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CERTAIN PROPERTIES OF THE GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. In this paper we give sufficient conditions for the generalized Mittag-Leffler functions $\mathcal{E}_{\alpha,\beta}$, to be included in the class of k-parabolic starlike respectively k-uniformly convex functions of order γ. We also give sufficient condition for a generalized class of k-parabolic starlike and k-uniformly convex functions of order γ, introduced in [7].

1. Introduction

Let $\mathbb{U}(r) = \{z \in \mathbb{C} : |z| < r\}$ be a disk in the complex plane $\mathbb{C}$, centered at zero, and $\mathbb{U} = \mathbb{U}(1)$ denote the open unit disk in the complex plane. We denote by $\mathcal{A}$ the class of the functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

defined in $\mathbb{U}$. Let r be a real number with $r \in (0, 1]$.

We say that f is starlike in $\mathbb{U}$, if $f : \mathbb{U} \to \mathbb{C}$ is univalent and $f(\mathbb{U})$ is a starlike domain in $\mathbb{C}$, with respect to origin. It is well-known that $f \in \mathcal{A}$ is starlike in $\mathbb{U}$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{for all } z \in \mathbb{U}.$$

The class of starlike functions with respect to origin is denoted by S^*.

The function $f \in \mathcal{A}$ is convex in $\mathbb{U}$, if and only if $f : \mathbb{U} \to \mathbb{C}$ is univalent and $f(\mathbb{U})$ is a convex domain in $\mathbb{C}$. The function $f \in \mathcal{A}$ is convex if and only if

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, \quad z \in \mathbb{U}.$$

We denoted by $\mathcal{K}$ the class of convex functions.

In [3] Goodman defined the class of uniformly convex functions, denoted by UCV as follows:

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Definition 1.1. [3] A function $f \in \mathcal{A}$ is said to be uniformly convex in $\mathbb{U}$, if $f \in \mathcal{K}$ and has the property that for every circular arc γ contained in $\mathbb{U}$, with center ζ, also in $\mathbb{U}$, the arc $f(\gamma)$ is convex.

Due to the analytic criterion for $f \in UCV$, given by Rønning [8]:
A function $f \in \mathcal{A}$ is uniformly convex in $\mathbb{U}$, if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathbb{U}. \tag{1}$$

The class of k-uniformly convex functions was introduced by Kanas and Wisniowska [5], as a generalization of uniform convexity. The class of k-uniformly convex functions are denoted by $k-UCV$. In [8] Rønning defined the class of parabolic starlike functions by the following way:

$$S_p = \{F \in S^* | F(z) = zf'(z), f \in \mathcal{UCV}\}.$$

Definition 1.2. [1] The class S_p of parabolic starlike functions consists of functions $f \in \mathcal{A}$, satisfying

$$\Re\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)}-1\right|,\ z \in \mathbb{U}.$$

The class of k-parabolic starlike functions, denoted by $k-S_p$ are related to the class $k-UCV$ by the well-known Alexander equivalence.

For $-1 < \gamma \le 1$ and $k \ge 0$ a function $f \in \mathcal{A}$ is said to be in the class of k-parabolic starlike functions of order γ, denoted by $k-S_p(\gamma)$, if

$$\Re\left(\frac{zf'(z)}{f(z)}-\gamma\right) > k\left|\frac{zf'(z)}{f(z)}-1\right|,\ z \in \mathbb{U}.$$

For the same conditions for the parameters γ and k, the function $f \in \mathcal{A}$ is said to be in the class of k - uniformly convex functions of order γ, if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right) > k\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathbb{U}.$$

We denote by $k-S_p(\gamma)$ the class of k-parabolic starlike functions of order γ and by $k-UCV(\gamma)$ the class of k-uniformly convex functions of order γ.

In [7] the authors generalized the classes of k-parabolic starlike, respectively k-uniformly convex functions, of order γ, for $0 \le \gamma < 1$.

