ON A $q$-ANALOGUE OF HILBERT'S INEQUALITY

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Abstract. In this article an expression relating $\sin_q(\pi, \alpha)$ and the $q$-analogue of the beta function $B_q(\alpha, \lambda - \alpha)$ is found. This is used to show that the $q$-analogue of Hilbert’s inequality

$$\int_0^\infty \int_0^\infty |f(x)g(y)| \frac{d_q x}{x+y} d_q y \leq \frac{2\pi_q}{(2)^{1/p'} (2)_q^{1/p} \sin_q(\pi/p')} \|f\|_p \|g\|_{p'},$$

holds and a generalization of this inequality is proved.

Hilbert’s inequality

$$\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} dxdy \leq \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_{p'},$$

(1)

where $1 < p < \infty$, $p' = \frac{p}{p-1}$ and $f \in L^p[0, \infty)$ and $g \in L^{p'}[0, \infty)$, has been heavily studied in mathematics. Of particular interest for us is the proof found in [3], where Schur’s test is applied to the homogeneous Hilbert kernel

$$k(x, y) = \frac{1}{x+y}$$

on $[0, \infty) \times [0, \infty)$ to prove the result. In this article, we prove the $q$-analogue of Hilbert’s inequality (see Theorem 4),

$$\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} d_q x d_q y \leq \frac{2\pi_q}{(2)_q^{1/p} (2)^{1/p} \sin_q(\pi/p')} \|f\|_p \|g\|_{p'},$$

(2)

is satisfied. To show this result, a special case of Schur’s test is proved in the quantum calculus setup (see Theorem 3).

For $p = p' = 2$, a generalization for the classical Hilbert’s inequality is proved in [1]. For those values of $p$ and $p'$ and $\lambda > 0$, they show that given two positive constants $a, b$ and two functions $f, g \in L^2([0, \infty), t^{1-\lambda} dt)$, the inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(ax+by)^{\lambda}} dxdy < \frac{B(\lambda/2, \lambda/2)}{(ab)^{\lambda}} \|f\|_2 \|g\|_2$$

(3)

is satisfied, with the norms on the right-hand-side are the norms induced by the measure $t^{1-\lambda} dt$. Here $B(\cdot, \cdot)$ is the classical beta function. To obtain a similar, but

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stronger, generalization for the $q$-analogue, we show that the $q$-analogue of the beta function satisfies:

$$B_q(x, \lambda - x) = \frac{\pi_q}{\sin_q(\pi_q x)} \prod_{k=1}^{\lambda-1} [1 + \frac{x}{k}] q^k.$$  

for $\lambda \in \mathbb{N}$ and $0 < x < \lambda$ (See Lemma 1). With this result, we are able to show that under certain conditions:

$$\int_0^\infty \int_0^\infty |f(x)g(y)| \frac{d_q x d_q y}{(ax + by)^2} \leq M_q(\lambda, \beta) \|f\|_p \|g\|_{p'}.$$  

is satisfied for some constant $M_q(\lambda, \beta)$ that is explicitly calculated (see Theorem 6).

1. Preliminaries

In this section, we will introduce the basic definitions of $q$-calculus that will be needed throughout this exposition. For the rest of the article, we will assume a deformation parameter $0 < q < 1$. The $q$-analogue of $x \in \mathbb{R}$ will be denoted and defined by:

$$[x] = \frac{1 - q^x}{1 - q}.$$  

Later it will be useful to consider the $q^k$ analogue of $x \in \mathbb{R}$ defined by:

$$[x]_{q^k} = \frac{1 - q^{kx}}{1 - q^k}.$$  

Integrals in this article will be understood as the Jackson integrals:

$$\int_0^b f(x) d_q x = (1 - q)b \sum_{j=0}^{\infty} q^j f(q^j b),$$

$$\int_0^\infty f(x) d_q x = (1 - q) \sum_{j=-\infty}^{\infty} q^j f(q^j).$$  

We remark that the Fundamental Theorem of Calculus is satisfied by the Jackson integral (see Theorem 20.1 in [5]) and for later use, we record the $q$-analogue of the power rule

$$\int x^\alpha d_q x = \frac{x^{\alpha+1}}{[\alpha + 1]} \quad \alpha \neq -1.$$  

