ON A COUPLED SYSTEM OF VOLTERRA-STIELTJES INTEGRAL EQUATIONS

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ABSTRACT. Volterra-Stieltjes integral equations have been studied in the space of continuous functions in many papers for example, (see [2]-[8]). Our aim here is to study the existence of at least one solution for a coupled system of nonlinear integral equations of Volterra-Stieltejs type in the space of continuous functions defined on a closed bounded interval. The main tool utilized in our considerations is the technique associated with certain Schauder fixed point theorem.

1. INTRODUCTION AND PRELIMINARIES

Let $I = [0, T]$ be a fixed interval. Denote by $C(I) = C[0, T]$ the class of all continuous functions defined on $I$ with the standard norm

$$\| x \|_I = \sup_{t \in I} |x(t)|.$$ 

Consider the nonlinear Riemann-Stieltjes integral equation

$$x(t) = p(t) + \int_0^t f(s, x(s)) \, ds, \quad t \in I$$

where $g : I \times I \to R$ and the symbol $ds$ indicates the integration with respect to $s$. Equations of type (1) and some of their generalizations were considered in several papers by J. Banaš (see [4]). The properties of the Volterra-Stieltjes integral operator were studies also by J. Banaš in [2]-[6]. 

Further facts concerning Stieltjes integrals and their properties (see Banaš [1]). The solvability of the coupled systems of integral equations in $C[0, T]$ was proved (see [12]-[14]).

In this paper, we generalize this result for the coupled system of Volterra-Stieltjes
integral equations
\[ x(t) = p_1(t) + \lambda_1 \int_0^t f_1(s, x(s), y(s)) \, ds \quad g_1(t, s), \quad t \in I \]
\[ y(t) = p_2(t) + \lambda_2 \int_0^t f_2(s, x(s), y(s)) \, ds \quad g_2(t, s), \quad t \in I \]

in the Banach space \( C(I) \), we study the existence of at least one solution for the coupled system (2).

2. Existence of solutions

In this section we study the existence of continuous solutions \( x, y \in C(I) \) for the coupled system of nonlinear integral equations of Volterra-Stieltjes type (2).

Now we formulate assumptions under which coupled system (2) will be considered. Namely, we shall assume that:

(i) \( p_i \in C(I), \lambda_i \in \mathbb{R}, i = 1, 2 \).
(ii) \( f_i : I \times \mathbb{R}^2 \to \mathbb{R}, (i = 1, 2) \) is continuous on \( I \), \( \forall x, y \in \mathbb{R}^2, t \in I \) such that there exist continuous functions \( k_i : I \to I \) and two positive constants \( b_i \) such that:
\[
| f_i(t, x, y)| \leq k_i(t) + b_i(\max\{|x|, |y|\})
\]

for \( t \in I \) and \( x, y \in \mathbb{R} \).
(iii) \( g_i : I \times I \to \mathbb{R}, i = 1, 2 \) and for all \( t_1, t_2 \in I \) with \( t_1 < t_2 \), the functions \( s \to g_i(t_2, s) - g_i(t_1, s) \) is nondecreasing on \( I \).
(iv) \( g_i(0, s) = 0 \) for any \( s \in I, i = 1, 2 \).
(v) The functions \( t \to g_i(t, t) \) and \( t \to g_i(t, 0) \) are continuous on \( I, i = 1, 2 \).

Put
\[
\mu = \sup_{t \in I} |g_i(t, t)| + \sup_{t \in I} |g_i(t, 0)| \quad \text{on} \quad I.
\]

Now, let \( X \) be the Banach space of all ordered pairs \((x, y), x, y \in C(I)\) with the norm
\[
\| (x, y) \|_X = \max\{\|x\|_{C(I)}, \|y\|_{C(I)}\}
\]
where
\[
\|x\| = \sup_{t \in I} |x(t)|, \quad \|y\| = \sup_{t \in I} |y(t)|.
\]

It is clear that \((X, \|(x, y)\|_X)\) is a Banach space.

**Theorem 1.** Let the assumptions (i)-(v) be satisfied, then the coupled system (2) has at least one solution in \( X \).

**Proof:** Define the operator \( T \) by putting
\[
T(x, y)(t) = (T_1x(t), T_2y(t))
\]
where
\[
T_1x(t) = p_1(t) + \lambda_1 \int_0^t f_1(s, u(s)) \, ds, \\
T_2y(t) = p_2(t) + \lambda_2 \int_0^t f_2(s, u(s)) \, ds
\]
We denoted operators in the space of continuous functions. We will prove a few results concerning the continuity and compactness of these operators in the space of continuous functions. We immediately that, for every $u \in X$, $t \in I$, $f_i(., u(\cdot))$ ($i = 1, 2$) is continuous on $I$. Observe that Assumptions (iii) and (iv) imply that the function $s \rightarrow g(t, s)$ is nondecreasing on the interval $I$, for any fixed $t \in I$. Indeed, putting $t_2 = t$, $t_1 = 0$ in (iii) and keeping in mind (iv), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \rightarrow g(t, s)$ is of bounded variation on $I$. Hence it follows that, $f_i(t, x(t), y(t))$ are Riemann-Stieltjes integrable on $I$ with respect to $s \rightarrow g_i(t, s)$. Thus $T_1$ make sense.

