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NUMERICAL TREATMENTS OF THE TRANSMISSION DYNAMICS OF WEST NILE VIRUS AND IT'S OPTIMAL CONTROL

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Abstract. In this paper, numerical studies for transmission dynamics of West Nile Virus mathematical model are presented. The nonstandard finite difference method is introduced to solve the posed model. Positivity, boundedness, and convergence of the nonstandard finite difference scheme are studied. Also, numerical stability analysis of fixed points is studied. An optimal control problem is formulated and studied theoretically using the Pontryagin's maximum principle. The obtained results by using nonstandard finite difference method are compared with standard finite difference method. It can be concluded that the nonstandard finite difference method is more efficient and preserves the stability and positivity of the solutions in large regions.

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1. Introduction

West Nile virus (WNV) is characterized as an arboviral encephalitis, a designation that refers to its mosquito (arthropod) vector, its viral pathogenic agent, and its encephalitic symptoms. The disease amplifies in a transmission cycle between vector mosquitoes and reservoir-host birds and is secondarily transmitted to mammals including humans ([5], [7]). WNV was first identified in Uganda in 1937 [38], and is widespread in Africa, Europe, the Middle East, west and central Asia, Oceania (subtype Kunjin), and North America, for more details see ([8], [36]). Nearly all human infections with WNV have resulted from mosquito bites; however, several novel modalities of transmission were recognized in 2002, for example, a pregnant woman was infected with WNV while in her second trimester, which was followed by transplacental transmission to the fetus [10].

The mathematical modeling of transmission dynamics of WNV has been developed in many publications recently. Thomas and Urena introduced a difference equation model for WNV targeting its effects on New York City and determined the amount of sparying (killing the mosquitoes) needed to eliminate the virus [40]. In 2004, there was a study by Wonham et al. on a single season model with a system of

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differential equations for WNV transmission in the mosquito-bird population [42]. But in 2005, Cruz-Pacheco et al. presented and analyzed a mathematical model for the transmission of WNV infection between mosquito and avain populations and by using experimental and field data as well as numerical simulations, they found the phenomena of damped oscillations of the infected bird population [13]. Lewis et al. studied the spatial spread of the virus in [26], but in [27] they introduced a comparative study of the discrete-time model in [40] and the continuous-time model in [43]. Kbenesh et al. determined the cost-effective strategies for combating the spread of WNV in a given population [23].

Numerical simulations, based on finite difference approximations, are widely used to predict the dynamics of the interacting populations. Unfortunately, their stability and accuracy depend strongly on the time step size [16]. Nonstandard finite difference (NSFD) techniques, developed by Mickens [28], to design elementary stable NSFD methods that preserve the local stability of equilibria of the approximated differential system for arbitrary time step sizes. It is important for constructing the positivity preserving schemes to avoid unrealistic negative values for the solution [4], [11], [15], [17], [18], [28], [33]. The explicit schemes are generally less expensive than other classical methods since larger step sizes can be taken without generating negative solutions [15], [28], [29]. NSFD methodology has been applied in many areas of science including biological and epidemic models [19], [28], [30].

In this paper, we introduced the transmission dynamics of WNV model which given in [23]. The aim is to study numerically the optimal control problem for the proposed model by the NSFD method. Many optimal control methods have been developed for studying the dynamics of some diseases such as vector-borne diseases, HIV and Mycobacterium tuberculosis ([1], [24]). This paper is organized as follows, in section 2, a mathematical model is presented. In section 3, NSFD for the WNV model is presented. The positivity and boundedness of the proposed scheme are studied in section 4. Existence and stability of equilibria are presented in section 5. In section 6, the optimal control problem is introduced and NSFD for this optimal control proplem is presented. In section 7, a numerical experiment is discussed. Finally, in section 8, conclusions are presented.

2. Mathematical Model

In this section, we consider the transmission dynamics of WNV model which given in [23]. This model consists of nine nonlinear ordinary differential equations (ODEs). The model is based on monitoring the temporal dynamics of susceptible mosquitoes $M_s(t)$, infected mosquitoes $M_i(t)$, susceptible birds $B_s(t)$, infected birds $B_i(t)$, susceptible humans $S(t)$, exposed humans $E(t)$, infectious humans $I(t)$, hospitalized humans H(t) and recoverd humans $R(t)$. Here, $N_M(t) = M_s(t)+M_i(t)$ is the total mosquito population at time t, $N_B(t) = B_s(t) + B_i(t)$ is the total bird population at time t and $N_H(t) = S(t) + E(t) + I(t) + H(t) + R(t)$ is the total human population at time t, as explained in Table 1. The list of parameters values and their interpretation are introduced in Tables 2 and 3, for more details see [23]. Then the WNV model can be formulated as follows:

$$
\begin{aligned}
\dot{M}_s(t) &= \lambda_M - \frac{b_1\beta_1 M_s B_i}{N_B} - \mu_M M_s,\\
\dot{M}_i(t) &= \frac{b_1\beta_1 M_s B_i}{N_B} - \mu_M M_i,\\
\dot{B}_s(t) &= \lambda_B - \frac{b_1\beta_2 M_i B_s}{N_B} - \Psi_B B_s - \mu_B B_s,\\
\dot{B}_i(t) &= \frac{b_1\beta_2 M_i B_s}{N_B} - d_B B_i - \Psi_B B_i - \mu_B B_i,\\
\dot{S}(t) &= \lambda_H - \frac{b_2\beta_3 M_i S}{N_H} - \mu_H S,\\
\dot{E}(t) &= \frac{b_2\beta_3 M_i S}{N_H} - \alpha E - \mu_H E,\\
\dot{I}(t) &= \alpha E - \gamma I - d_I I - rI - \mu_H I,\\
\dot{H}(t) &= \gamma I - d_H H - \tau H - \mu_H H,\\
\dot{R}(t) &= \tau H + rI - \mu_H R, \qquad (1)
\end{aligned}
$$

with the following initial conditions:

$$
\begin{aligned}
&M_s(0) = M_{s_0}, \quad M_i(0) = M_{i_0}, \quad B_s(0) = B_{s_0}, \quad B_i(0) = B_{i_0}, \quad S(0) = S_0,\\
&E(0) = E_0, \quad I(0) = I_0, \quad H(0) = H_0, \quad R(0) = R_0. \qquad (2)
\end{aligned}
$$

Table 1. All variables in the system (1) and their definitions.

Variable	Definition
$M_s(t)$	The population of susceptible mosquitoes.
$M_i(t)$	The population of infected mosquitoes.
$N_M(t)$	The total population of mosquitoes $N_M(t) = M_s(t) + M_i(t)$.
$B_s(t)$	The population of susceptible birds.
$B_i(t)$	The population of infected birds.
$N_B(t)$	The total population of birds $N_B(t) = B_s(t) + B_i(t)$.
$S(t)$	The population of suscepible humans.
$E(t)$	The population of exposed humans.
$I(t)$	The population of infected humans.
$H(t)$	The population of hospitalized humans.
$R(t)$	The population of recovered humans.
$N_H(t)$	The total population of humans $N_H(t) = S(t) + E(t) + I(t) + H(t) + R(t)$.

2.1. **The Basic Reproduction Number R_0.** The basic reproduction number [41], R_0, is presented for a general compartmental disease transmission model based on a system of ODEs. These models have a disease-free equilibrium (DFE) at which the population remains in the absence of disease. Thus, R_0 is a threshold parameter for the model. It is the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual. The DFE is locally asymptotically stable if $R_0 < 1$, the average of an infected individual produces less than one new infected individual over the course of his infectious period and the infection cannot grow. But the DFE is unstable and invasion is always possible if $R_0 > 1$, the average of each infected individual produces, more than one

TABLE 2. All parameters in the system (1) and their interpretation.

Parameter	Interpretation
λ_M	The recruitment rate of mosquitoes (assumed susceptible).
μ_M	The natural death rate of mosquitoes.
λ_B	The recruitment rate of birds (assumed susceptible).
μ_B	The natural death rate of birds.
Ψ_B	The migration rate of birds.
d_B	The WNV-induced death rate of birds.
λ_H	The recruitment rate of humans (assumed susceptible).
μ_H	The natural death rate of humans.
d_I	The WNV-induced death rate of humans.
d_H	The death rate of hospitalized humans.
β_1	The probability of WNV transmissionfrom from an infected bird to a susceptible mosquito.
β_2	The probability of WNV transmissionfrom from an infected mosquitoes to a susceptible bird.
β_3	The probability of WNV transmissionfrom from an mosquitos to humans.
b_1	The per capita bitting rate of mosquitoes on the primary host (birds) and $b_1 = \frac{bN_B}{N_B+N_H}$.
b_2	The per capita bitting rate of mosquitoes on the humans and $b_2 = \frac{bN_H}{N_B+N_H}$.
b	The average bitting rate of mosquitoes.
α	The rate of development of clinical symptoms of WNV.
r	The natural recovery rate.
τ	The treatment-induced recovery rate.
γ	The hospitalized rate of infectious humans.

TABLE 3. Parameters values used in the system (1).

Parameter	Value	Parameter	Value
μ_H	$(3.91 \times 10^{-5} - 0.005)$	d_H	$(5 \times 10^{-5} - 0.015)$
μ_B	$(0.0001 - 0.0003)$	d_B	$(0.06 - 0.2)$
μ_M	$(0.016 - 0.07)$	λ_M	51.1×10^{-3}
λ_B	2.1	λ_H	5×10^{-2}
d_I	$d_H + 10^{-5}$	Ψ_B	5.2×10^{-2}
β_1	0.4	β_2	0.1
β_3	10^{-2}	b	3
α	0.1	r	2×10^{-4}
τ	0.05	γ	9×10^{-4}

new infection, and the disease can invade the population, for more details see [23]. In [23], Kbenesh et al. introduced the DFE to be $(\hat{M}_s, \hat{M}_i, \hat{B}_s, \hat{B}_i, \hat{S}, \hat{E}, \hat{I}, \hat{H}, \hat{R})$. Thus, the reproduction number R_0 for the system (1), is given as follows:

$$R_0 = b \ \frac{\sqrt{\mu_M(\mu_B + \Psi_B + d_B)\beta_1\beta_2\hat{M}_s\hat{B}_s}}{\mu_M(\mu_B + \Psi_B + d_B)(\hat{B}_s + \hat{S})}. \tag{3}$$

3. NSFD for WNV Model

In this section, we introduce the NSFD schemes to obtain numerical solutions of the transmission dynamics of WNV model (1). Mickens introduced NSFD schemes in 1980, as a powerful numerical method that preserve some of the main essential physical properties of the solution, such as, monotonicity or convergence towards a stable steady state [4], [18], [28]. The NSFD schemes were defined as follows:

Definition 1 [3] A numerical scheme is called NSFD discretization if at least one of the following conditions is satisfied:

(1) nonlocal approximation is used [3], [28], [31].
(2) the discretization of derivative is not traditional and a nonnegative function $\varphi(\Delta t) = \Delta t + O((\Delta t)^2)$, called a denominator function is used [30].