For $0 \le \lambda < 1$, $0 \le \gamma < 1$ and $k \ge 0$, the function $f \in \mathcal{A}$ belongs to the class $k-S_p(\lambda,\gamma)$ if

$$\Re\left\{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-\gamma\right\} > k\left|\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right|,\ z \in \mathbb{U}. \tag{2}$$

For the same conditions to the parameters λ, γ and k, the function $f \in \mathcal{A}$ belongs to the class $k-UCV(\lambda,\gamma)$ if

$$\Re\left\{\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf''(z)}-\gamma\right\} > k\left|\frac{f'(z)+zf''(z)}{f'(z)+\lambda zf'(z)}-1\right|,\ z \in \mathbb{U}. \tag{3}$$

It is easily seen that, $k-S_p(0,\gamma) = k-S_p(\gamma)$, $k-S_p(0,0) = k-S_p$, $k-UCV(0,\gamma) = k-UCV(\gamma)$ and $k-UCV(0,0) = k-UCV$, where $0 \le \gamma < 1$.

The function of the form

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},$$

where $\Re(\alpha) > 0$ and $z \in \mathbb{C}$, was introduced by Mittag-Leffler in 1903 and is called the Mittag-Leffler function.

The generalized Mittag-Leffer function has the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{4}$$

where $z, \alpha, \beta \in \mathbb{C}$ and $\Re(\alpha) > 0$ was studied by Wiman [10]. Several well-known special cases of the Mittag-Leffler and the generalized Mittag-Leffler functions were presented for example in the papers [2], [4] as follows:

$$E_0(z) = \frac{1}{1-z},\ E_1(z) = e^z,\ E_2(z) = \cosh(\sqrt{z}),$$

$$E_{0,1}(z) = \frac{z}{1-z},\ E_{1,1}(z) = ze^z,\ E_{1,2}(z) = e^z - 1.$$

Because the Mittag-Leffler function $E_{\alpha,\beta}$ does not belong to the family $\mathcal{A}$, it is natural to consider the following normalization of the Mittag-Leffler function:

$$\mathcal{E}_{\alpha,\beta}(z) = \frac{zE_{\alpha,\beta}(z)}{E_{\alpha,\beta}(0)} = z + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}z^2 + \frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}z^3 + ...,$$

which is equivalent to

$$\mathcal{E}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]} z^n.$$

In [2] the authors have proved that if $\alpha \geq 1$ and $\beta \geq (3+\sqrt{17})/2$ then $\mathcal{E}_{\alpha,\beta}$ is starlike in U respectively convex in $U_{1/2}$. In this paper we find sufficient conditions so that, the generalized Mittag-Leffler function $\mathcal{E}_{\alpha,\beta}$ to be in the classes S^*, $\mathcal{K}$, S_p, UCV, $k - S_p(\gamma)$, $k - UCV(\gamma)$, respectively in $k - S_p(\lambda, \gamma)$ and $k - UCV(\lambda, \gamma)$.

2. Main Results

Theorem 2.1. *Let $\alpha, \beta > 0$, $k \geq 0$ and $0 < \gamma \leq 1$. If*

$$\sum_{n=2}^{\infty} \frac{(n-1)(k+1) + 1 - \gamma}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1-\gamma}{\Gamma(\beta)}, \tag{5}$$

then $\mathcal{E}_{\alpha,\beta} \in k - S_p(\gamma)$.

Proof. It is sufficient to show that

$$k\left|\frac{z\mathcal{E}'_{\alpha,\beta}(z)}{\mathcal{E}_{\alpha,\beta}(z)} - 1\right| - \Re\left(\frac{z\mathcal{E}'_{\alpha,\beta}(z)}{\mathcal{E}_{\alpha,\beta}(z)} - 1\right) \leq 1 - \gamma.$$

Now we have

$$k\left|\frac{z\mathcal{E}'_{\alpha,\beta}(z)}{\mathcal{E}_{\alpha,\beta}(z)} - 1\right| - \Re\left(\frac{z\mathcal{E}'_{\alpha,\beta}(z)}{\mathcal{E}_{\alpha,\beta}(z)} - 1\right) \leq (1+k)\left|\frac{z\mathcal{E}'_{\alpha,\beta}(z)}{\mathcal{E}_{\alpha,\beta}(z)} - 1\right| \leq$$

$$\leq (1+k)\frac{\displaystyle\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}|z|^n}{1 - \displaystyle\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}|z|^n}.$$

