

$$h = (b-a)/N, \qquad \triangle t = T/M.$$

The numerical value of u at the grid point $(x_n, t_m) = (nh, m\triangle t)$ is denoted by u_n^m and the nonstandard differences approximations are given as the following:

$$(u_t)_n^m = \frac{u_n^{m+1} - u_n^{m-1}}{2\varphi(\triangle t)} + O((\varphi(\triangle t))^2), \tag{4a}$$

$$\begin{aligned} (u_{xxt})_n^m &= \frac{u_{xx}^{m+1} - u_{xx}^{m-1}}{2\varphi(\triangle t)} + O((\varphi(\triangle t))^2), \\ &= \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1} - u_{n+1}^{m-1} + 2u_n^{m-1} - u_{n-1}^{m-1}}{2\varphi(\triangle t)\cdot(\phi(h))^2} + O((\phi(h))^2 + (\varphi(\triangle t))^2), \end{aligned} \tag{4b}$$

$$\begin{aligned} (u_{xx})_n^m &= \frac{u_{xx}^{m+1} + u_{xx}^{m-1}}{2}, \\ &= \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1} + u_{n+1}^{m-1} - 2u_n^{m-1} + u_{n-1}^{m-1}}{2(\phi(h))^2} + O((\phi(h))^2), \end{aligned} \tag{4c}$$

$$\begin{aligned} ((\beta+u)u_x)_n^m &= (\beta + \frac{u_n^{m+1} + u_n^m}{2}) \cdot \frac{u_x^{m+1} + u_x^m}{2} \\ &= (\beta + \frac{u_n^{m+1} + u_n^m}{2}) \cdot \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1} + u_{n+1}^m - u_{n-1}^m}{4(\phi(h))} + O((\phi(h))^2). \end{aligned} \tag{4d}$$

Substituting these equations (4) into (1), the resulting equations take the form:

$$\begin{aligned} \frac{u_n^{m+1} - u_n^{m-1}}{2\varphi(\triangle t)} &- \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1} - u_{n+1}^{m-1} + 2u_n^{m-1} - u_{n-1}^{m-1}}{2\varphi(\triangle t)\cdot(\phi(h))^2} \\ &- \alpha\frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1} + u_{n+1}^{m-1} - 2u_n^{m-1} + u_{n-1}^{m-1}}{2(\phi(h))^2} \\ &+ (\beta + \frac{u_n^{m+1} + u_n^m}{2}) \cdot \frac{u_{n+1}^{m+1} - u_{n-1}^{m+1} + u_{n+1}^m - u_{n-1}^m}{4(\phi(h))} = T_n^m. \end{aligned} \tag{5}$$

The above replacements give rise to an error, the truncation error, denoted here by T_n^m. Its value will be discussed in Section 4.2. Neglecting the truncation error, the resulting computable difference scheme takes the form:

$$
\begin{aligned}
u_{n+1}^{m+1}\Big[&\frac{1}{2\varphi(\triangle t)\cdot(\phi(h))^2}+\frac{\alpha}{2(\phi(h))^2}-\frac{1}{4(\phi(h))}(\beta+\frac{u_n^{m+1}+u_n^m}{2})\Big]=\\
&\frac{u_n^{m+1}-u_n^{m-1}}{2\varphi(\triangle t)}-\frac{-2u_n^{m+1}+u_{n-1}^{m+1}-u_{n+1}^{m-1}+2u_n^{m-1}-u_{n-1}^{m-1}}{2\varphi(\triangle t)\cdot(\phi(h))^2}\\
&-\alpha\frac{-2u_n^{m+1}+u_{n-1}^{m+1}+u_{n+1}^{m-1}-2u_n^{m-1}+u_{n-1}^{m-1}}{2(\phi(h))^2}\\
&+(\beta+\frac{u_n^{m+1}+u_n^m}{2})\cdot\frac{-u_{n-1}^{m+1}+u_{n+1}^m-u_{n-1}^m}{4(\phi(h))}. \qquad (6)
\end{aligned}
$$

Scheme (6) with the boundary conditions (2) and the initial condition (3) construct a nonlinear algebraic system of $(N+1)(M+1)$ equation with the unknown u_n^m, $(n=0,\ 1,\ 2,\ ...,\ N,\quad m=0,\ 1,\ 2,\ ...,\ M)$. This system will be solved in this work using Newton's iteration methods [26].