If $f(t) \in C^1(\mathbb{R})$, the first derivative $\frac{df(t)}{dt}$ can be defined as $\frac{df(t)}{dt} = \frac{f(t+\Delta t)-f(t)}{\varphi(\Delta t)}$, where $\varphi(\Delta t)$ is a real-valued function on $\mathbb{R}$. The above definition focused on the nonlocal approximation strategy for the construction of NSFD schemes (*i.e.*, if there is nonlinear term such as $X(t)Y(t)$ in the differential equation, it can be replaced by $X(t)Y(t+\Delta t)$ or $X(t+\Delta t)Y(t)$, for more details see [30]) and the renormalization of the denominator. The scheme is defined as finite difference or standard finite difference (SFD) method if $\varphi(\Delta t) = \Delta t$, where Δt is the time step size of the scheme, for more details see [39]. Let us denote by M_s^n, M_i^n, B_s^n, B_i^n, S^n, E^n, I^n, H^n and R^n the values of the approximations of $M_s(n\Delta t)$, $M_i(n\Delta t)$, $B_s(n\Delta t)$, $B_i(n\Delta t)$, $S(n\Delta t)$, $E(n\Delta t)$, $I(n\Delta t)$, $H(n\Delta t)$ and $R(n\Delta t)$ respectively, for $n = 0, 1, 2, \ldots$ and Δt is the time step of the scheme. All sequences M_s^n, M_i^n, B_s^n, B_i^n, S^n, E^n, I^n, H^n and R^n should be nonnegative in order to be consistent with the biological nature of the model. The discretization of the system (1) is given as follows:

$$\begin{aligned}
\frac{M_s^{n+1} - M_s^n}{\varphi(\Delta t)} &= \lambda_M - \frac{b_1\beta_1 M_s^n B_i^n}{N_B^n} - \mu_M M_s^n,\\
\frac{M_i^{n+1} - M_i^n}{\varphi(\Delta t)} &= \frac{b_1\beta_1 M_s^n B_i^n}{N_B^n} - \mu_M M_i^n,\\
\frac{B_s^{n+1} - B_s^n}{\varphi(\Delta t)} &= \lambda_B - \frac{b_1\beta_2 M_i^n B_s^n}{N_B^n} - \Psi_B B_s^n - \mu_B B_s^n,\\
\frac{B_i^{n+1} - B_i^n}{\varphi(\Delta t)} &= \frac{b_1\beta_2 M_i^n B_s^n}{N_B^n} - d_B B_i^n - \Psi_B B_i^n - \mu_B B_i^n,\\
\frac{S^{n+1} - S^n}{\varphi(\Delta t)} &= \lambda_H - \frac{b_2\beta_3 M_i^n S^n}{N_H^n} - \mu_H S^n,\\
\frac{E^{n+1} - E^n}{\varphi(\Delta t)} &= \frac{b_2\beta_3 M_i^n S^n}{N_H^n} - \alpha E^n - \mu_H E^n,\\
\frac{I^{n+1} - I^n}{\varphi(\Delta t)} &= \alpha E^n - \gamma I^n - d_I I^n - r I^n - \mu_H I^n,\\
\frac{H^{n+1} - H^n}{\varphi(\Delta t)} &= \gamma I^n - d_H H^n - \tau H^n - \mu_H H^n,\\
\frac{R^{n+1} - R^n}{\varphi(\Delta t)} &= \tau H^n + r I^n - \mu_H R^n.
\end{aligned} \tag{4}$$

The discretizations for N_M, N_B and N_H are given as follows:

$$\begin{aligned}
N_M^n &= M_s^n + M_i^n,\\
N_B^n &= B_s^n + B_i^n,\\
N_H^n &= S^n + E^n + I^n + H^n + R^n.
\end{aligned} \tag{5}$$

The local approximations are used for the nonlinear terms. We use the denominator function of the form $\varphi(\Delta t) = 1 - e^{-\Delta t}$. Then we can obtain:

$$\begin{aligned}
M_s^{n+1} &= M_s^n + \varphi(\Delta t)\left[\lambda_M - \frac{b_1\beta_1 M_s^n B_i^n}{N_B^n} - \mu_M M_s^n\right],\\
M_i^{n+1} &= M_i^n + \varphi(\Delta t)\left[\frac{b_1\beta_1 M_s^n B_i^n}{N_B^n} - \mu_M M_i^n\right],\\
B_s^{n+1} &= B_s^n + \varphi(\Delta t)\left[\lambda_B - \frac{b_1\beta_2 M_i^n B_s^n}{N_B^n} - \Psi_B B_s^n - \mu_B B_s^n\right],\\
B_i^{n+1} &= B_i^n + \varphi(\Delta t)\left[\frac{b_1\beta_2 M_i^n B_s^n}{N_B^n} - d_B B_i^n - \Psi_B B_i^n - \mu_B B_i^n\right],\\
S^{n+1} &= S^n + \varphi(\Delta t)\left[\lambda_H - \frac{b_2\beta_3 M_i^n S^n}{N_H^n} - \mu_H S^n\right],\\
E^{n+1} &= E^n + \varphi(\Delta t)\left[\frac{b_2\beta_3 M_i^n S^n}{N_H^n} - \alpha E^n - \mu_H E^n\right],\\
I^{n+1} &= I^n + \varphi(\Delta t)\left[\alpha E^n - \gamma I^n - d_I I^n - r I^n - \mu_H I^n\right],\\
H^{n+1} &= H^n + \varphi(\Delta t)\left[\gamma I^n - d_H H^n - \tau H^n - \mu_H H^n\right],\\
R^{n+1} &= R^n + \varphi(\Delta t)\left[\tau H^n + r I^n - \mu_H R^n\right].
\end{aligned} \tag{6}$$

4. Positivity and Boundedness of NSFD Scheme

Theorem 1 [Positivity] Assume that in the system (6) if $M_s^0 > 0$, $M_i^0 > 0$, $B_s^0 > 0$, $B_i^0 > 0$, $S^0 > 0$, $E^0 > 0$, $I^0 > 0$, $H^0 > 0$, $R^0 > 0$, $\mu_H > 0$, $d_H > 0$, $\mu_B > 0$, $d_B > 0$, $\mu_M > 0$, $\lambda_M > 0$, $\lambda_B > 0$, $\lambda_H > 0$, $d_I > 0$, $\Psi_B > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$, $b > 0$, $\alpha_w > 0$, $r > 0$, $\tau > 0$, and $\gamma > 0$, then $M_s^n > 0$, $M_i^n > 0$, $B_s^n > 0$, $B_i^n > 0$, $S^n > 0$, $E^n > 0$, $I^n > 0$, $H^n > 0$ and $R^n > 0$ hold for all $n = 0, 1, 2, \cdots$.

Proof. According to (6), let us have for $n = 0$

$$\begin{aligned}
M_s^{1} &= M_s^0 + \varphi(\Delta t)\left[\lambda_M - \frac{b_1\beta_1 M_s^0 B_i^0}{N_B^0} - \mu_M M_s^0\right],\\
M_i^{1} &= M_i^0 + \varphi(\Delta t)\left[\frac{b_1\beta_1 M_s^0 B_i^0}{N_B^0} - \mu_M M_i^0\right],\\
B_s^{1} &= B_s^0 + \varphi(\Delta t)\left[\lambda_B - \frac{b_1\beta_2 M_i^0 B_s^0}{N_B^0} - \Psi_B B_s^0 - \mu_B B_s^0\right],\\
B_i^{1} &= B_i^0 + \varphi(\Delta t)\left[\frac{b_1\beta_2 M_i^0 B_s^0}{N_B^0} - d_B B_i^0 - \Psi_B B_i^0 - \mu_B B_i^0\right],\\
S^{1} &= S^0 + \varphi(\Delta t)\left[\lambda_H - \frac{b_2\beta_3 M_i^0 S^0}{N_H^0} - \mu_H S^0\right],\\
E^{1} &= E^0 + \varphi(\Delta t)\left[\frac{b_2\beta_3 M_i^0 S^0}{N_H^0} - \alpha E^0 - \mu_H E^0\right],\\
I^{1} &= I^0 + \varphi(\Delta t)\left[\alpha E^0 - \gamma I^0 - d_I I^0 - r I^0 - \mu_H I^0\right],\\
H^{1} &= H^0 + \varphi(\Delta t)\left[\gamma I^0 - d_H H^0 - \tau H^0 - \mu_H H^0\right],\\
R^{1} &= R^0 + \varphi(\Delta t)\left[\tau H^0 + r I^0 - \mu_H R^0\right].
\end{aligned} \tag{7}$$

Thus $M_s^1 > 0$, $M_i^1 > 0$, $B_s^1 > 0$, $B_i^1 > 0$, $S^1 > 0$, $E^1 > 0$, $I^1 > 0$, $H^1 > 0$ and $R^1 > 0$, we assume that for $1, 2, \cdots, n$, $M_s^n > 0$, $M_i^n > 0$, $B_s^n > 0$, $B_i^n > 0$, $S^n > 0$, $E^n > 0$, $I^n > 0$, $H^n > 0$ and $R^n > 0$.

Theorem 2 [Boundedness] Let us suppose that, if we have $N_M^0 = M_s^0 + M_i^0$, $N_B^0 = B_s^0 + B_i^0$, $N_H^0 = S^0 + E^0 + I^0 + H^0 + R^0$, and $\mu_H > 0$, $d_H > 0$, $\mu_B > 0$, $d_B > 0$, $\mu_M > 0$, $\lambda_M > 0$, $\lambda_B > 0$, $\lambda_H > 0$, $d_I > 0$, $\Psi_B > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$, $b > 0$, $\alpha_w > 0$, $r > 0$, $\tau > 0$, $\gamma > 0$, then the numerical NSFD scheme given by the system (6), such that, $N_M^n = M_s^n + M_i^n < (1 - \mu_M \varphi(\Delta t))\, N_M^{N_0 - 1}$, $N_B^n = B_s^n + B_i^n < (1 - (\Psi_B + \mu_B)\, \varphi(\Delta t))\, N_B^{N_0-1}$, and $N_H^n = S^n + E^n + I^n + H^n + R^n < (1 - \mu_H \varphi(\Delta t))\, N_H^{N_0 - 1}$ for all $n = 0, 1, 2, \cdots, N_0$.

Proof. Firstly, for the total mosquito population N_M we have for $n = 0$

$$M_s^1 = M_s^0 + \varphi(\Delta t)\left[\lambda_M - \frac{b_1 \beta_1 M_s^0 B_i^0}{N_B^0} - \mu_M M_s^0\right], \tag{8}$$

$$M_i^1 = M_i^0 + \varphi(\Delta t)\left[\frac{b_1 \beta_1 M_s^0 B_i^0}{N_B^0} - \mu_M M_i^0\right], \tag{9}$$

$$N_M^1 = M_s^1 + M_i^1 < (1 - \mu_M \varphi(\Delta t))\, N_M^0. \tag{10}$$

Next for $n = 1$

$$M_s^2 = M_s^1 + \varphi(\Delta t)\left[\lambda_M - \frac{b_1 \beta_1 M_s^1 B_i^1}{N_B^1} - \mu_M M_s^1\right], \tag{11}$$

$$M_i^2 = M_i^1 + \varphi(\Delta t)\left[\frac{b_1 \beta_1 M_s^1 B_i^1}{N_B^1} - \mu_M M_i^1\right], \tag{12}$$

$$N_M^2 = M_s^2 + M_i^2 < (1 - \mu_M \varphi(\Delta t))\, N_M^1. \tag{13}$$

Next for $n = 2$

$$N_M^3 = M_s^3 + M_i^3 < (1 - \mu_M \varphi(\Delta t))\, N_M^2. \tag{14}$$

Now we assume that for $n = 3, \cdots, N_0$, is

$$N_M^{N_0} = M_s^{N_0} + M_i^{N_0} < (1 - \mu_M \varphi(\Delta t))\, N_M^{N_0 - 1}. \tag{15}$$

Secondly, for the total bird population N_B we have for $n = 0$

$$B_s^1 = B_s^0 + \varphi(\Delta t)\left[\lambda_B - \frac{b_1 \beta_2 M_i^0 B_s^0}{N_B^0} - \Psi_B B_s^0 - \mu_B B_s^0\right], \tag{16}$$

$$B_i^1 = B_i^0 + \varphi(\Delta t)\left[\frac{b_1 \beta_2 M_i^0 B_s^0}{N_B^0} - d_B B_i^0 - \Psi_B B_i^0 - \mu_B B_i^0\right], \tag{17}$$

$$N_B^1 = B_s^1 + B_i^1 < (1 - (\Psi_B + \mu_B)\, \varphi(\Delta t))\, N_B^0. \tag{18}$$

Next for $n = 1$ and $n = 2$, we have

$$N_B^2 = B_s^2 + B_i^2 < (1 - (\Psi_B + \mu_B)\, \varphi(\Delta t))\, N_B^1, \tag{19}$$

$$N_B^3 = B_s^3 + B_i^3 < (1 - (\Psi_B + \mu_B)\, \varphi(\Delta t))\, N_B^2. \tag{20}$$

Thus for $n = 3, \cdots, N_0$, is

$$N_B^{N_0} = B_s^{N_0} + B_i^{N_0} < (1 - (\Psi_B + \mu_B)\, \varphi(\Delta t))\, N_B^{N_0 - 1}. \tag{21}$$