An important estimate used in this article uses the $q$-analogues of the gamma (c.f. [4]) and beta functions (c.f. [2]). For $s, t > 0$:

$$\Gamma_q(t) = \frac{(1 - q)^{t-1}}{(1 - q)^{t-1}}$$

$$B_q(t, s) = \frac{\Gamma_q(s) \Gamma_q(t)}{\Gamma_q(s + t)}.$$  

(4)

(5)
where,
\[
(a + b)^q_n = \prod_{j=0}^{n-1} (a + q^j b) \quad \text{for } n \in \mathbb{N},
\]
\[
(1 + a)^q = \prod_{j=0}^{\infty} (1 + q^j a) \quad \text{and}
\]
\[
(1 + a)^t_q = \frac{(1 + a)^{\infty}_q}{(1 + aq^t)^{\infty}_q} \quad \text{for } t \in \mathbb{C}.
\]
These \(q\)-binomials satisfy many properties of which we mention the following, for later use, (c.f. [2])
\[
(1 + x)^{s+t}_q = (1 + x)^s_q (1 + q^s x)^t_q,
\]
for \(s, t \in \mathbb{R}\). Integral representations of these functions
\[
\Gamma_q(t) = K(A, t) \int_0^{\infty / (A(1-q))} x^{t-1} e_q^{-x} d_q x
\]
\[
B_q(t, s) = K(A, t) \int_0^{\infty / A} x^{t-1} \frac{(1 + x)^{s+t}_q}{(1 + x)^{\infty}_q} d_q x,
\]
are proved in [2], where
\[
K(A, t) = \frac{1}{1 + A} A^t \left(1 + \frac{1}{A} \right)_q^t (1 + A)^{1-t}_q
\]
and
\[
\int_0^{\infty / A} f(x) d_q x = (1 - q)b \sum_{j=0}^{\infty} q^j A^q f(q^j / A).
\]
In particular, (7) will be especially important to our exposition. One last fact regarding the \(q\)-gamma functions will be useful for our purposes. In [6], they obtain:
\[
\Gamma_q(t) \Gamma_q(1 - t) = \frac{\pi_q}{\sin_q(\pi_q t)},
\]
where
\[
\sin_q(x) = \frac{e_q(ix) - e_q(-ix)}{2i}
\]
and \(\pi_q\) is the analogue of \(\pi\) defined via the equation
\[
\sin_q(\pi_q x) = \pi_q[x] \prod_{k=1}^{\infty} \left[1 + \frac{x}{k} \right]_{q^k}^\lambda_1 \left[1 - \frac{x}{k} \right]_{q^k}.
\]
In the following lemma, we generalize equation (9).

**Lemma 1.** For \(\lambda \in \mathbb{N}\) and \(0 < x < \lambda\) we have
\[
B_q(x, \lambda - x) = \frac{\pi_q}{\sin_q(\pi_q x)} \prod_{k=1}^{\lambda-1} \left[1 + \frac{x}{k} \right]_{q^k}.
\]
*Proof.* It is known that (c.f. Eq. 3, [6]):
\[
\Gamma_q(x)^{-1} = [x] \prod_{k=1}^{\infty} \left[1 + \frac{x}{k} \right]_{q^k} \left[1 + \frac{1}{k} \right]_{q^k}^{x},
\]
we calculate:

\[
B_q(x, \lambda - x) = \frac{[\lambda]}{[x][\lambda - x]} \prod_{k=1}^{\infty} \frac{[1 + \frac{\lambda}{k}]}{[1 + \frac{x}{k}][1 + \frac{\lambda - x}{k}]} \frac{[1 - q^k]}{[1 - q^{k+\lambda - x}][1 - q^{k+\lambda}]}
\]