4. Stability analysis and truncating error

4.1. **Stability analysis.** To investigate the stability of scheme (6) we apply the Jon von Neumann method after linearizing this scheme by considering the term $(\beta+\frac{u_n^{m+1}+u_n^m}{2})$ as a constant D ([27], [28]). Scheme (6) can be written in the following form:

$$
u_{n+1}^{m+1}P_1=u_n^{m+1}P_2+u_{n-1}^{m+1}P_3+u_{n+1}^{m}P_4+u_{n-1}^{m}P_5+u_{n+1}^{m-1}P_6+u_n^{m-1}P_7+u_{n-1}^{m-1}P_8, \quad (7)
$$

where
$P_1=\frac{1}{2\varphi(\triangle t)\cdot(\phi(h))^2}+\frac{\alpha}{2(\phi(h))^2}-\frac{D}{4\phi(h)}$, $P_2=\frac{1}{2\varphi(\triangle t)}+\frac{1}{\varphi(\triangle t)\cdot(\phi(h))^2}+\frac{\alpha}{(\phi(h))^2}$,
$P_3=-\frac{1}{2\varphi(\triangle t)\cdot(\phi(h))^2}-\frac{\alpha}{2(\phi(h))^2}-\frac{D}{4\phi(h)}$, $P_4=\frac{D}{4\phi(h)}$, $P_5=-\frac{D}{4\phi(h)}$,
$P_6=\frac{1}{2\varphi(\triangle t)\cdot(\phi(h))^2}-\frac{\alpha}{2(\phi(h))^2}$, $P_7=\frac{-1}{2\varphi(\triangle t)}-\frac{1}{\varphi(\triangle t)\cdot(\phi(h))^2}+\frac{\alpha}{(\phi(h))^2}$,
$P_8=\frac{1}{2\varphi(\triangle t)\cdot(\phi(h))^2}-\frac{\alpha}{2(\phi(h))^2}$.

According to the Jon von Neumann technique, we have

$$
u_n^m=\xi^m e^{inqh}, \quad (8)
$$

where $i=\sqrt{-1},\ q\in\mathbb{R},\ \xi\in\mathbb{R}$ is the amplification factor. Substituting Eq. (8) in Eq. (7) then we obtain the following equation:

$$
\begin{aligned}
\xi^{m+1}e^{i(n+1)qh}P_1 &= \xi^{m+1}e^{inqh}P_2+\xi^{m+1}e^{i(n-1)qh}P_3+\xi^m e^{i(n+1)qh}P_4+\xi^m e^{i(n-1)qh}P_5\\
&+ \xi^{m-1}e^{i(n+1)qh}P_6+\xi^{m-1}e^{inqh}P_7+\xi^{m-1}e^{i(n-1)qh}P_8, \qquad (9)
\end{aligned}
$$

Dividing both sides of (9) by $\xi^m e^{inqh}$, we can write the following equation:

$$
\xi=\frac{\xi^{-1}(P_6e^{iqh}+P_7+P_8e^{-iqh})+P_4e^{iqh}+P_5e^{-iqh}}{P_1e^{iqh}-P_2-P_3e^{-iqh}}. \quad (10)
$$

The necessary condition for stability of the difference system (7) is $|\xi|\leq 1$ for all q i.e.,

$$
\Big|\frac{\xi^{-1}(P_6e^{iqh}+P_7+P_8e^{-iqh})+P_4e^{iqh}+P_5e^{-iqh}}{P_1e^{iqh}-P_2-P_3e^{-iqh}}\Big|\leq 1,
$$

Using the Eulers formula $e^{i\theta} = cos\theta + isin\theta$, and considering the time-independent limit value $\xi = -1$ as in ([29], [30]) and under some simplifications we have

$$|\frac{A + B\ cos(qh) + iC\ sin(qh)}{A + B\ cos(qh) - iC\ sin(qh)}| \leq 1,$$

where $A = \frac{-1}{\varphi(\triangle t)\cdot(\phi(h))^2} + \frac{\alpha}{(\phi(h))^2} - \frac{1}{2\varphi(\triangle t)},\ B = \frac{1}{\varphi(\triangle t)\cdot(\phi(h))^2} - \frac{\alpha}{(\phi(h))^2},\ C = \frac{1}{4\varphi(\triangle t)}$.