Finally, the total human population N_H we have for $n = 0$

$$S^1 = S^0 + \varphi(\Delta t)\left[\lambda_H - \frac{b_2\beta_3 M_i^0 S^0}{N_H^0} - \mu_H S^0\right], \tag{22}$$

$$E^1 = E^0 + \varphi(\Delta t)\left[\frac{b_2\beta_3 M_i^0 S^0}{N_H^0} - \alpha E^0 - \mu_H E^0\right], \tag{23}$$

$$I^1 = I^0 + \varphi(\Delta t)\left[\alpha E^0 - \gamma I^0 - d_I I^0 - rI^0 - \mu_H I^0\right], \tag{24}$$

$$H^1 = H^0 + \varphi(\Delta t)\left[\gamma I^0 - d_H H^0 - \tau H^0 - \mu_H H^0\right], \tag{25}$$

$$R^1 = R^0 + \varphi(\Delta t)\left[\tau H^0 + rI^0 - \mu_H R^0\right], \tag{26}$$

$$N_H^1 = S^1 + E^1 + I^1 + H^1 + R^1 < (1 - \mu_H\varphi(\Delta t))\, N_H^0. \tag{27}$$

Next for $n = 1$ and $n = 2$, we have

$$N_H^2 = S^2 + E^2 + I^2 + H^2 + R^2 < (1 - \mu_H\varphi(\Delta t))\, N_H^1, \tag{28}$$

$$N_H^3 = S^3 + E^3 + I^3 + H^3 + R^3 < (1 - \mu_H\varphi(\Delta t))\, N_H^2. \tag{29}$$

Thus for $n = 3, \cdots, N_0$, is

$$N_H^{N_0} = S^{N_0} + E^{N_0} + I^{N_0} + H^{N_0} + R^{N_0} < (1 - \mu_H\varphi(\Delta t))\, N_H^{N_0-1}. \tag{30}$$

5. Existence and Stability of Equilibria

5.1. **Disease Free Equilibrium.** In this section, the stability and convergence properties of the DFE point of the proposed NSFD scheme will be studied. In [23], Kbenesh et al. established the DFE point and its stability. We determine this DFE point by considering $\mathcal{D}^* = (\hat{M}_s, \hat{M}_i, \hat{B}_s, \hat{B}_i, \hat{S}, \hat{E}, \hat{I}, \hat{H}, \hat{R})$ to be the fixed point of the system (6). The DFE point $\mathcal{D}^*$ can be found by solving the following system:

$$\begin{aligned}
&\mathcal{K}_1 = \hat{M}_s, \quad \mathcal{K}_2 = \hat{M}_i, \quad \mathcal{K}_3 = \hat{B}_s,\\
&\mathcal{K}_4 = \hat{B}_i, \quad \mathcal{K}_5 = \hat{S}, \quad \mathcal{K}_6 = \hat{E},\\
&\mathcal{K}_7 = \hat{I}, \quad \mathcal{K}_8 = \hat{H}, \quad \mathcal{K}_9 = \hat{R},
\end{aligned}$$

where,

$$\mathcal{K}_j = \mathcal{K}_j(\hat{M}_s, \hat{M}_i, \hat{B}_s, \hat{B}_i, \hat{S}, \hat{E}, \hat{I}, \hat{H}, \hat{R}), \quad \forall j = 1, 2, \ldots, 9. \tag{31}$$

Thus, the system (6) can be written as follows:

$$
\begin{aligned}
&\mathcal{K}_1(\hat{M}_s,\hat{M}_i,\hat{B}_s,\hat{B}_i,\hat{S},\hat{E},\hat{I},\hat{H},\hat{R}) = \hat{M}_s + \varphi(\Delta t)\left[\lambda_M - \frac{b_1\beta_1\hat{M}_s\hat{B}_i}{\hat{N}_B} - \mu_M\hat{M}_s\right],\\
&\mathcal{K}_2(\hat{M}_s,\hat{M}_i,\hat{B}_s,\hat{B}_i,\hat{S},\hat{E},\hat{I},\hat{H},\hat{R}) = \hat{M}_i + \varphi(\Delta t)\left[\frac{b_1\beta_1\hat{M}_s\hat{B}_i}{\hat{N}_B} - \mu_M\hat{M}_i\right],\\
&\mathcal{K}_3(\hat{M}_s,\hat{M}_i,\hat{B}_s,\hat{B}_i,\hat{S},\hat{E},\hat{I},\hat{H},\hat{R}) = \hat{B}_s + \varphi(\Delta t)\left[\lambda_B - \frac{b_1\beta_2\hat{M}_i\hat{B}_s}{\hat{N}_B} - \Psi_B\hat{B}_s - \mu_B\hat{B}_s\right],\\
&\mathcal{K}_4(\hat{M}_s,\hat{M}_i,\hat{B}_s,\hat{B}_i,\hat{S},\hat{E},\hat{I},\hat{H},\hat{R}) = \hat{B}_i + \varphi(\Delta t)\left[\frac{b_1\beta_2\hat{M}_i\hat{B}_s}{\hat{N}_B} - d_B\hat{B}_i - \Psi_B\hat{B}_i - \mu_B\hat{B}_i\right],\\
&\mathcal{K}_5(\hat{M}_s,\hat{M}_i,\hat{B}_s,\hat{B}_i,\hat{S},\hat{E},\hat{I},\hat{H},\hat{R}) = \hat{S} + \varphi(\Delta t)\left[\lambda_H - \frac{b_2\beta_3\hat{M}_i\hat{S}}{\hat{N}_H} - \mu_H\hat{S}\right],\\
&\mathcal{K}_6(\hat{M}_s,\hat{M}_i,\hat{B}_s,\hat{B}_i,\hat{S},\hat{E},\hat{I},\hat{H},\hat{R}) = \hat{E} + \varphi(\Delta t)\left[\frac{b_2\beta_3\hat{M}_i\hat{S}}{\hat{N}_H} - \alpha\hat{E} - \mu_H\hat{E}\right],\\
&\mathcal{K}_7(\hat{M}_s,\hat{M}_i,\hat{B}_s,\hat{B}_i,\hat{S},\hat{E},\hat{I},\hat{H},\hat{R}) = \hat{I} + \varphi(\Delta t)\left[\alpha\hat{E} - \gamma\hat{I} - d_I\hat{I} - r\hat{I} - \mu_H\hat{I}\right],\\
&\mathcal{K}_8(\hat{M}_s,\hat{M}_i,\hat{B}_s,\hat{B}_i,\hat{S},\hat{E},\hat{I},\hat{H},\hat{R}) = \hat{H} + \varphi(\Delta t)\left[\gamma\hat{I} - d_H\hat{H} - \tau\hat{H} - \mu_H\hat{H}\right],\\
&\mathcal{K}_9(\hat{M}_s,\hat{M}_i,\hat{B}_s,\hat{B}_i,\hat{S},\hat{E},\hat{I},\hat{H},\hat{R}) = \hat{R} + \varphi(\Delta t)\left[\tau\hat{H} + r\hat{I} - \mu_H\hat{R}\right], \qquad (32)
\end{aligned}
$$

where,

$$\hat{N}_M = \hat{M}_s + \hat{M}_i, \quad \hat{N}_B = \hat{B}_s + \hat{B}_i, \quad \hat{N}_H = \hat{S} + \hat{E} + \hat{I} + \hat{H} + \hat{R}. \qquad (33)$$

If we put $\hat{M}_i = 0$, $\hat{B}_i = 0$ and $\hat{I} = 0$ in the above system (32), then the DFE point $\mathcal{D}^*$ of the system (6) is given by $\mathcal{D}^* = (\frac{\lambda_M}{\mu_M}, 0, \frac{\lambda_B}{\Psi_B+\mu_B}, 0, \frac{\lambda_H}{\mu_H}, 0, 0, 0, 0)$. Let us consider the initial conditions for the WNV model (1) as follows ($M_s(0)$, $M_i(0)$, $B_s(0)$, $B_i(0)$, $S(0)$, $E(0)$, $I(0)$, $H(0)$, $R(0)$) = (10000, 1000, 1000, 0, 1000, 0, 0, 0, 0). For determining the stability properties of the system (1), we calculate the Jacobian matrix at the DFE point $\mathcal{D}^* = (\frac{\lambda_M}{\mu_M}, 0, \frac{\lambda_B}{\Psi_B+\mu_B}, 0, \frac{\lambda_H}{\mu_H}, 0, 0, 0, 0)$. It will take the following form:

$$
J = \begin{pmatrix}
a_{11} & 0 & 0 & a_{14} & 0 & 0 & 0 & 0 & 0\\
0 & a_{22} & 0 & a_{24} & 0 & 0 & 0 & 0 & 0\\
0 & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_{42} & 0 & a_{44} & 0 & 0 & 0 & 0 & 0\\
0 & a_{52} & 0 & 0 & a_{55} & 0 & 0 & 0 & 0\\
0 & a_{62} & 0 & 0 & 0 & a_{66} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a_{67} & a_{77} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & a_{78} & a_{88} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & a_{79} & a_{89} & a_{99}
\end{pmatrix},
$$

where:

$$
\begin{aligned}
a_{11} &= 1 - \mu_M \varphi(\Delta t), & a_{14} &= -\frac{b_1\beta_1\lambda_M(\mu_B + \Psi_B)}{\mu_M\lambda_B}\varphi(\Delta t),\\
a_{22} &= 1 - \mu_M \varphi(\Delta t), & a_{24} &= \frac{b_1\beta_1\lambda_M(\mu_B + \Psi_B)}{\mu_M\lambda_B}\varphi(\Delta t),\\
a_{32} &= -b_1\beta_2\varphi(\Delta t), & a_{33} &= 1 - (\Psi_B + \mu_B)\varphi(\Delta t),\\
a_{42} &= b_1\beta_2\varphi(\Delta t), & a_{44} &= 1 - (d_B + \Psi_B + \mu_B)\varphi(\Delta t),\\
a_{52} &= -b_2\beta_3\varphi(\Delta t), & a_{55} &= 1 - \mu_H\varphi(\Delta t),\\
a_{62} &= b_2\beta_3\varphi(\Delta t), & a_{66} &= 1 - (\alpha + \mu_H)\varphi(\Delta t),\\
a_{67} &= \alpha\varphi(\Delta t), & a_{77} &= 1 - (\gamma + d_I + r + \mu_H)\varphi(\Delta t),\\
a_{78} &= \gamma\varphi(\Delta t), & a_{88} &= 1 - (d_H + \tau + \mu_H)\varphi(\Delta t),\\
a_{79} &= r\varphi(\Delta t), & a_{89} &= \tau\varphi(\Delta t),\\
a_{99} &= 1 - \mu_H\varphi(\Delta t).
\end{aligned}
$$

Now, we determine the stability of the fixed points of the system (6) numerically by reporting the spectral radii ρ of the Jacobian matrix J corresponding to the DFE point of NSFD scheme when $R_0 < 1$. It can be seen that all the spectral radii ρ in Table 4. are less than one in magnitude irrespective of the time step size Δt used in simulations. Hence, we have the DFE point $\mathcal{D}^* = (\frac{\lambda_M}{\mu_M}, 0, \frac{\lambda_B}{\Psi_B+\mu_B}, 0, \frac{\lambda_H}{\mu_H}, 0, 0, 0, 0)$ of the system (1) is unconditionally locally asymptotically stable if $R_0 < 1$.

Table 4. The spectral radii of the Jacobian matrix corresponding to the DFE point of NSFD scheme when $R_0 < 1$.