\[
= \frac{[\lambda]}{[x][\lambda - x]} \prod_{k=1}^{\infty} \frac{(1 - q^{k-x})(1 - q^k)}{(1 - q^{k+\lambda - x})(1 - q^{k+\lambda})}
\]

\[
= \frac{[\lambda]}{[x][\lambda - x]} \prod_{k=1}^{\lambda} \frac{(1 - q^{k-x})}{(1 - q^k)} \prod_{k=1}^{\infty} \frac{1}{[1 + \frac{x}{k}][1 + \frac{\lambda - x}{k}][1 - q^k]}
\]

\[
= \frac{\pi_q}{\sin_q(\pi_q t)} \prod_{k=1}^{\lambda - 1} \frac{1 - q^{k-x}}{1 - q^k},
\]

which is the desired result.

Notice that when \(\lambda = 1\), Equation (11) reduces to (9), thus generalizing this result. With all these tools at hand, we are now able to tackle the proof of the \(q\)-analogue of Hilbert’s inequality.

2. \(q\)-Hilbert’s Inequality

The first step in our proof will be to obtain the value of the integral of \(x^{-\alpha}k(x, y)\) where \(k(x, y)\) is the Hilbert kernel \(k(x, y) = 1/(x + y)\).

**Lemma 2.** For \(0 < \alpha < 1\),

\[
\int_0^\infty \frac{1}{(x+y)x^{\alpha}} d_q x = \frac{2\pi_q}{y^\alpha(2)^q(2)^{1-\alpha}\sin_q(\alpha\pi_q)}.
\]

In particular, when \(\alpha = 1/p'\),

\[
\int_0^\infty \frac{1}{(x+y)x^{1/p'}} d_q x = \frac{2\pi_q}{y^{1/p'}(2)^{1/p'}(2)^{1/p}\sin_q(\pi_q/p')}.
\]

**Proof.** Equation (13) follows from (12). So, using (7) with \(t = 1 - \alpha\) and \(s = \alpha\) we obtain:

\[
\int_0^\infty \frac{1}{(1+u)u^{\alpha}} d_q u = \frac{B_q(1, \alpha)}{K(1, 1-\alpha)}.
\]

Now we use (5) and (8) to obtain:

\[
\int_0^\infty \frac{1}{(1+u)u^{\alpha}} d_q u = \frac{2\Gamma_q(\alpha)\Gamma_q(1-\alpha)}{(2)^{\alpha}(2)^{1-\alpha}}.
\]

In particular, (9) implies:

\[
\int_0^\infty \frac{1}{(1+u)u^{\alpha}} d_q u = \frac{2\pi_q}{(2)^{\alpha}(2)^{1-\alpha}\sin_q(\pi_q\alpha)}.
\]

To obtain our result, notice that

\[
\int_0^\infty \frac{1}{(x+y)x^{\alpha}} d_q x = \int_0^\infty \frac{1}{(1+x/y)^{\alpha+1}(x/y)^{\alpha}} d_q x,
\]

and now the substitution \(u = x/y\) gives the result (c.f. Eq. 19.15 [5]).
To show Hilbert's inequality we will need a $q$-analogue of Schur's test in the quantum calculus setup. We show this in the following theorem.