From last inequality the Jon von Neumann's sufficient condition for stability, $max_q|\xi(q)| \leq 1$, is satisfied for all real q. Hence, the difference scheme (6) is unconditionally stable.

4.2. **Truncating error.** From the definition of truncating error given by Eq. (5), and depending on relations (4) one gets

$$T_n^m = O((\varphi(\triangle t))^2 + (\phi(h))^2),$$

but we have $\quad \varphi(\triangle t) = \triangle t + O(\triangle t)^2$ and $\phi(h) = h + O(h)^2$,
so

$$T_n^m = O((\triangle t)^2 + h^2).$$

Depending on Lax equivalence theorem [27] the propsed NSFDS is convergent to the exact solution of BBMB equation when h and $\triangle t \longrightarrow 0$.

5. Numerical simulations

To illustrate the effectiveness of the proposed method in the present paper for solving BBMB, two test examples are carried out in this section. To justify the accuracy of the present method in comparison with the other methods, we report L_∞ error using formula $L_\infty = max|u_{exact}(x_n, t_m) - u_{approximation}(x_n, t_m)|$ for all $n,\ m$.

Example 1. [10] *We consider equation (1) with $\alpha = 0$, $\beta = 1$, $a = -40$ and $b = 60$ where*

$$f(x) = 3c\ sech^2(k(x - x_0)),$$

and the exact solution is

$$u(x,t) = 3c\ sech^2(k(x - vt - x_0)).$$

We compare our results with the cubic B-spline collocation method [10] which was the most accurate method for solving the BBMB in the literature. For this purpose, we consider the same parameter values for Eq. (1) as considered in [10], namely $c = 0.1$, $v = 1 + c$, $x_0 = 0$, $k = \sqrt{\frac{c}{4v}}$,

Take

$$\phi(h) = (e^{(\sqrt{2}h/3)} - 2 + e^{(-\sqrt{2}h/3)})/(2/9), \quad \varphi(\triangle t) = sinh(dt)$$

, tables (1, 2) give a comparisons between the proposed NSFDS in this work and the method in [10].

Figure (1) shows the exact solution and the obtained numerical results by means of the proposed NSFDS when $N = 200,\ T = 1$ and $\triangle t = 0.05$.

Table 1. Comparison of the maximum errors calculated by the NSFDS and by the method in [10] with $\triangle t = 0.1$ for example (1).

T	$N = 100$		$N = 300$	
	Method in [10]	Our scheme	Method in [10]	Our scheme
1	1.7726e-04	8.5290e-06	1.8456e-05	4.9879e-07
2	2.2262e-04	1.6605e-07	2.1289e-05	9.5561e-07
3	2.7444e-04	2.7551e-07	2.5022e-05	1.5831e-07
4	3.3417e-04	4.5642e-07	2.9085e-05	2.3844e-07

Table 2. Comparison of the maximum errors calculated by the NSFDS and by the method in [10] with $N = 400$ for example (1).

T	$\triangle t = 0.5$		$\triangle t = 0.1$	
	Method in [10]	Our scheme	Method in [10]	Our scheme
2	5.8464e-05	7.1542e-07	1.0607e-05	2.3482e-07
3	9.6939e-05	9.9297e-07	1.1653e-05	1.5191e-07
5	1.9834e-04	1.8465e-06	1.4465e-05	3.2664e-07
7	2.8452e-04	3.2538e-06	1.7653e-05	6.0026e-07