Δt	ρ (NSFD)
0.05	0.9998(*convergent*)
0.1	0.9995(*convergent*)
0.5	0.9980(*convergent*)
1	0.9968(*convergent*)
10	0.9950(*convergent*)
25	0.9950(*convergent*)

5.2. **Endemic Equilibria.** In this section, we present a study for the existence and uniqueness, stability and convergence properties of the endemic equilibrium for the WNV model (1) (with b_1, b_2 constants). In the following we compute the endemic equilibria. Consider the first four equations in system (1):

$$\dot{M}_s(t) = \lambda_M - \frac{b_1\beta_1 M_s B_i}{N_B} - \mu_M M_s, \tag{34}$$

$$\dot{M}_i(t) = \frac{b_1\beta_1 M_s B_i}{N_B} - \mu_M M_i, \tag{35}$$

$$\dot{B}_s(t) = \lambda_B - \frac{b_1\beta_2 M_i B_s}{N_B} - \Psi_B B_s - \mu_B B_s, \tag{36}$$

$$\dot{B}_i(t) = \frac{b_1\beta_2 M_i B_s}{N_B} - d_B B_i - \Psi_B B_i - \mu_B B_i. \tag{37}$$

Since we have $N_M = M_s + M_i$ and $N_B = B_s + B_i$, given by:

$$\dot{N_M}(t) = \dot{M_s}(t) + \dot{M_i}(t) = \lambda_M - \mu_M N_M, \tag{38}$$

$$\dot{N_B}(t) = \dot{B_s}(t) + \dot{B_i}(t) = \lambda_B - d_B B_i - (\mu_B + \Psi_B) N_B, \tag{39}$$

Adding equations (34) and (35) gives, at steady state,

$$M_s = \frac{\lambda_M}{\mu_M} - M_i. \tag{40}$$

Similarly, adding equations (36) and (37)

$$B_s = \frac{\lambda_B}{\mu_B + \Psi_B} - \left(1 + \frac{d_B}{\mu_B + \Psi_B}\right) B_i. \tag{41}$$

From (40) and (41), we have $M_s \geq 0$ and $B_s \geq 0$ if $M_i \leq \frac{\lambda_M}{\mu_M}$ and $B_i \leq \frac{\lambda_B}{\mu_B + \Psi_B + d_B} = \tilde{B}_{i2}$, respectively. Thus all state variables in equations (34)-(37) are non-negative if $(M_i, B_i) \in \left[0, \frac{\lambda_M}{\mu_M}\right] \times \left[0, \tilde{B}_{i2}\right]$. At steady state, equation (38) can be written as:

$$N_B = \frac{\lambda_B}{\mu_B + \Psi_B} - \frac{d_B}{\mu_B + \Psi_B} B_i, \tag{42}$$

substituting from (40) and (42) to (35), at steady state, we get

$$M_i = \frac{b_1 \beta_1 \frac{\lambda_M}{\mu_M} B_i}{\mu_M \frac{\lambda_B}{\mu_B + \Psi_B} + \left(b_1 \beta_1 - \mu_M \frac{d_B}{\mu_B + \Psi_B}\right) B_i} = \Theta_1(B_i). \tag{43}$$

It is clear that $\Theta_1(0) = 0$ and $m\Theta_1 = \frac{b_1 \beta_1 \lambda_M (\mu_B + \Psi_B)}{\lambda_B \mu_M^2}$ is the slope of Θ_1 at (M_i, B_i). If $b_1 \beta_1 (\mu_B + \Psi_B) \neq \mu_M d_B$, then Θ_1 has a vertical asymptote given by $B_i = \frac{\mu_M \lambda_B}{\mu_M d_B - b_1 \beta_1 (\mu_B + \Psi_B)} = \tilde{B}_{i1}$. substituting from (41) and (42) to (37), at steady state, we get

$$M_i = \frac{(\lambda_B - d_B B_i) B_i}{b_1 \beta_2 \left(\frac{\lambda_B}{\mu_B + \Psi_B + d_B} - B_i\right)} = \Theta_2(B_i), \tag{44}$$

where Θ_2 has a vertical asymptote at $B_i = \frac{\lambda_B}{\mu_B + \Psi_B + d_B} = \tilde{B}_{i2}$. We need to verify that $M_i \leq \frac{\lambda_M}{\mu_M}$, since we have $B_i = \tilde{B}_{i2} \leq \frac{\lambda_B}{\mu_B + \Psi_B + d_B} \leq \frac{\lambda_B}{d_B}$. Therefore, from equation (43) we have

$$M_i = \frac{b_1 \beta_1 \frac{\lambda_M}{\mu_M} B_i}{b_1 \beta_1 B_i + \frac{\mu_M d_B}{\mu_B + \Psi_B}\left(\frac{\lambda_B}{d_B} - B_i\right)} \leq \frac{b_1 \beta_1 \frac{\lambda_M}{\mu_M} B_i}{b_1 \beta_1 B_i} = \frac{\lambda_M}{\mu_M}. \tag{45}$$

The subsystem (34)-(37) has a unique endemic equilibrium at $B_i = \tilde{\tilde{B}}_i \in \left(0, \tilde{B}_{i2}\right)$. It can be obtained by substituting from (43) to (44) to get

$$a_0 \left(\tilde{\tilde{B}}_i\right)^2 + a_1 \left(\tilde{\tilde{B}}_i\right) + a_2 = 0, \tag{46}$$

where

$$a_0 = \frac{d_B\left(\mu_M d_B - b_1\beta_1\left(\mu_B + \Psi_B\right)\right)}{b_1\beta_2\left(\mu_B + \Psi_B\right)}, \tag{47}$$

$$a_1 = b_1\beta_1\frac{\lambda_M}{\mu_M} - \left(\frac{2\mu_M d_B - b_1\beta_1\left(\mu_B + \Psi_B\right)}{b_1\beta_2\left(\mu_B + \Psi_B\right)}\right)\lambda_B, \tag{48}$$

$$a_2 = \frac{\mu_M\lambda_B^2}{b_1\beta_2\left(\mu_B + \Psi_B\right)} - \frac{b_1\beta_1\lambda_M\lambda_B}{\mu_M\left(\mu_B + \Psi_B + d_B\right)}. \tag{49}$$

We can get $\tilde{\tilde{B}}_i$ by solving equation (46) and also the variables $\tilde{\tilde{M}}_i$, $\tilde{\tilde{B}}_s$ and $\tilde{\tilde{M}}_s$ can be computed. Secondly, we consider the last five equations in system (1):

$$\dot{S}(t) = \lambda_H - \frac{b_2\beta_3 M_i S}{N_H} - \mu_H S, \tag{50}$$

$$\dot{E}(t) = \frac{b_2\beta_3 M_i S}{N_H} - \alpha E - \mu_H E, \tag{51}$$

$$\dot{I}(t) = \alpha E - \gamma I - d_I I - rI - \mu_H I, \tag{52}$$

$$\dot{H}(t) = \gamma I - d_H H - \tau H - \mu_H H, \tag{53}$$

$$\dot{R}(t) = \tau H + rI - \mu_H R. \tag{54}$$

At equilibrium, the variables S, E, I, H and R in equations (50)-(54) can be expressed in terms of the variable E, *i.e.* $S = \frac{\lambda_H}{\mu_H} - \left(\frac{\mu_H+\alpha}{\mu_H}\right)E,\ I = \frac{\alpha}{\gamma+d_I+r+\mu_H}E,\ H = \frac{\alpha\gamma}{(d_H+\tau+\mu_H)(\gamma+d_I+r+\mu_H)}E$, and $R = \frac{\tau\alpha\gamma+r\alpha(d_H+\tau+\mu_H)}{\mu_H(d_H+\tau+\mu_H)(\gamma+d_I+r+\mu_H)}E$, where $E = \tilde{\tilde{E}}$ in the biologically meaningful range $0 < \tilde{\tilde{E}} < \frac{\lambda_H}{\mu_H+\alpha}$. Thus the unique endemic equilibrium for the full system (1) is $E_1 = \left(\tilde{\tilde{M}}_s,\ \tilde{\tilde{M}}_i,\ \tilde{\tilde{B}}_s,\ \tilde{\tilde{B}}_i,\ \tilde{\tilde{S}},\ \tilde{\tilde{E}},\ \tilde{\tilde{I}},\ \tilde{\tilde{H}},\ \tilde{\tilde{R}}\right)$.

Theorem 3 The endemic equilibrium E_1 is asymptotically stable if $R_0 > 1$ for the model (1).

Proof. Let us consider the mosquito-bird cycle, described by the subsystem (34)-(37). We evaluate the Jacobian of (34)-(37) at E_1:

$$J = \begin{pmatrix} -\frac{b_1\beta_1\tilde{\tilde{B}}_i}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)} - \mu_M & 0 & \frac{b_1\beta_1\tilde{\tilde{M}}_s\tilde{\tilde{B}}_i}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)^2} & -\frac{b_1\beta_1\tilde{\tilde{M}}_s}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)} + \frac{b_1\beta_1\tilde{\tilde{M}}_s\tilde{\tilde{B}}_i}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)^2} \\ \frac{b_1\beta_1\tilde{\tilde{B}}_i}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)} & -\mu_M & -\frac{b_1\beta_1\tilde{\tilde{M}}_s\tilde{\tilde{B}}_i}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)^2} & \frac{b_1\beta_1\tilde{\tilde{M}}_s}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)} - \frac{b_1\beta_1\tilde{\tilde{M}}_s\tilde{\tilde{B}}_i}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)^2} \\ 0 & -\frac{b_1\beta_2\tilde{\tilde{B}}_s}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)} & -\frac{b_1\beta_2\tilde{\tilde{M}}_i}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)} + \frac{b_1\beta_2\tilde{\tilde{M}}_i\tilde{\tilde{B}}_s}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)^2} - (\mu_B + \Psi_B) & \frac{b_1\beta_2\tilde{\tilde{M}}_i\tilde{\tilde{B}}_s}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)^2} \\ 0 & \frac{b_1\beta_2\tilde{\tilde{B}}_s}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)} & \frac{b_1\beta_2\tilde{\tilde{M}}_i}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)} - \frac{b_1\beta_2\tilde{\tilde{M}}_i\tilde{\tilde{B}}_s}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)^2} & -\frac{b_1\beta_2\tilde{\tilde{M}}_i\tilde{\tilde{B}}_s}{(\tilde{\tilde{B}}_s+\tilde{\tilde{B}}_i)^2} - (\mu_B + \Psi_B + d_B) \end{pmatrix},$$

where the eigenvalues of J are $-\mu_M$ and the roots of the following equation

$$\Lambda^3 + a_1\Lambda^2 + a_2\Lambda + a_3 = 0,$$

where

$$
\begin{aligned}
a_1 &= \frac{1}{(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)}[b_1\beta_2\tilde{\tilde{M}}_i + (b_1\beta_1 + (\mu_M + d_B) + 2(\mu_B + \Psi_B))\tilde{\tilde{B}}_i + ((\mu_M + d_B) \\
&+ 2(\mu_B + \Psi_B))\tilde{\tilde{B}}_s], \\
a_2 &= \frac{1}{(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^2}[(\mu_B^2 + \Psi_B^2 + \mu_B d_B + \mu_M d_B + \Psi_B d_B + 2\mu_B\mu_M + 2\mu_M\Psi_B \\
&+ 2\mu_B\Psi_B)\tilde{\tilde{B}}_s^2 + (b_1\beta_1 d_B + 2b_1\beta_1(\mu_B + \Psi_B) + \mu_B^2 + \Psi_B^2 + d_B(\mu_B + \mu_M + \Psi_B) \\
&+ 2\mu_M(\mu_B + \Psi_B) + 2\mu_B\Psi_B)\tilde{\tilde{B}}_i^2 + (b_1\beta_1 d_B + 2b_1\beta_1(\mu_B + \Psi_B) + 2\mu_B^2 + 2\Psi_B^2 \\
&+ 2d_B(\mu_B + \mu_M + \Psi_B) + 4\mu_M(\mu_B + \Psi_B) + 4\mu_B\Psi_B)\tilde{\tilde{B}}_s\tilde{\tilde{B}}_i + b_1\beta_2(\mu_B + \mu_M \\
&+ \Psi_B)\tilde{\tilde{M}}_i\tilde{\tilde{B}}_s + b_1\beta_2(\mu_B + \mu_M + \Psi_B + d_B)\tilde{\tilde{M}}_i\tilde{\tilde{B}}_i + b_1^2\beta_1\beta_2(\tilde{\tilde{B}}_i - \tilde{\tilde{M}}_s)\tilde{\tilde{M}}_i] \\
&+ \frac{1}{(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^3}[b_1^2\beta_1\beta_2(\tilde{\tilde{M}}_i - \tilde{\tilde{B}}_s)\tilde{\tilde{M}}_s], \\
a_3 &= \frac{1}{(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^3}[\mu_M(\mu_B + \Psi_B)(\mu_B + \Psi_B + d_B)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^3 + (\mu_B + \Psi_B \\
&+ d_B)(b_1\beta_1(\mu_B + \Psi_B)\tilde{\tilde{B}}_i + b_1\beta_2\mu_M\tilde{\tilde{M}}_i)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^2 + b_1\beta_2\tilde{\tilde{M}}_i(b_1\beta_1(d_B + \Psi_B \\
&+ \mu_B)\tilde{\tilde{B}}_i - b_1\beta_1(\mu_B + \Psi_B)\tilde{\tilde{M}}_s + d_B\mu_M\tilde{\tilde{B}}_s + b_1^2\beta_1\beta_2\tilde{\tilde{M}}_s(\tilde{\tilde{M}}_i + \tilde{\tilde{B}}_s))(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i) \\
&+ b_1^2\beta_1\beta_2(b_1\beta_2\tilde{\tilde{M}}_s\tilde{\tilde{M}}_i(\tilde{\tilde{B}}_s - \tilde{\tilde{M}}_i) - \beta_1\tilde{\tilde{B}}_s\tilde{\tilde{B}}_i((\mu_B + \Psi_B + d_B)\tilde{\tilde{M}}_s + d_B\tilde{\tilde{M}}_i) \\
&+ \beta_1(\mu_B + \Psi_B)\tilde{\tilde{M}}_s\tilde{\tilde{M}}_i\tilde{\tilde{B}}_i)].
\end{aligned}
$$