**Theorem 3** ($q$-Schur's test). For $k = k(x, y)$ and $1 < p < \infty$, suppose that there exist functions $s \in L^p(0, \infty)$, $t \in L^p(0, \infty)$ and constants $A$ and $B$ such that such that the integrals exist and satisfy:

$$\int_0^\infty k(x, y)t(y)^{p'} \, dq_y \leq (As(x))^{p'}$$

for every $x \in [0, \infty)$, and

$$\int_0^\infty k(x, y)s(x)^p \, dx \leq (Bt(y))^p$$

for every $y \in [0, \infty)$. Then, if $f \in L^p(0, \infty)$,

$$T(f)(x) = \int_0^\infty k(x, y)f(y) \, dq_y$$

exists for every $x \in [0, \infty)$, $T(f) \in L^p(0, \infty)$ and

$$\|T(f)\|_p \leq AB\|f\|_p.$$

**Proof.** By the $q$-Hölder's inequality, it suffices to show that for every non-negative functions $h \in L^p(0, \infty)$ and $g \in L^p(0, \infty)$, the following inequality is satisfied

$$|h \cdot T(g)||_1 \leq AB\|g\|_p\|h\|_{p'}.$$

Then, by the $q$-Hölder's inequality and the hypothesis of the theorem we obtain:

$$\begin{align*}
\int_0^\infty k(x, y)g(y) \, dq_y &= \int_0^\infty \left( t(y)k(x, y)\frac{g(y)}{t(y)} \right) \, dq_y \\
&= \int_0^\infty \frac{t(y)k(x, y)^{1/p'} g(y)k(x, y)^{1/p}}{t(y)} \, dq_y \\
&\leq \left( \int_0^\infty t(y)^{p'} k(x, y) \, dq_y \right)^{1/p'} \left( \int_0^\infty \frac{g(y)^p k(x, y)}{t(y)^p} \, dq_y \right)^{1/p} \\
&\leq As(x) \left( \int_0^\infty \frac{k(x, y)g(y)^p}{t(y)^p} \, dq_y \right)^{1/p}.
\end{align*}$$

(14)

Now, we consider the norm:

$$\begin{align*}
|h \cdot T(g)||_1 &\leq A \int_0^\infty h(x)s(x) \left( \int_0^\infty \frac{k(x, y)g(y)^p}{t(y)^p} \, dq_y \right)^{1/p} \, dx \\
&\leq A\|h\|_{p'} \left( \int_0^\infty \int_0^\infty s(x)^p \frac{k(x, y)g(y)^p}{t(y)^p} \, dq_y \, dx \right)^{1/p} \\
&= A\|h\|_{p'} \left( \int_0^\infty s(x)^p k(x, y) \, dx \int_0^\infty \frac{g(y)^p}{t(y)^p} \, dq_y \right)^{1/p} \\
&\leq AB\|h\|_{p'} \left( \int_0^\infty g(y)^p \, dq_y \right)^{1/p},
\end{align*}$$

(15) (16) (17) (18)

where the (15) follows from (14), (16) follows from the $q$-Hölder’s inequality, (17) follows from Fubini-Tonelli’s theorem, and (18) follows from the hypotheses of the theorem.

$\square$
We are now able to show the $q$-analogue of Hilbert’s inequality.

**Theorem 4** ($q$-Hilbert’s inequality). Let $1 < p < \infty$ and let $k(x, y) = \frac{1}{x + y}$. If $f \in L^p[0, \infty)$ and $g \in L^{p'}[0, \infty)$. Then,

$$
\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x + y} d_qx d_qy \leq \frac{2\pi q}{(2\frac{1}{p})(2\frac{1}{p'}) \sin_q(\pi_q/p)} \|f\|_p \|g\|_{p'}.
$$

(19)

Furthermore, the constant $\frac{2\pi q}{(2\frac{1}{p})(2\frac{1}{p'}) \sin_q(\pi_q/p)}$ is the best possible.