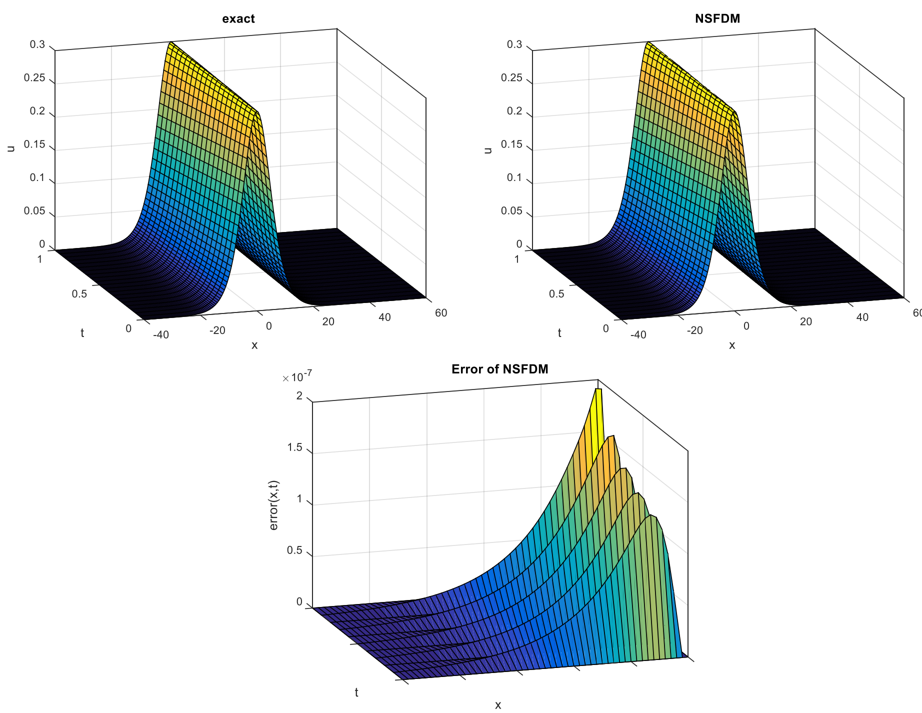

Figure 1. Comparison between, the numerical solution using our proposed scheme and the exact solution for example (1)

Example 2. *In this example we will introduce numerical solutions of equation (1) in the interval* $[-10, 10]$ *with* $\alpha = 1$ *and* $\beta = 1$ *and the initial condition is* $u(x, 0) = e^{-x^2}$.

Take $\phi(h) = (e^{(\sqrt{2}h/3)} - 2 + e^{(-\sqrt{2}h/3)})/(2/9)$, $\varphi(\triangle t) = sinh(dt)$, the behavior of the approximate solution with $\triangle t = 0.01$ and $N = 200$ is presented in Fig. 2.

In order to numerically check the error we consider the inhomogeneous BBMB equation

$$u_t - u_{xxt} - u_{xx} + u_x + uu_x = g(x), \quad x \in [0, 1], \quad t \in [0, T], \tag{11}$$

where

$$g(x) = (1 + \pi^2)sin(\pi x) + \pi t(\pi sin(\pi x) + cos(\pi x) + 0.5\ t\ sin(2\pi x)).$$

The exact solution of (11) is:

$$u(x, t) = t\ sin(\pi x).$$

In table (3) we find the maximum error of using the proposed scheme for solving Eq. (11) with different values of N, M.

TABLE 3. The maximum error for (11).

$N = M$	Max-error
10	1.0606e-04
20	5.9621e-05
40	1.8978e-05
80	4.7084e-06

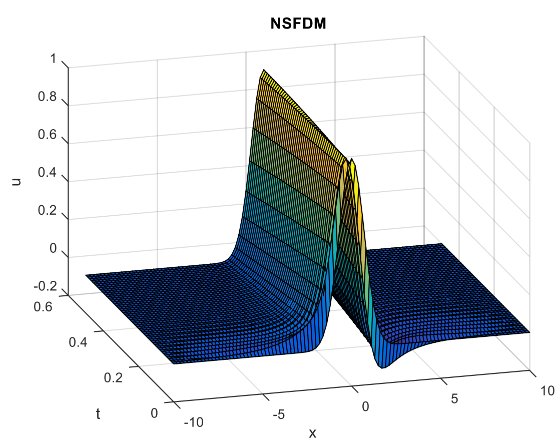

FIGURE 2. Approximate solution for example (2) when $T = 0.5$.

6. Conclusion

A new numerical unconditional stable scheme is constructed to introduce an approximate solutions of the BBMB equation. The proposed algorithm is based on the NSFDM. Stability analysis of this scheme is studied using John von Neumann technique, also the truncation error is studied. To confirm the efficiently and the accuracy of the proposed scheme we introduced comparisons between our numerical solutions of the problem with its exact solutions and with the approximate solutions that achieved by other methods.

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