It is clear that $a_1 > 0$. But for a_2 and a_3 are both positive if $\tilde{\tilde{B}}_i < \frac{\lambda_B}{\mu_B+\Psi_B+d_B} < \frac{\lambda_B}{\Psi_B+d_B}$ (this condition is required for B_s) and $\mu_B > \Psi_B + d_B$ (this condition is biologically reasonable). We will use Routh-Hurwitz criteria to show that $a_1a_2 - a_3 > 0$. Since $a_1a_2 - a_3$ can be written as:

$$Z_3b_1^3 + Z_2b_1^2 + Z_1b_1 + Z_0, \tag{55}$$

where

$$
\begin{aligned}
Z_3 =\ & \beta_2(\tilde{\tilde{B}}_i + \tilde{\tilde{M}}_i)(\beta_1\beta_2\tilde{\tilde{M}}_i(\tilde{\tilde{B}}_i - \tilde{\tilde{M}}_s)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^2 - \beta_1\beta_2\tilde{\tilde{M}}_s\tilde{\tilde{B}}_i * (\tilde{\tilde{B}}_s - \tilde{\tilde{M}}_i)(\tilde{\tilde{B}}_s \\
& + \tilde{\tilde{B}}_i)^3)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i) - (\beta_1\beta_2^2\tilde{\tilde{M}}_s\tilde{\tilde{M}}_i(\tilde{\tilde{B}}_s - \tilde{\tilde{M}}_i) + \beta_1\beta_2^2\tilde{\tilde{M}}_s\tilde{\tilde{M}}_i(\tilde{\tilde{B}}_s + \tilde{\tilde{M}}_i)(\tilde{\tilde{B}}_s \\
& + \tilde{\tilde{B}}_i))(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^3, \\
Z_2 =\ & (\beta_1\beta_2(\beta_1\tilde{\tilde{B}}_i\tilde{\tilde{B}}_s(d_B\tilde{\tilde{M}}_i + \tilde{\tilde{M}}_s(d_B + \mu_B + \Psi_B)) - \beta_1\tilde{\tilde{M}}_s\tilde{\tilde{M}}_i\tilde{\tilde{B}}_i(\mu_B + \Psi_B)) \\
& + \beta_2\tilde{\tilde{M}}_i(\beta_1\tilde{\tilde{M}}_s(\mu_B + \Psi_B) - \beta_1\tilde{\tilde{B}}_i(d_B + \mu_B + \Psi_B))(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i))(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^3 \\
& + (\tilde{\tilde{B}}_i(d_B + 2\mu_B + \mu_M + 2\Psi_B) + \tilde{\tilde{B}}_s(d_B + 2\mu_B + \mu_M + 2\Psi_B))(\beta_1\beta_2\tilde{\tilde{M}}_i(\tilde{\tilde{B}}_i \\
& - \tilde{\tilde{M}}_s)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^2 - \beta_1\beta_2\tilde{\tilde{M}}_s\tilde{\tilde{B}}_i(\tilde{\tilde{B}}_s - \tilde{\tilde{M}}_i)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^3)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i) + (\beta_2\tilde{\tilde{B}}_i \\
& + \beta_2\tilde{\tilde{M}}_i)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^3(\tilde{\tilde{B}}_i^2(\beta_1 d_B + 2\beta_1(\mu_B + \Psi_B)) + \tilde{\tilde{B}}_s\tilde{\tilde{B}}_i(\beta_1 d_B + 2\beta_1(\mu_B + \Psi_B)) \\
& + \beta_2\tilde{\tilde{M}}_i\tilde{\tilde{B}}_i(d_B + \mu_B + \mu_M + \Psi_B) + \beta_2\tilde{\tilde{B}}_s\tilde{\tilde{M}}_i(\mu_B + \mu_M + \Phi_B)), \\
Z_1 =\ & (\tilde{\tilde{B}}_i(d_B + 2\mu_B + \mu_M + 2\Psi_B) + \tilde{\tilde{B}}_s(d_B + 2\mu_B + \mu_M + 2\Psi_B))(\tilde{\tilde{B}}_s \\
& + \tilde{\tilde{B}}_i)^3(\tilde{\tilde{B}}_i^2(\beta_1 d_B + 2\beta_1(\mu_B + \Psi_B)) + \tilde{\tilde{B}}_s\tilde{\tilde{B}}_i(\beta_1 d_B + 2\beta_1(\mu_B + \Psi_B)) \\
& + \beta_2\tilde{\tilde{M}}_i\tilde{\tilde{B}}_i(d_B + \mu_B + \mu_M + \Psi_B) + \beta_2\tilde{\tilde{M}}_i\tilde{\tilde{B}}_s(\mu_B + \mu_M + \Psi_B)) - ((\beta_1\tilde{\tilde{B}}_i(\mu_B \\
& + \Psi_B) + \beta_2\mu_M\tilde{\tilde{M}}_i)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^2(d_B + \mu_B + \Psi_B) + \beta_2 d_B\mu_M\tilde{\tilde{M}}_i\tilde{\tilde{B}}_s(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i))(\tilde{\tilde{B}}_s \\
& + \tilde{\tilde{B}}_i)^3 + (\beta_2\tilde{\tilde{B}}_i + \beta_2\tilde{\tilde{M}}_i)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^3((2\mu_M(\mu_B + \Psi_B)) + 2\mu_B\Psi_B + d_B(\mu_B + \mu_M \\
& + \Psi_B) + \mu_B^2 + \Psi_B^2)\tilde{\tilde{B}}_i^2 + (4\mu_M(\mu_B + \Psi_B) + 4\mu_B\Psi_B + 2d_B(\mu_B + \mu_M + \Psi_B) + 2\mu_B^2 \\
& + 2\Psi_B^2\tilde{\tilde{B}}_s\tilde{\tilde{B}}_i + (d_B\mu_B + d_B\mu_M + d_B\Psi_B + 2\mu_B\mu_M + 2\mu_B\Psi_B + 2\mu_M\Psi_B + \mu_B^2 \\
& + \Psi_B^2)\tilde{\tilde{B}}_s^2), \\
Z_0 =\ & (\tilde{\tilde{B}}_i(d_B + 2\mu_B + \mu_M + 2\Psi_B) + \tilde{\tilde{B}}_s(d_B + 2\mu_B + \mu_M + 2\Psi_B))(\tilde{\tilde{B}}_s \\
& + \tilde{\tilde{B}}_i)^3((2\mu_M(\mu_B + \Psi_B) + 2\mu_B\Psi_B + d_B(\mu_B + \mu_M + \Psi_B) + \mu_B^2 + \Psi_B^2)\tilde{\tilde{B}}_i^2 \\
& + (4\mu_M(\mu_B + \Psi_B) + 4\mu_B\Psi_B + 2d_B(\mu_B + \mu_M + \Psi_B) + 2\mu_B^2 + 2\Psi_B^2)\tilde{\tilde{B}}_s\tilde{\tilde{B}}_i + (d_B\mu_B \\
& + d_B\mu_M + d_B\Psi_B + 2\mu_B\mu_M + 2\mu_B\Psi_B + 2\mu_M\Psi_B + \mu_B^2 + \Psi_B^2)\tilde{\tilde{B}}_s^2) - \mu_M(\mu_B \\
& + \Psi_B)(\tilde{\tilde{B}}_s + \tilde{\tilde{B}}_i)^6(d_B + \mu_B + \Psi_B).
\end{aligned}
$$

If the inequaliltis $\tilde{\tilde{B}}_i < \frac{\lambda_B}{\mu_B+\Psi_B+d_B} < \frac{\lambda_B}{\Psi_B+d_B}$ and $\mu_B > \Psi_B + d_B$ are satisfied, then equation (55) is positive. Thus $a_1a_2 - a_3 > 0$ provided the above inequalities hold. . For the human subsystem described by equations (50)-(54)(since we have $\tilde{\tilde{N}}_H = \tilde{\tilde{S}} + \tilde{\tilde{E}} + \tilde{\tilde{I}} + \tilde{\tilde{H}} + \tilde{\tilde{R}}$), the Jacobian of (50)-(54) at E_1

$$
J = \begin{pmatrix}
-\frac{b_2\beta_3\tilde{\tilde{M}}_i}{\tilde{\tilde{N}}_H} + \frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H^2} - \mu_H & \frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H^2} & \frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H^2} & \frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H^2} & \frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H^2} \\
\frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H} - \frac{b_2\beta_3\tilde{\tilde{M}}_i}{\tilde{\tilde{N}}_H^2} & -\frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H^2} - (\alpha + \mu_H) & -\frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H^2} & -\frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H^2} & -\frac{b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{S}}}{\tilde{\tilde{N}}_H^2} \\
0 & \alpha & -(\gamma + d_I + r + \mu_H) & 0 & 0 \\
0 & 0 & \gamma & -(d_H + \tau + \mu_H) & 0 \\
0 & 0 & r & \tau & -\mu_H
\end{pmatrix},
$$

where the eigenvalues of J are $-\mu_H$ and the roots of the polynomial

$$\Lambda^4 + G_1\Lambda^3 + G_2\Lambda^2 + G_3\Lambda + G_4 = 0,$$

where

$$\begin{aligned}
G_1 = {}& \frac{1}{\tilde{\tilde{N}}_H^2}(d_H\tilde{\tilde{N}}_H^2 + d_I\tilde{\tilde{N}}_H^2 + \gamma\tilde{\tilde{N}}_H^2 + 3\mu_H\tilde{\tilde{N}}_H^2 + r\tilde{\tilde{N}}_H^2 + \tau\tilde{\tilde{N}}_H^2 + b_2\beta_3\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H),\\
G_2 = {}& \frac{1}{\tilde{\tilde{N}}_H^2}(\mu_H^2\tilde{\tilde{N}}_H^2 + d_Hd_I\tilde{\tilde{N}}_H^2 + d_H\gamma\tilde{\tilde{N}}_H^2 + 2d_H\mu_H\tilde{\tilde{N}}_H^2 + 2d_I\mu_H\tilde{\tilde{N}}_H^2 + 2\gamma\mu_H\tilde{\tilde{N}}_H^2\\
& + d_Hr\tilde{\tilde{N}}_H^2 + d_I\tau\tilde{\tilde{N}}_H^2 + 2\mu_Hr\tilde{\tilde{N}}_H^2 + 2\mu_H\tau\tilde{\tilde{N}}_H^2 + r\tau\tilde{\tilde{N}}_H^2 + b_2\beta_3d_H\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H\\
& + b_2\beta_3d_I\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\gamma\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + 2b_2\beta_3\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3r\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H\\
& + b_2\beta_3\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\tau\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H),\\
G_3 = {}& \frac{1}{\tilde{\tilde{N}}_H^2}(\mu_H\tilde{\tilde{N}}_H^2 + d_H\mu_H^2\tilde{\tilde{N}}_H^2 + d_I\mu_H^2\tilde{\tilde{N}}_H^2 + \gamma\mu_H^2\tilde{\tilde{N}}_H^2 + r\mu_H^2\tilde{\tilde{N}}_H^2 + \tau\mu_H^2\tilde{\tilde{N}}_H^2\\
& + d_Hd_I\mu_H\tilde{\tilde{N}}_H^2 + d_H\gamma\mu_H\tilde{\tilde{N}}_H^2 + d_H\mu_Hr\tilde{\tilde{N}}_H^2 + d_I\mu_H\tau\tilde{\tilde{N}}_H^2 + \gamma\mu_H\tau\tilde{\tilde{N}}_H^2 + \mu_Hr\tau\tilde{\tilde{N}}_H^2\\
& + b_2\beta_3\mu_H^2\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + 2b_2\beta_3\mu_H^2\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3d_Hd_I\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3d_H\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}}\\
& + b_2\beta_3d_H\gamma\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3\alpha\gamma\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3d_H\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3d_I\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H\\
& + 2b_2\beta_3\mu_H\alpha\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3\gamma\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3d_Hr\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3d_H\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{S}}\\
& + b_2\beta_3d_I\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3d_I\tau\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3\gamma\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\tau\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}}\\
& + b_2\beta_3\gamma\tau\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3\mu_Hr\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3\mu_H\tau\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3\mu_Hr\tilde{\tilde{M}}_i\tilde{\tilde{S}}\\
& + b_2\beta_3r\tau\tilde{\tilde{M}}_i\tilde{\tilde{N}}_H + b_2\beta_3\mu_H\tau\tilde{\tilde{M}}_i\tilde{\tilde{S}}),\\
G_4 = {}& \frac{1}{\tilde{\tilde{N}}_H^2}(b_2\beta_3\mu_H^3\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\mu_H^2r\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\mu_H^2\tau\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\mu_H^2\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}}\\
& + b_2\beta_3d_H\mu_H^2\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3d_I\mu_H^2\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\gamma\mu_H^2\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\gamma\tau\tilde{\tilde{M}}_i\tilde{\tilde{S}}\\
& + b_2\beta_3\mu_Hr\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3d_I\mu_H\tau\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3r\tau\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\mu_H\gamma\tau\tilde{\tilde{M}}_i\tilde{\tilde{S}}\\
& + b_2\beta_3\mu_Hr\tau\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3d_H\mu_H\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3d_Hd_I\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3\gamma\mu_H\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}}\\
& + b_2\beta_3d_Hr\alpha\tilde{\tilde{M}}_i\tilde{\tilde{S}} + b_2\beta_3d_H\gamma\mu_H\tilde{\tilde{M}}_i\tilde{\tilde{S}}).
\end{aligned}$$

All constants in the above polynomial are positive. Thus, E_1 is locally asymptotically stable provided that $\mu_B > \Psi_B + d_B$.