**Proof.** Let

$$
s(x) = t(x) = \frac{1}{x^{1/(pp')}}.
$$

Then, by Lemma 2,

$$
\int_0^\infty s(x)^p k(x, y)d_dx = \int_0^\infty \frac{1}{(x + y)x^{1/p'}}d_qx = \frac{2\pi q}{(2\frac{1}{p})(2\frac{1}{p'}) \sin_q(\pi_q/p')} t(y)^p = \frac{2\pi q}{(2\frac{1}{p})(2\frac{1}{p'}) \sin_q(\pi_q/p')} t(y)^p
$$

and

$$
\int_0^\infty t(y)^p k(x, y)dy = \int_0^\infty \frac{1}{(x + y)y^{1/p}}d_qy = \frac{2\pi q}{(2\frac{1}{p})(2\frac{1}{p'}) \sin_q(\pi_q/p')} s(x)^{p'} = \frac{2\pi q}{(2\frac{1}{p})(2\frac{1}{p'}) \sin_q(\pi_q/p')} s(x)^{p'}.
$$

Hence,

$$
AB = \left(\frac{2\pi q}{(2\frac{1}{p})(2\frac{1}{p'}) \sin_q(\pi_q/p)}\right)^{1/p+1/p'} = \frac{2\pi q}{(2\frac{1}{p})(2\frac{1}{p'}) \sin_q(\pi_q/p)}
$$

in Theorem 3. This completes the proof of the inequality.

Now, we proceed to show that the constant is the best possible. Let $\chi_S : [0, \infty) \to \{0, 1\}$ denote the characteristic function of the set $S$ and define

$$
f_\lambda(x) = |\lambda - 1|^{1/p} x^{-\lambda/p} \chi_{[1, \infty)}(x)
$$

and

$$
g_\lambda(y) = |\lambda - 1|^{1/p'} y^{-\lambda/p'} \chi_{[1, \infty)}(y).
$$
Then, if \(1 < q < p\), we have

\[
\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x + y} d_qx d_qy = [\lambda - 1] \int_1^\infty \frac{1}{y^{\lambda/p'}} \left( \int_1^\infty \frac{1}{x^{\lambda/p}(x + y)} d_qx \right) d_qy \\
= [\lambda - 1] \int_1^\infty \frac{1}{y^{\lambda/p'}} \left( \int_0^1 \frac{1}{x^{\lambda/p}(x + y)} d_qx \right) d_qy \\
- [\lambda - 1] \int_1^\infty \frac{1}{y^{\lambda/p'}} \left( \int_0^1 \frac{1}{x^{\lambda/p}(x + y)} d_qx \right) d_qy \\
= \frac{2\pi q^{\lambda-1}}{(2)_q^{1-\lambda/p}(2)_q^{\lambda/p} \sin_q(\lambda\pi q/p)} \\
- [\lambda - 1] \int_1^\infty \frac{1}{y^\lambda} \left( \int_0^1 \frac{1}{u^{\lambda/p}(1 + u)} d_qu \right) d_qy
\]

by (12). For the last integral, notice that \(0 \leq x \leq 1\) and \(1 < y < \infty\). Thus, \(1 + u \geq 1\). So, we obtain:

\[
\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x + y} d_qx d_qy \geq \frac{2\pi q^{\lambda-1}}{(2)_q^{1-\lambda/p}(2)_q^{\lambda/p} \sin_q(\lambda\pi q/p)} \\
- [\lambda - 1] \int_1^\infty \frac{1}{y^\lambda} \left( \int_0^1 \frac{1}{u^{\lambda/p}(1 + u)} d_qu \right) d_qy
\]

Observe that

\[
\int_0^{1/y} u^{-\lambda/p} d_qu = y^{(\lambda/p) - 1} \frac{1}{[1 - \lambda/p]}
\]

Then,

\[
\int_1^\infty \frac{1}{y^\lambda} \left( \int_0^{1/y} \frac{1}{u^{\lambda/p}} d_qu \right) d_qy = \frac{1}{[1 - \lambda/p]} \int_1^\infty y^{-(\lambda/p') - 1} d_qy = -\frac{1}{[1 - \lambda/p][\lambda/p'] q^{\lambda/p'}}.
\]

Therefore,

\[
\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x + y} d_qx d_qy \geq \frac{2\pi q^{\lambda-1}}{(2)_q^{1-\lambda/p}(2)_q^{\lambda/p} \sin_q(\lambda\pi q/p)} \\
+ \frac{[\lambda - 1]}{[1 - \lambda/p][\lambda/p'] q^{\lambda/p'}} \\
\]