Considering $z \to 1^-$ along to the real axis, we get:

$$(1+k)\frac{\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}}{1-\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}} \leq 1-\gamma. \tag{6}$$

The inequality (6) is equivalent to

$$\sum_{n=2}^{\infty}\frac{(n-1)\Gamma(\beta)+\frac{1-\gamma}{1+k}\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1-\gamma}{1+k},$$

and finally we obtain

$$\sum_{n=2}^{\infty}\frac{(n-1)(k+1)+1-\gamma}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1-\gamma}{\Gamma(\beta)},$$

which is the (5) condition. □

Theorem 2.2. *Let* $\alpha, \beta > 0$*,* $k \geq 0$ *and* $0 < \gamma \leq 1$*. If*

$$\sum_{n=2}^{\infty} n\frac{(n-1)(k+1)+1-\gamma}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1-\gamma}{\Gamma(\beta)}, \tag{7}$$

then $\mathcal{E}_{\alpha,\beta} \in k-UCV(\gamma)$*.*

Proof. It is sufficient to show that

$$k\left|\frac{z\mathcal{E}''_{\alpha,\beta}(z)}{\mathcal{E}'_{\alpha,\beta}(z)}\right| - \Re\left(\frac{z\mathcal{E}''_{\alpha,\beta}(z)}{\mathcal{E}'_{\alpha,\beta}(z)}\right) \leq 1-\gamma.$$

Now we have

$$k\left|\frac{z\mathcal{E}''_{\alpha,\beta}(z)}{\mathcal{E}'_{\alpha,\beta}(z)}\right| - \Re\left(\frac{z\mathcal{E}''_{\alpha,\beta}(z)}{\mathcal{E}'_{\alpha,\beta}(z)}\right) \leq (k+1)\left|\frac{z\mathcal{E}''_{\alpha,\beta}(z)}{\mathcal{E}'_{\alpha,\beta}(z)}\right| \leq 1-\gamma.$$

The above inequality is equivalent to

$$\left|\frac{z\mathcal{E}''_{\alpha,\beta}(z)}{\mathcal{E}'_{\alpha,\beta}(z)}\right| \leq \frac{1-\gamma}{1+k}.$$

Considering the first and the second order derivative of the generalized Mittag-Leffler funcion we obtain

$$\frac{\sum_{n=2}^{\infty}\frac{n(n-1)\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}|z|^n}{1-\sum_{n=2}^{\infty}\frac{n\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}|z|^n} \leq \frac{1-\gamma}{1+k}.$$

Letting $z \to 1$ along the real axis, we obtain

$$\sum_{n=2}^{\infty} n\Gamma(\beta)\frac{n-1+\frac{1-\gamma}{1+k}}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1-\gamma}{1+k}. \tag{8}$$

The inequality (8) is equivalent to

$$\sum_{n=2}^{\infty} n \frac{(n-1)(k+1)+1-\gamma}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1-\gamma}{\Gamma(\beta)},$$

which is the (9) condition. □

For $k=1$ and $\gamma=0$ we obtain the following characterization properties for the classes S_p and UCV.

Corollary 2.1. *Let* $\alpha, \beta > 0$. *If*

$$\sum_{n=2}^{\infty} \frac{2n-1}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1}{\Gamma(\beta)},$$

then $\mathcal{E}_{\alpha,\beta} \in S_p$.

Corollary 2.2. *Let* $\alpha, \beta > 0$. *If*

$$\sum_{n=2}^{\infty} \frac{n(2n-1)}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1}{\Gamma(\beta)}, \tag{9}$$

then $\mathcal{E}_{\alpha,\beta} \in UCV$.

Theorem 2.3. *Let* $\alpha, \beta > 0$, $k \geq 0$ *and* $0 < \gamma \leq 1$. *If*

$$\sum_{n=2}^{\infty} \frac{n-1}{\Gamma[\alpha(n-1)+\beta]} \leq \frac{1-\gamma}{\Gamma(\beta)[1-\lambda\gamma+k(1-\lambda)]}, \tag{10}$$

where $0 \leq \lambda < 1$ *then* $\mathcal{E}_{\alpha,\beta} \in k - S_p(\lambda, \gamma)$.