6. The Optimal Control Problem

In this section, an optimal control problem for the transmission dynamics of WNV is introduced. This optimal control problem is described by two control functions u_k, $k = 1, 2$, (u_1 represents the level of larvacide and adulticide used for mosquito control administered at mosquito breeding sites and u_2 measures the level of successful prevention (personal protection) efforts). We need to minimize the exposed and infected human populations, the total number of mosquitos and the cost of implementing the control. The optimal control problem in transmission dynamics of WNV model in [23] is given by

$$\breve{J}(u_1, u_2) = \int_0^T \left(A_1 E(t) + A_2 I(t) + A_3 N_M(t) + B_1 u_1^2 + B_2 u_2^2\right) dt, \tag{56}$$

where A_1, A_2, and A_3 represent, respectively, the weight constants of the exposed, infected human and the total mosquito populations and B_1 and B_2 are weights constants for mosquito control and personal protection (prevention of mosquito-human contacts) [23].

Subject to the constraints,

$$\begin{aligned}
\frac{dM_s}{dt} &= \lambda_M N_M(1 - u_1(t)) - \frac{b_1\beta_1 M_s B_i}{N_B} - \mu_M M_s - r_0 u_1(t) M_s,\\
\frac{dM_i}{dt} &= \frac{b_1\beta_1 M_s B_i}{N_B} - \mu_M M_i - r_0 u_1(t) M_i,\\
\frac{dB_s}{dt} &= \lambda_B + \rho N_B - \frac{b_1\beta_2 M_i B_s}{N_B} - \Psi_B B_s - \mu_B B_s,\\
\frac{dB_i}{dt} &= \frac{b_1\beta_2 M_i B_s}{N_B} - d_B B_i - \Psi_B B_i - \mu_B B_i,\\
\frac{dS}{dt} &= \lambda_H + \gamma_H N_H - \frac{b_2\beta_3 M_i S(1 - u_2(t))}{N_H} - \mu_H S,\\
\frac{dE}{dt} &= \frac{b_2\beta_3 M_i S(1 - u_2(t))}{N_H} - \alpha E - \mu_H E,\\
\frac{dI}{dt} &= \alpha E - \gamma I - d_I I - rI - \mu_H I,\\
\frac{dH}{dt} &= \gamma I - d_H H - \tau H - \mu_H H,\\
\frac{dR}{dt} &= \tau H + rI - \mu_H R,
\end{aligned} \tag{57}$$

where the initial conditions are given in (2). Since, the factor of the term $(1-u_1(t))$ reduces the reproduction rate of the mosquito population and in human population, the associated force of infection is reduced by a factor of $(1-u_2(t))$. Let us consider

$$\begin{aligned}
\dot{M}_s &= \Im_1, \quad \dot{M}_i = \Im_2, \quad \dot{B}_s = \Im_3,\\
\dot{B}_i &= \Im_4, \quad \dot{S} = \Im_5, \quad \dot{E} = \Im_6,\\
\dot{I} &= \Im_7, \quad \dot{H} = \Im_8, \quad \dot{R} = \Im_9,
\end{aligned}$$

where,

$$\Im_j = \Im_j(M_s, M_i, B_s, B_i, S, E, I, H, R, u_1, u_2, t), \quad \forall j = 1, 2, \ldots, 9.$$

Now, we define the Hamiltonian function $\mathcal{H}_a(M_s, M_i, B_s, B_i, S, E, I, H, R, u_1, u_2, t)$ as follows:

$$\mathcal{H}_a(M_s, M_i, B_s, B_i, S, E, I, H, R, u_1, u_2, t) = \Upsilon(M_s, M_i, B_s, B_i, S, E, I, H, R, u_1, u_2, t) + \sum_{j=1}^{9} \lambda_j \Im_j(M_s, M_i, B_s, B_i, S, E, I, H, R, u_1, u_2, t), \tag{58}$$

where $\lambda_j,\ j = 1, 2, \ldots, 9$, are Lagrange multipliers. Thus, a modified objective function can be expressed by

$$\check{J} = \int_0^T [\mathcal{H}_a(M_s, M_i, B_s, B_i, S, E, I, H, R, u_1, u_2, t) - \sum_{j=1}^{9} \lambda_j \Im_j(M_s, M_i, B_s, B_i, S, E, I, H, R, u_1, u_2, t)]dt. \tag{59}$$

According to Pontryagin's maximum principle [37], the necessary conditions for the optimal control problem (56) and (57) are

$$\begin{aligned} \dot{\lambda}_1 &= \frac{\partial \mathcal{H}_a}{\partial M_s}, \quad \dot{\lambda}_2 = \frac{\partial \mathcal{H}_a}{\partial M_i}, \quad \dot{\lambda}_3 = \frac{\partial \mathcal{H}_a}{\partial B_s}, \\ \dot{\lambda}_4 &= \frac{\partial \mathcal{H}_a}{\partial B_i}, \quad \dot{\lambda}_5 = \frac{\partial \mathcal{H}_a}{\partial S}, \quad \dot{\lambda}_6 = \frac{\partial \mathcal{H}_a}{\partial E}, \\ \dot{\lambda}_7 &= \frac{\partial \mathcal{H}_a}{\partial I}, \quad \dot{\lambda}_8 = \frac{\partial \mathcal{H}_a}{\partial H}, \quad \dot{\lambda}_9 = \frac{\partial \mathcal{H}_a}{\partial R}, \end{aligned} \tag{60}$$

$$\frac{\partial \mathcal{H}_a}{\partial u_k} = 0, \quad \forall k = 1, 2, \tag{61}$$

and also we have

$$\lambda_j(T) = 0, \quad j = 1, 2, 3, ..., 9. \tag{62}$$

From the necessary conditions (60) and (61), the Lagrange multipliers λ_j and the control variables u_k, $k = 1, 2$, can be written as follows [23]:

$$\begin{aligned} \dot{\lambda}_1 &= -A_3 - (\lambda_2 - \lambda_1) b\beta_1 B_i \xi - \lambda_1 [\lambda_M (1 - u_1) - (\mu_1 + \mu_2 N_M) \\ &\quad - \mu_2 M_s - r_0 u_1] + \lambda_2 \mu_2 M_i, \\ \dot{\lambda}_2 &= -A_3 - \lambda_1 \left[\lambda_M (1 - u_1) - \mu_2 M_s\right] + \lambda_2 \left[\mu_1 + \mu_2 N_M + \mu_2 M_i + r_0 u_1\right] \\ &\quad - (\lambda_4 - \lambda_3) b\beta_2 B_s \xi - b\beta_3 S(1 - u_2)\xi(\lambda_6 - \lambda_5), \\ \dot{\lambda}_3 &= (\lambda_2 - \lambda_1) b\beta_1 B_i M_s \xi^2 - \lambda_3 \left[\rho - b\beta_2 M_i(\xi - B_s \xi^2) - \varPsi_B - \mu_B\right] \\ &\quad - \lambda_4 b\beta_2 M_i(\xi - B_s \xi^2) - (\lambda_5 - \lambda_6) b\beta_3 M_i S(1 - u_2)\xi^2, \\ \dot{\lambda}_4 &= -(\lambda_2 - \lambda_1) b\beta_1 M_s(\xi - B_i \xi^2) - \lambda_3 [\rho + b\beta_2 M_i B_s \xi^2] \\ &\quad + \lambda_4 [b\beta_2 M_i B_s \xi^2 + (d_B + \varPsi_B + \mu_B)] - (\lambda_5 - \lambda_6) b\beta_3 M_i S(1 - u_2)\xi^2, \end{aligned}$$

$$\begin{aligned}
\dot{\lambda_5} &= -(\lambda_1-\lambda_2)b\beta_1 B_i M_s \xi^2 - (\lambda_3-\lambda_4)b\beta_2 M_i B_s \xi^2 \\
&- \lambda_5[\gamma_H - b\beta_3 M_i(1-u_2)(\xi - S\xi^2) - \mu_4 S - (\mu_3+\mu_4 N_H)] \\
&- \lambda_6[b\beta_3 M_i(1-u_2)(\xi - S\xi^2) - \mu_4 E] + \lambda_7\mu_4 I + \lambda_8\mu_4 H + \lambda_9\mu_4 R, \\
\dot{\lambda_6} &= -A_1 - (\lambda_1-\lambda_2)b\beta_1 B_i M_s \xi^2 - (\lambda_3-\lambda_4)b\beta_2 M_i B_s \xi^2 \\
&- \lambda_5[\gamma_H + b\beta_3 M_i B_s(1-u_2)\xi^2 - \mu_4 S] \\
&+ \lambda_6[b\beta_3 M_i S(1-u_2)\xi^2 + \alpha + \mu_4 E + \mu_3 + \mu_4 N_H] \\
&- \lambda_7[\alpha - \mu_4 I] + \lambda_8\mu_4 H + \lambda_9\mu_4 R, \\
\dot{\lambda_7} &= -A_2 - (\lambda_1-\lambda_2)b\beta_1 B_i M_s \xi^2 - (\lambda_3-\lambda_4)b\beta_2 M_i B_s \xi^2 \\
&- \lambda_5[\gamma_H + b\beta_3 M_i S(1-u_2)\xi^2 - \mu_4 S] + \lambda_6[b\beta_3 M_i S(1-u_2)\xi^2 + \mu_4 E] \\
&+ \lambda_7[\gamma + d_I + r + \mu_4 I + \mu_3 + \mu_4 N_H] - \lambda_8[\gamma - \mu_4 H] - \lambda_9[r - \mu_4 R], \\
\dot{\lambda_8} &= -(\lambda_1-\lambda_2)b\beta_1 M_s B_i \xi^2 - (\lambda_3-\lambda_4)b\beta_2 M_i B_s \xi^2 \\
&- \lambda_5[\gamma_H + b\beta_3 M_i S(1-u_2)\xi^2 - \mu_4 S] + \lambda_6[b\beta_3 M_i S(1-u_2)\xi^2 + \mu_4 E] \\
&+ \lambda_7\mu_4 I + \lambda_8[d_H + \tau + \mu_4 N_H + \mu_3 + \mu_4 H] - \lambda_9[\tau - \mu_4 R], \\
\dot{\lambda_9} &= -(\lambda_1-\lambda_2)b\beta_1 M_s B_i \xi^2 - (\lambda_3-\lambda_4)b\beta_2 M_i B_s \xi^2 \\
&- \lambda_5[\gamma_H + b\beta_3 M_i S(1-u_2)\xi^2 - \mu_4 S] + \lambda_6[b\beta_3 M_i S(1-u_2)\xi^2 + \mu_4 E] \\
&+ \lambda_7\mu_4 I + \lambda_8\mu_4 H + \lambda_9[\mu_4 R + \mu_3 + \mu_4 N_H], \qquad (63)
\end{aligned}$$

$$\begin{aligned}
u_1 &= \max\{0, \min\{1, \frac{1}{2B_1}[\lambda_1(\lambda_M N_M + r_0 M_s) + \lambda_2 r_0 M_i]\}\}, \\
u_2 &= \max\{0, \min\{1, \frac{1}{2B_2} b\beta_3 M_i S\xi(\lambda_6 - \lambda_5)\}\}, \qquad (64)
\end{aligned}$$

where $\xi = \frac{1}{N_B+N_H}$. Thus, we have the following theorem:
Theorem 1 The optimal controls u_1 and u_2 of the optimal control problem (56) and (57) satisfy the necessary conditions (60) and (61) and the Lagrange multipliers $\lambda_j(T) = 0$, $\forall j = 1, 2, \ldots, 9$.