Now, as \(\lambda \to 1\),

\[
\frac{2\pi q^{\lambda-1}}{(2)_q^{1-\lambda/p}(2)_q^{\lambda/p} \sin_q(\lambda\pi q/p)} + \frac{[\lambda - 1]}{[1 - \lambda/p][\lambda/p'] q^{\lambda/p'}} \to 2\pi q
\]

and this implies equality for these particular functions \(f\) and \(g\). This completes the proof. \(\Box\)

3. A Generalization of \(q\)-Hilbert’s Inequality

In this section, we show the \(q\)-analogue of (3) for \(\lambda \in \mathbb{N}\). We start this section with a generalization of Lemma 2.
Lemma 5. For $\lambda \in \mathbb{N}$ and $0 < \alpha < 1$,

$$
\int_0^\infty \frac{1}{(ax + by)^\lambda y^\alpha} \, dqy
= b^{\alpha-1}(ax)^{1-\alpha-\lambda} \left(\frac{2\pi q}{2^\alpha(2^\alpha - 2\alpha)^{1-\alpha}} \sin_q(\alpha \pi q)\right)^{\frac{\lambda - 1}{k}} q^{\frac{\lambda - 1}{k}} \prod_{k=1}^{\lambda - 1} \left[ 1 + \frac{1 - \alpha}{k} \right] q^k.
$$

(20)

and

$$
\int_0^\infty \frac{1}{(ax + by)^\lambda x^\alpha} \, dqx
= a^{\alpha-1}(by)^{1-\alpha-\lambda} \left(\frac{2\pi q}{2^\alpha(2^\alpha - 2\alpha)^{1-\alpha}} \sin_q(\alpha \pi q)\right)^{\frac{\lambda - 1}{k}} q^{\frac{\lambda - 1}{k}} \prod_{k=1}^{\lambda - 1} \left[ 1 + \frac{1 - \alpha}{k} \right] q^k.
$$

(21)

Proof. The proof of (20) is similar to the proof of (12) noting that

$$(ax + by)^\lambda = \prod_{j=0}^{\lambda-1} (ax + q^j by) = (ax)^\lambda \prod_{j=0}^{\lambda-1} (1 + q^j (by)/(ax)) = (ax)^\lambda (1 + (by)/(ax))^\lambda$$

and using (11) instead of (9) at the end. Hence, it is omitted. However, the proof of (21) is inherently different due to the fact that $(a + b)^\lambda \neq (b + a)^\lambda$ for $\lambda > 1$. For this proof we will use the following property of the $q$-integrals (c.f [2]):

$$
\int_0^A f(x) \, dq x = \int_0^A \frac{1}{x^2} f \left( \frac{1}{x} \right) \, dq x.
$$

(22)

We start by rewriting the integral:

$$
\int_0^\infty \frac{1}{(ax + by)^\lambda x^\alpha} \, dq x = \int_0^\infty \frac{1}{x^{\alpha+\lambda} (1 + by/(ax))^\lambda} \, dq x.
$$

Now we let $u = ax/(by)$ to obtain:

$$
\int_0^\infty \frac{1}{(ax + by)^\lambda x^\alpha} \, dq x = a^{\alpha-1}(by)^{1-\alpha-\lambda} \int_0^\infty \frac{1}{u^{\alpha+\lambda} (1 + 1/u)^\lambda} \, dq u.
$$

To use (22), we rewrite this as:

$$
\int_0^\infty \frac{1}{(ax + by)^\lambda x^\alpha} \, dq x = a^{\alpha-1}(by)^{1-\alpha-\lambda} \int_0^\infty \frac{1}{u^{\alpha+\lambda} (1 + 1/u)^\lambda} \, dq u.
$$