Proof. In view of Theorem 2.1 demonstration, we need to prove

$$(1+k)\left|\frac{z\mathcal{E}'_{\alpha,\beta}(z)}{(1-\lambda)\mathcal{E}_{\alpha,\beta}(z)+\lambda z\mathcal{E}'_{\alpha,\beta}(z)} - 1\right| \leq 1-\gamma. \tag{11}$$

The inequality (11) is equivalent to

$$(1+k)(1-\lambda)\left|\frac{z\mathcal{E}'_{\alpha,\beta}(z)-\mathcal{E}_{\alpha,\beta}(z)}{(1-\lambda)\mathcal{E}_{\alpha,\beta}(z)+\lambda z\mathcal{E}'_{\alpha,\beta}(z)}\right| \leq 1-\gamma.$$

From the above inequality we get

$$(1+k)(1-\lambda)\left|\frac{z\mathcal{E}'_{\alpha,\beta}(z)-\mathcal{E}_{\alpha,\beta}(z)}{(1-\lambda)\mathcal{E}_{\alpha,\beta}(z)+\lambda z\mathcal{E}'_{\alpha,\beta}(z)}\right| \leq$$

$$\leq (1+k)(1+\lambda)\frac{\displaystyle\sum_{n=2}^{\infty} \frac{(n-1)\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}|z|^n}{1-\displaystyle\sum_{n=2}^{\infty} \frac{(\lambda n-\lambda+1)\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}|z|^n} \leq 1-\gamma.$$

Considering $z \to 1^-$ along to the real axis, we obtain:

$$\frac{(1+k)(1-\lambda)}{1-\gamma}\sum_{n=2}^{\infty} \frac{(n-1)\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]} \leq 1 - \sum_{n=2}^{\infty} \frac{[\lambda(n-1)+1]\Gamma(\beta)}{\Gamma[\alpha(n-1)+\beta]}. \tag{12}$$

From (12) we have

$$\Gamma(\beta)\sum_{n=2}^{\infty}\frac{(n-1)[1-\lambda\gamma+k(1-\lambda)]}{(1-\gamma)\Gamma[\alpha(n-1)+\beta]}\leq 1,$$

which is equivalent to

$$\sum_{n=2}^{\infty}\frac{n-1}{\Gamma[\alpha(n-1)+\beta]}\leq\frac{1-\gamma}{\Gamma(\beta)[1-\lambda\gamma+k(1-\lambda)]}. \tag{13}$$

The inequality (13) is the (10) condition and the proof is done. □

Putting $k=\lambda=\gamma=0$ in the above theorem we obtain the analytic criteria for the class S^*.

Corollary 2.3. *Let* $\alpha,\beta>0$. *If*

$$\sum_{n=2}^{\infty}\frac{n-1}{\Gamma[\alpha(n-1)+\beta]}\leq\frac{1}{\Gamma(\beta)}, \tag{14}$$

then $\mathcal{E}_{\alpha,\beta}\in S^*$.

We give a similary theorem for the class $k-UCV(\lambda,\gamma)$, without proof.

Theorem 2.4. *Let* $\alpha,\beta>0$, $k\geq 0$ *and* $0<\gamma\leq 1$. *If*

$$\sum_{n=2}^{\infty}n\frac{(n-1)[1-\lambda\gamma+k(1-\lambda)]+1-\gamma}{\Gamma[\alpha(n-1)+\beta]}\leq\frac{1-\gamma}{\Gamma(\beta)}, \tag{15}$$

where $0\leq\lambda<1$, *then* $\mathcal{E}_{\alpha,\beta}\in k-UCV(\lambda,\gamma)$.

Putting $k=\lambda=\gamma=0$ in the Theorem 2.4 we obtain the anlytic criteria for the class $\mathcal{K}$.

Corollary 2.4. *Let* $\alpha,\beta>0$. *If*

$$\sum_{n=2}^{\infty}\frac{n^2}{\Gamma[\alpha(n-1)+\beta]}\leq\frac{1}{\Gamma(\beta)},$$

then $\mathcal{E}_{\alpha,\beta}\in\mathcal{K}$.

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