6.1. **NSFD for the Optimal Control Problem.** In this section, the numerical scheme for optimal control problem classified into two steps. Firsty, the state system under control (57) is discretized by using local approximation for the nonlinear terms, see section 3. Secondly, the adjoint system (63) will be discretized by using nonlocal approximation as follows:

$$\begin{aligned}
\frac{\lambda_1^n - \lambda_1^{n+1}}{\varphi(\Delta t)} &= -A_3 - (\lambda_2^n - \lambda_1^n)b\beta_1 B_i^{n+1}\xi^n - \lambda_1^n[\lambda_M(1-u_1^{n+1}) - (\mu_1 + \mu_2 N_M^n) \\
&- \mu_2 M_s^{n+1} - r_0 u_1^{n+1}] + \lambda_2^n \mu_2 M_i^{n+1}, \\
\frac{\lambda_2^n - \lambda_2^{n+1}}{\varphi(\Delta t)} &= -A_3 - \lambda_1^n\left[\lambda_M(1-u_1^{n+1}) - \mu_2 M_s^{n+1}\right] + \lambda_2^n\left[\mu_1 + \mu_2 N_M^n + \mu_2 M_i^{n+1}\right. \\
&\left. + r_0 u_1^{n+1}\right] - (\lambda_4^n - \lambda_3^n)b\beta_2 B_s^{n+1}\xi^n - b\beta_3 S^{n+1}(1-u_2^{n+1})\xi^n(\lambda_6^n - \lambda_5^n),
\end{aligned}$$

$$
\begin{aligned}
\frac{\lambda_3^n-\lambda_3^{n+1}}{\varphi(\Delta t)} &= (\lambda_2^n-\lambda_1^n)b\beta_1 B_i^{n+1}M_s^{n+1}(\xi^n)^2 \\
&- \lambda_3^n\left[\rho - b\beta_2 M_i^{n+1}(\xi^n - B_s^{n+1}(\xi^n)^2) - \Psi_B - \mu_B\right] \\
&- \lambda_4^n b\beta_2 M_i^{n+1}(\xi^n - B_s^{n+1}(\xi^n)^2) - (\lambda_5^n-\lambda_6^n)b\beta_3 M_i^{n+1}S^{n+1}(1-u_2^{n+1})(\xi^n)^2, \\
\frac{\lambda_4^n-\lambda_4^{n+1}}{\varphi(\Delta t)} &= -(\lambda_2^n-\lambda_1^n)b\beta_1 M_s^{n+1}(\xi^n - B_i^{n+1}(\xi^n)^2) - \lambda_3^n[\rho + b\beta_2 M_i^{n+1}B_s^{n+1}(\xi^n)^2] \\
&+ \lambda_4^n[b\beta_2 M_i^{n+1}B_s^{n+1}(\xi^n)^2 + (d_B + \Psi_B + \mu_B)] \\
&- (\lambda_5^n-\lambda_6^n)b\beta_3 M_i^{n+1}S^{n+1}(1-u_2^{n+1})(\xi^n)^2, \\
\frac{\lambda_5^n-\lambda_5^{n+1}}{\varphi(\Delta t)} &= -(\lambda_1^n-\lambda_2^n)b\beta_1 B_i^{n+1}M_s^{n+1}(\xi^n)^2 - (\lambda_3^n-\lambda_4^n)b\beta_2 M_i^{n+1}B_s^{n+1}(\xi^n)^2 \\
&- \lambda_5^n[\gamma_H - b\beta_3 M_i^{n+1}(1-u_2^{n+1})(\xi^n - S^{n+1}(\xi^n)^2) - \mu_4 S^{n+1} - (\mu_3 + \mu_4 N_H^n)] \\
&- \lambda_6^n[b\beta_3 M_i^{n+1}(1-u_2^{n+1})(\xi^n - S^{n+1}(\xi^n)^2) - \mu_4 E^{n+1}] + \lambda_7^n \mu_4 I^{n+1} \\
&+ \lambda_8^n \mu_4 H^{n+1} + \lambda_9^n \mu_4 R^{n+1}, \\
\frac{\lambda_6^n-\lambda_6^{n+1}}{\varphi(\Delta t)} &= -A_1 - (\lambda_1^n-\lambda_2^n)b\beta_1 B_i^{n+1}M_s^{n+1}(\xi^n)^2 - (\lambda_3^n-\lambda_4^n)b\beta_2 M_i^{n+1}B_s^{n+1}(\xi^n)^2 \\
&- \lambda_5^n[\gamma_H + b\beta_3 M_i^{n+1}B_s^{n+1}(1-u_2^{n+1})(\xi^n)^2 - \mu_4 S^{n+1}] \\
&+ \lambda_6^n[b\beta_3 M_i^{n+1}S^{n+1}(1-u_2^{n+1})(\xi^n)^2 + \alpha + \mu_4 E^{n+1} + \mu_3 + \mu_4 N_H^{n+1}] \\
&- \lambda_7^n[\alpha - \mu_4 I^{n+1}] + \lambda_8^n \mu_4 H^{n+1} + \lambda_9^n \mu_4 R^{n+1}, \\
\frac{\lambda_7^n-\lambda_7^{n+1}}{\varphi(\Delta t)} &= -A_2 - (\lambda_1^n-\lambda_2^n)b\beta_1 B_i^{n+1}M_s^{n+1}(\xi^n)^2 - (\lambda_3^n-\lambda_4^n)b\beta_2 M_i^{n+1}B_s^{n+1}(\xi^n)^2 \\
&- \lambda_5^n[\gamma_H + b\beta_3 M_i^{n+1}S^{n+1}(1-u_2^{n+1})(\xi^n)^2 - \mu_4 S^{n+1}] \\
&+ \lambda_6^n[b\beta_3 M_i^{n+1}S^{n+1}(1-u_2^{n+1})(\xi^n)^2 + \mu_4 E^{n+1}] \\
&+ \lambda_7^n[\gamma + d_I + r + \mu_4 I^{n+1} + \mu_3 + \mu_4 N_H^n] - \lambda_8^n[\gamma - \mu_4 H^{n+1}] - \lambda_9^n[r \\
&- \mu_4 R^{n+1}], \\
\frac{\lambda_8^n-\lambda_8^{n+1}}{\varphi(\Delta t)} &= -(\lambda_1^n-\lambda_2^n)b\beta_1 M_s^{n+1}B_i^{n+1}(\xi^n)^2 - (\lambda_3^n-\lambda_4^n)b\beta_2 M_i^{n+1}B_s^{n+1}(\xi^n)^2 \\
&- \lambda_5^n[\gamma_H + b\beta_3 M_i^{n+1}S^{n+1}(1-u_2^{n+1})(\xi^n)^2 - \mu_4 S^{n+1}] \\
&+ \lambda_6^n[b\beta_3 M_i^{n+1}S^{n+1}(1-u_2^{n+1})(\xi^n)^2 + \mu_4 E^{n+1}] \\
&+ \lambda_7^n \mu_4 I^{n+1} + \lambda_8^n[d_H + \tau + \mu_4 N_H^n + \mu_3 + \mu_4 H^{n+1}] - \lambda_9^n[\tau - \mu_4 R^{n+1}], \\
\frac{\lambda_9^n-\lambda_9^{n+1}}{\varphi(\Delta t)} &= -(\lambda_1^n-\lambda_2^n)b\beta_1 M_s^{n+1}B_i^{n+1}(\xi^n)^2 - (\lambda_3^n-\lambda_4^n)b\beta_2 M_i^{n+1}B_s^{n+1}(\xi^n)^2 \\
&- \lambda_5^n[\gamma_H + b\beta_3 M_i^{n+1}S^{n+1}(1-u_2^{n+1})(\xi^n)^2 - \mu_4 S^{n+1}] \\
&+ \lambda_6^n[b\beta_3 M_i^{n+1}S^{n+1}(1-u_2^{n+1})(\xi^n)^2 + \mu_4 E^{n+1}] \\
&+ \lambda_7^n \mu_4 I^{n+1} + \lambda_8^n \mu_4 H^{n+1} + \lambda_9^n[\mu_4 R^{n+1} + \mu_3 + \mu_4 N_H^n]. \qquad (65)
\end{aligned}
$$

7. Numerical Experiment

In this section, two numerical methods are introduced to solve the system (1) and the optimality system (57) and (63); NSFD method and SFD method. These methods are applied at different time step sizes Δt. Firstly, NSFD and SFD methods are used for obtaining the approximate solutions for the system (1) as provided in the previous sections. The initial conditions are (10000,1000,1000,0,1000,0,0,0,0). In Table 5, the convergence behavior of these proposed methods is introduced. It can be seen that, the SFD method is convergent at time step sizes $\Delta t = 0.05$, $\Delta t = 0.1$ and $\Delta t = 0.5$, otherwise it is divergent. But NSFD method is convergent at all time step sizes Δt. Figures 1., 2. and 3., respectively, describe the numerical simulations of the system (1) at different time step sizes Δt. Figure 1. describes the numerical comparisons between NSFD and SFD methods of the system (1) at time step size $\Delta t = 0.5$. But the numerical simulations of the system (1) using NSFD method at time step size $\Delta t = 1$ is displayed in Figure 2. It is clear from Figure 3. that the SFD method is divergent at time step size $\Delta t = 1$. From the numerical results presented in Table 5., it can be concluded that NSFD preserves the positivity of the solution and numerical stability in large regions. Secondly,

Table 5. Comparisons between NSFD and SFD methods for the system (1) with different time step size Δt when $R_0 > 1$.

Δt	SFD	NSFD
0.05	*convergent*	*convergent*
0.1	*convergent*	*convergent*
0.5	*convergent*	*convergent*
1	*divergent*	*convergent*
5	*divergent*	*convergent*
10	*divergent*	*convergent*
25	*divergent*	*convergent*

we present different optimal control strategies for the optimality system (57) and (63) under the parameter values are given in Table 1. The following strategies are explored:

- **Strategy 1**, which implements measures for the level of larvacide and adulticide used for mosquito control administered at mosquito breeding sites (*control* u_1 *only*),
- **Strategy 2**, which implements measures for the level of successful prevention (personal protection) efforts (*control* u_2 *only*),
- **Strategy 3**, which represents measures for the level of larvacide and adulticide used for mosquito control administered at mosquito breeding sites and measures for the level of successful prevention (personal protection) efforts (*controls* u_1 *and* u_2). More than one approach is used for obtaining and confirming the numerical results.

The weights $A_1 = A_2 = 1$, $A_3 = 10^{-4}$ in the cost functional (56), (*i.e.*, the minimization of the number of exposed and infected humans, is given more importance than the reduction of the total number of mosquito). We use the upper bound of 0.8 and 0.5 on u_1 and u_2, respectively. The convergence behavior of numerical comparisons between NSFD and SFD methods of the optimality system (57) and (63) at different time step sizes Δt is presented in Table 6. Also, we observe that NSFD method is convergent at large time step sizes Δt but SFD method is divergent. Numerical comparisons between strategy 1 (describes *control* u_1 *only*), strategy 2 (describes *control* u_2 *only*) and strategy 3 (describes *controls* u_1 *and* u_2) of the optimality system (57) and (63) by using NSFD method are provided in Figures 4. and 6. at time step size $\Delta t = 1$ and $\Delta t = 4$, respectively. By applying strategy 1, we observe that the optimal control u_1 stays at the upper bound for 19 days when $\Delta t = 1$ and for 16 days when $\Delta t = 4$ (see Figures 4. and 6.), respectively. When the control u_1 is considered, we see the level of the infected human population $I(t)$ is about 383 when $\Delta t = 1$ and about 328 when $\Delta t = 4$. If strategy 2 is considered, we observe that the optimal control u_2 stays at the upper bound for almost the same duration when $\Delta t = 1$ and for 96 days when $\Delta t = 4$. In this strategy, we see the level of $I(t)$ is about 749 when $\Delta t = 1$ and about 503 when $\Delta t = 4$. This implies a higher value of the cost functional $\check{J}(u_1, u_2)$ associated strategy 1, and strategy 2, as clear in Table 7. The best choice to use is strategy 3. Indeed, with strategy 3, there is a lower value of the cost functional $\check{J}(u_1, u_2)$. Numerical comparison between strategy 3 (using NSFD method) and SFD method of the optimality system (57) and (63) at time step size $\Delta t = 1$ is provided in Figure 5. In Figure 7. it can be observed that the SFD method is divergent of the optimality system (57) and (63) at time step size $\Delta t = 4$. The cost function $\check{J}(u_1, u_2)$ and the sum of numerical values of E and I at $T = 100$ days at different time step sizes Δt are computed by these implemented methods in Table 8.