So that, the integral becomes

$$
\int_0^\infty \frac{1}{(ax + by)^\lambda x^\alpha} \, dq x = a^{\alpha-1}(by)^{1-\alpha-\lambda} \int_0^\infty \frac{u^{\alpha+\lambda-2}}{(1 + 1/u)^\lambda} \, dq u.
$$

Now, using (7) we obtain

$$
\int_0^\infty \frac{1}{(ax + by)^\lambda x^\alpha} \, dq x = a^{\alpha-1}(by)^{1-\alpha-\lambda} B_q(\lambda - (1-\alpha), 1-\alpha) \frac{K(1, \lambda - (1-\alpha))}{K(1, \lambda - (1-\alpha))}.
$$

Now, (11) and the fact that $\sin_q(\pi_1(1-\alpha)) = \sin_q(\pi_1\alpha)$ give the desired result. $\square$
It must be noted that when \(a, b, \lambda = 1\) we recover the result in Lemma 2. To show our generalization of (23), let \(0 < \beta < \min\{1/p, 1/p'\}\) and we consider

\[
I := \int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{(ax + by)^\lambda} d_q x d_q y = \int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{(ax)^\lambda(1 + (by)/(ax))^\lambda} q^\lambda d_q x d_q y
\]

\[
= \int_0^\infty \int_0^\infty \frac{|f(x)|(x/y)^\beta}{(ax)^\lambda p(1 + (by)/(ax))^\lambda} \cdot \frac{|g(y)|(y/x)^\beta}{(ax)^\lambda p'(1 + q^\lambda p(by)/(ax))^\lambda} q^{\lambda/p} d_q x d_q y.
\]

By Hölder’s inequality, Equation (6), and Lemma 5, we have:

\[
I \leq \left( \int_0^\infty \int_0^\infty \frac{|f(x)|^{p_1} q^{\lambda/p(1-p'\beta-\lambda)}}{(ax + by)^{1-\beta}q^{\lambda/p(1-p'\beta-\lambda)}} d_q x d_q y \right)^{1/p} \left( \int_0^\infty \int_0^\infty \frac{|g(y)|^{p'} q^{\lambda/p} q^{\lambda/p'}}{(ax + q^{\lambda/p}by)^{1-\beta}q^{\lambda/p}} d_q x d_q y \right)^{1/p'}
\]

\[
\leq M_q^{\lambda, \beta}(a, b) \left( \int_0^\infty |f(x)|^{p_1-\lambda} d_q x \right)^{1/p} \left( \int_0^\infty |g(y)|^{p'-\lambda} d_q y \right)^{1/p'},
\]

where

\[
M_q^{\lambda, \beta}(a, b) := \frac{2\pi q(ab)^{-\lambda} q^{\lambda/p(1-p'\beta-\lambda)}}{(2\varpi q^2)^{1-\beta}q^{\lambda/p(1-p'\beta-\lambda)}} \cdot \Pi_{k=1}^{\lambda-1} \left[ 1 + \frac{1 - p\beta}{k} \right]^{1/p} \cdot \Pi_{k=1}^{\lambda-1} \left[ 1 + \frac{1 - p'\beta}{k} \right]^{1/p'}.
\]

This proves the following theorem:

**Theorem 6** (Generalization of the \(q\)-Hilbert’s inequality). Let \(1 < p < \infty, a, b > 0\), and \(\lambda \in \mathbb{N}\). Let \(\beta \in \mathbb{R}\) such that \(0 < \beta < \min\{1/p, 1/p'\}\). Then, if \(f \in L^p([0, \infty), t^{1-\lambda} d_q t)\) and \(g \in L^{p'}([0, \infty), t^{1-\lambda} d_q t)\), then,

\[
\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{(ax + by)^{\lambda}} q^{\lambda} d_q x d_q y \leq M_q^{\lambda, \beta}(a, b) \|f\|_p \|g\|_{p'}.
\]

**REFERENCES**


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