We will prove a few results concerning the continuity and compactness of these operators in the space of continuous functions. We denoted $K := \max\{k_i(t) : t \in I, \ i = 1, 2\}$, and we define the set $U$ by

$$U := \{u = (x, y) \mid (x, y) \in R^2 : \|x, y\| \leq r, r = \left\|p_0\right\| + \lambda K \mu \over 1 - \lambda b_i \mu\}$$

The remainder of the proof will be given in four steps.

**Step 1**: The operator $T$ transforms $X$ into $X$.

For $u = (x, y) \in U$, for all $\epsilon > 0$, $\delta > 0$ and for each $t_1, t_2 \in I$, $t_1 < t_2$ such that $|t_2 - t_1| < \delta$, we have

$$\left|T_1 x(t_2) - T_1 x(t_1)\right| \leq |p_1(t_2) - p_1(t_1)| + \lambda_1 \int_0^{t_1} f_1(s, x(s), y(s)) \left|d_s g_1(t_2, s) - d_s g_1(t_1, s)\right|$$

$$\leq |p_1(t_2) - p_1(t_1)| + \lambda_1 \left|\int_0^{t_2} f_1(s, x(s), y(s)) \left|d_s g_1(t_2, s) - d_s g_1(t_1, s)\right|\right|$$

$$\leq |p_1(t_2) - p_1(t_1)| + \lambda_1 \int_1^{t_2} f_1(s, x(s), y(s)) \left|d_s g_1(t_2, s) - g_1(t_1, s)\right| ds \left(\sup_{z=0} g_1(t_2, z)\right)$$

$$+ \lambda_1 \left|\int_0^{t_1} f_1(s, x(s), y(s)) \left|d_s (\sup_{z=0} g_1(t_2, z) - g_1(t_1, z))\right|\right|$$

$$\leq |p_1(t_2) - p_1(t_1)| + \lambda \int_0^{t_2} (k_1(s) + b_1(\max\{|x(s)|, |y(s)|\})) \left|d_s (\sup_{z=0} g_1(t_2, z) - g_1(t_1, z))\right|$$

$$+ \lambda \int_0^{t_1} (k_1(s) + b_1(\max\{|x(s)|, |y(s)|\})) \left|d_s (\sup_{z=0} g_1(t_2, z) - g_1(t_1, z))\right|$$
As done above we can obtain maps $C_x, y$

Step 2: Also, note that the class of $T$

Therefore, by our assumption (v), we see that $T$

Hence, from the continuity of the functions $g_1$ assumption (v), we deduce that $T_1$ maps $C(I)$ into $C(I)$.

As done above we can obtain

Also, by our assumption (v), we see that $T_2$ maps $C(I)$ into $C(I)$.

Now, from the definition of the operator $T$ we get

Therefore, $T$ maps $X$ into $X$.

Also, note that the class of $\{Tu(t)\}$ is equi-continuous on $I$.

**Step 2:** The operator $T$ map $U$ into $U$.

for $(x, y) \in U$, we have

$$
|T_1 x(t)| \leq |p_1(t)| + |\lambda_1 \int_0^t f_1(s, x(s), y(s)) \, ds | g_1(t, s) | \\
$$

$$
\leq |p_1(t)| + |\lambda_1| \int_0^t \left| f_1(s, x(s), y(s)) \right| \, ds \left( \max_{z=0}^{s} g_1(t, z) \right) \\
\leq \|p_1\| + \lambda \int_0^t \left( k_1(s) + b_1 \left( \max_{s=0}^{t} |x(s)|, |y(s)| \right) \right) \, ds \left( \max_{z=0}^{s} g_1(t, z) \right)
$$
where

Firstly, we prove that

Thus for every

Therefore,

By a similar way can deduce that

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Therefore,

By a similar way can deduce that

Therefore,

Thus for every \( u = (x, y) \in U \), we have \( Tu \in U \) and hence \( TU \subset U \), (i.e. \( T : U \rightarrow U \)). This means that the functions of \( TU \) are uniformly bounded on \( I \).

**Step 3:** The operator \( T \) is compact.

The compactness of the operator \( T \) is a consequence of the estimates of the quantities \(|T_1x(t_2) - T_1x(t_1)|, |T_2y(t_2) - T_2y(t_1)|\) conducted in Step 1, assumption (v) and the Arzelà-Ascoli theorem.

**Step 4:** The operator \( T \) is continuous.

Firstly, we prove that \( T_1 \) is continuous. Let \( \epsilon^* > 0 \), the continuity of \( f_i \) yields \( \exists \delta = \delta(\epsilon^*) \) such that \( |f_i(t, x, y) - f_i(t, u, v)| < \epsilon^* \) whenever \( \|x - u\| \leq \delta \), thus if \( \|x - u\| \leq \delta \), we arrive at:

\[
|T_1x(t) - T_1u(t)| \leq \lambda_1 \int_0^t |f_1(s, x(s), y(s)) dx_1(t, s) - \lambda_1 \int_0^t f_1(s, u(s), y(s)) ds dx_1(t, s) |
\]

\[
\leq |\lambda_1| \int_0^t |f_1(s, x(s), y(s)) - f_1(s, u(s), y(s))| ds(\sqrt{g_1(t, z)})
\]

\[
\leq \epsilon^* \lambda \int_0^t ds(\sqrt{g_1(t, z)})
\]

\[
\leq \epsilon^* \lambda \int_0^t dx_1(t, s)
\]

\[
\leq \epsilon^* \lambda [g_1(t, t) - g_1(t, 0)]
\]

\[
\leq \epsilon^* \lambda \|g_1(t, t)\| + \|g_1(t, 0)\|
\]

\[
\leq \epsilon^* \lambda \sup_{t \in I} |g_1(t, t)| + \|g_1(t, 0)\| \leq \epsilon
\]

where \( \epsilon := \epsilon^* \lambda \mu \).

Therefore,

\[
|T_1x(t) - T_1u(t)| \leq \epsilon.
\]
This means that the operator \( T_1 \) is continuous.
By a similar way as done above we can prove that for any \( y, v \in C[0, T] \) and 
\[ \| y - v \| < \delta, \]
we have
\[ | T_2 y(t) - T_2 v(t) | \leq \epsilon. \]
Hence \( T_2 \) is continuous operator.
The operators \( T_i (i = 1, 2) \) is continuous operator it imply that \( T \) is continuous operator.
Since all conditions of Schauder fixed point theorem are satisfied, then \( T \) has at least one fixed point \( u = (x, y) \in U \), which completes the proof.

In what follows, we provide an example illustrating the above obtained results.

**Example :** Consider the functions \( g_i : I \times I \rightarrow R \) defined by the formula
\[
g_1(t, s) = \begin{cases} 
  t \ln \frac{t}{s}, & \text{for } t \in (0, 1], \ s \in I, \\
  0, & \text{for } t = 0, \ s \in I.
\end{cases}
\]
\[
g_2(t, s) = t(t + s - 1), \ t \in I.
\]
It can be easily seen that the functions \( g_1(t, s) \) and \( g_2(t, s) \) satisfies assumptions (iii)-(v) given in Theorem 1, and \( g_1(t, s) \) is function of bounded variation but it is not continuous on \( I \). In this case, the coupled system of Volterra-Stieltjes integral equations (2) has the form
\[
x(t) = p_1(t) + \lambda_1 \int_0^t \frac{t}{s} f_1(s, x(s), y(s)) \ ds, \ t \in I \tag{3}
\]
\[
y(t) = p_2(t) + \lambda_2 \int_0^t f_2(s, x(s), y(s)) \ ds, \ t \in I.
\]
Also, consider the functions \( f_i : I \times R^2 \rightarrow R \) defined by the formula
\[
f_1(t, x, y) = t + x + y,
\]
\[
f_2(t, x, y) = t + x^2 - y^2.
\]
Now, it can be easily seen that the functions \( f_1 \) and \( f_2 \) satisfies assumptions (ii) given in Theorem 1:
\[
| f_1(t, x, y) | \leq | t + x + y | \\
\leq | t | + | x | + | y | \\
\leq T + 2 \max \{| x |, | y |\}
\]
And
\[
| f_2(t, x, y) | \leq | t + x^2 - y^2 | \\
\leq | t | + | x^2 - y^2 | \\
\leq T + | (x - y)(x + y) | \\
\leq T + 2 \max \{| x |, | y |\}
\]
Hence, $k_i(t) = T$, and $b_i = 2$.
Therefore, the functions $f_i$ satisfies the assumption
\[ |f_i(t, x, y)| \leq k_i(t) + b_i(\max\{|x|, |y|\}). \]
Therefore, the coupled system (3) has at least one solution $x, y \in C[0, T]$.

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References

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