Table 6. Comparisons between NSFD and SFD methods for the optimality system (57) and (63) with different time step size Δt when $R_0 > 1$.

Δt	SFD	NSFD
0.05	*convergent*	*convergent*
0.1	*convergent*	*convergent*
0.5	*convergent*	*convergent*
1	*convergent*	*convergent*
5	*divergent*	*convergent*
10	*divergent*	*convergent*
25	*divergent*	*convergent*

Table 7. Comparisons between different strategies of NSFD method for the optimality system (57) and (63) with different time step size Δt, where the total simulation time $T = 100$ days.

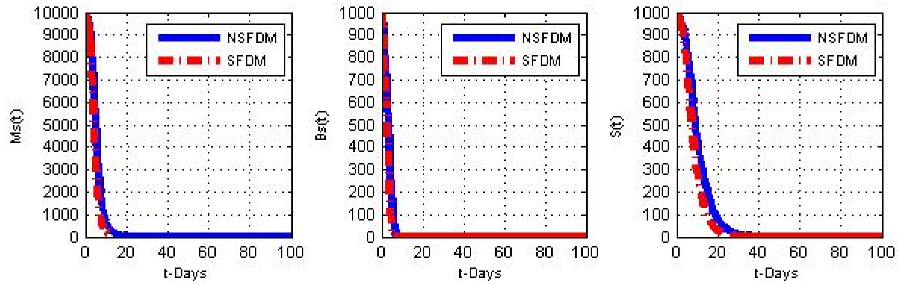
Δt	Methods	$\check{J}(u_1, u_2)$
0.5	NSFD-strategy1	18404
	NSFD-strategy2	33766
	NSFD-strategy3	12440
1	NSFD-strategy1	18254
	NSFD-strategy2	32581
	NSFD-strategy3	12265
2	NSFD-strategy1	17464
	NSFD-strategy2	29012
	NSFD-strategy3	11633
4	NSFD-strategy1	14943
	NSFD-strategy2	21312
	NSFD-strategy3	9804.3
5	NSFD-strategy1	13083
	NSFD-strategy2	17274
	NSFD-strategy3	8511
10	NSFD-strategy1	6918.4
	NSFD-strategy2	6898.3
	NSFD-strategy3	4353.9
25	NSFD-strategy1	1678.3
	NSFD-strategy2	1146.9
	NSFD-strategy3	1044.8

Table 8. Comparisons between NSFD-strategy3 and SFD methods for the optimality system (57) and (63) with different time step size Δt, where the total simulation time $T = 100$ days.

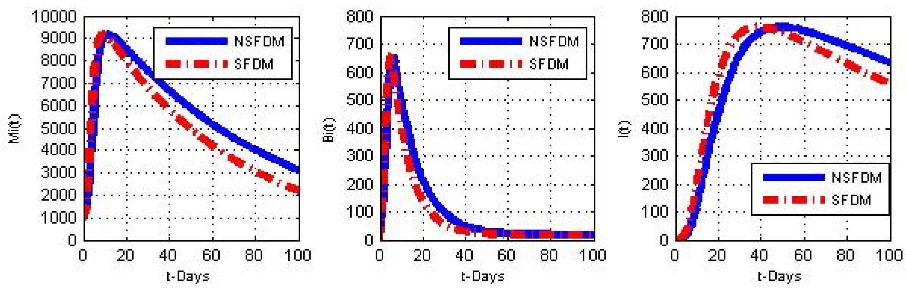
Δt	Methods	$\check{J}(u_1, u_2)$	$E(100) + I(100)$
4	NSFD-strategy3	9804.3	286.2127
	SFD	NaN	NaN
5	NSFD-strategy3	8511	280.6345
	SFD	NaN	NaN
10	NSFD-strategy3	4353.9	202.4972
	SFD	5.4681×10^{54}	3.2809×10^{54}
25	NSFD-strategy3	1044.8	48.8770
	SFD	-7.7748×10^{14}	-1.8659×10^{14}

8. Conclusion

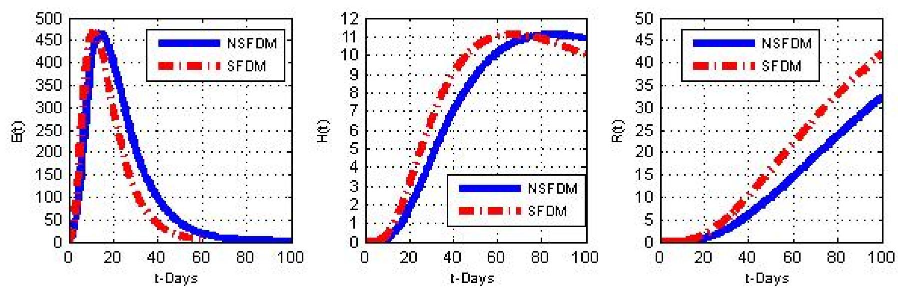
In this paper, numerical studies for the transmission dynamics of WNV mathematical model and it's optimal control are presented. It can be concluded from the numerical results provided that NSFD scheme is more efficient than SFD scheme. It preserves the positivity of the solutions and numerical stability in large regions. The optimal control problem is described by two control functions u_1 and u_2. The



(a) Susceptibles



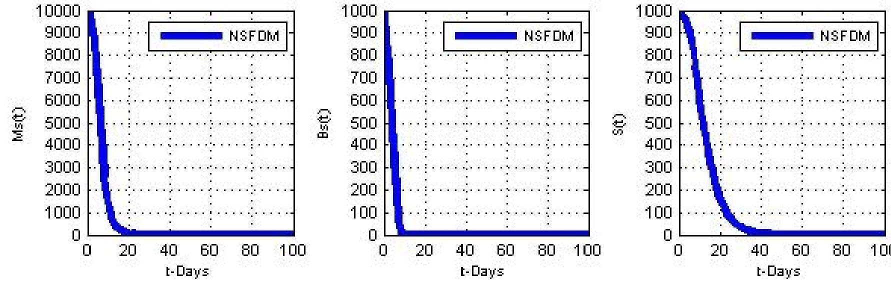
(b) Infected



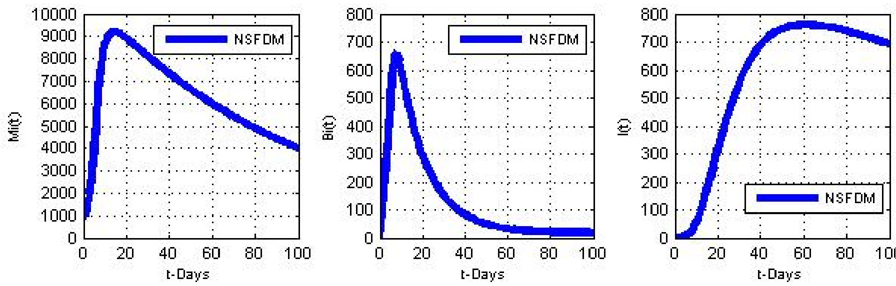
(c) Exposed, Hospitalized and Recovered Humans

FIGURE 1. Numerical simulations of the system (1) when $R_0 > 1$ with time step size $\Delta t = 0.5$ by using NSFD and SFD methods.

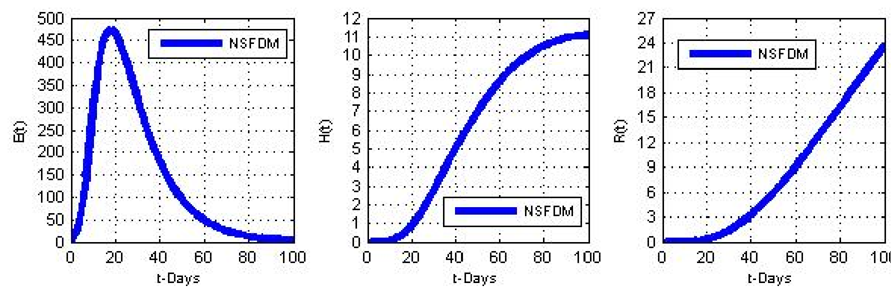
measures for the level of larvacide and adulticide used for mosquito control administered at mosquito breeding sites is represented by u_1 and the measures for the level of successful prevention (personal protection) efforts is represented by u_2. Three optimal control strategies are presented. If we considered only one control, then we have strategy 1 for the first control u_1 and strategy 2 for the second control u_2. When the two controls u_1 and u_2 are considered, this means that we have strategy 3. According to the numerical results, we have the best choice to use strategy 3 Indeed, with strategy 3, there is a lower value of the cost functional $\check{J}(u_1, u_2)$.



(a) Susceptibles



(b) Infected



(c) Exposed, Hospitalized and Recovered Humans

FIGURE 2. Numerical simulations of the system (1) when $R_0 > 1$ with time step size $\Delta t = 1$ by using NSFD method.

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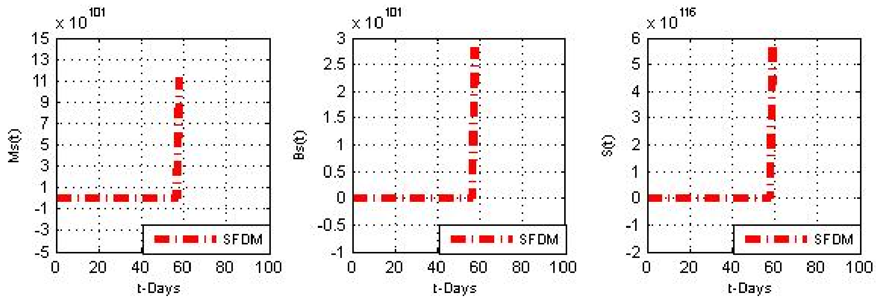
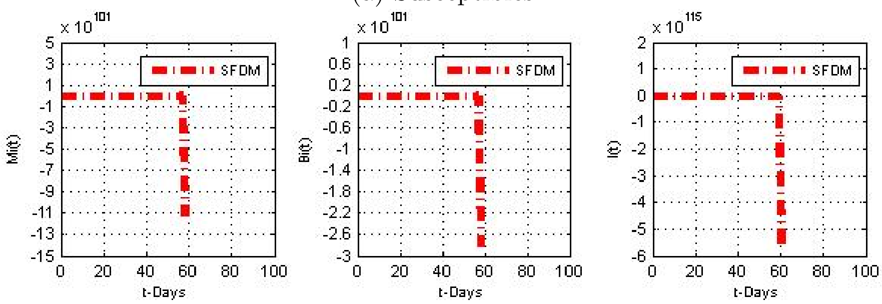
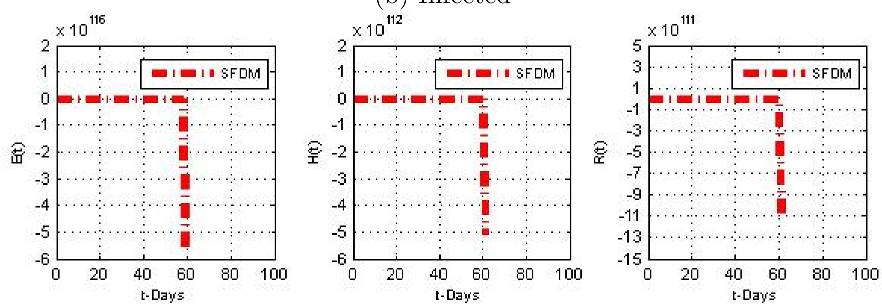


(a) Susceptibles



(b) Infected



(c) Exposed, Hospitalized and Recovered Humans

FIGURE 3. Numerical simulations of the system (1) when $R_0 > 1$ with time step size $\Delta t = 1$ by using SFD method.

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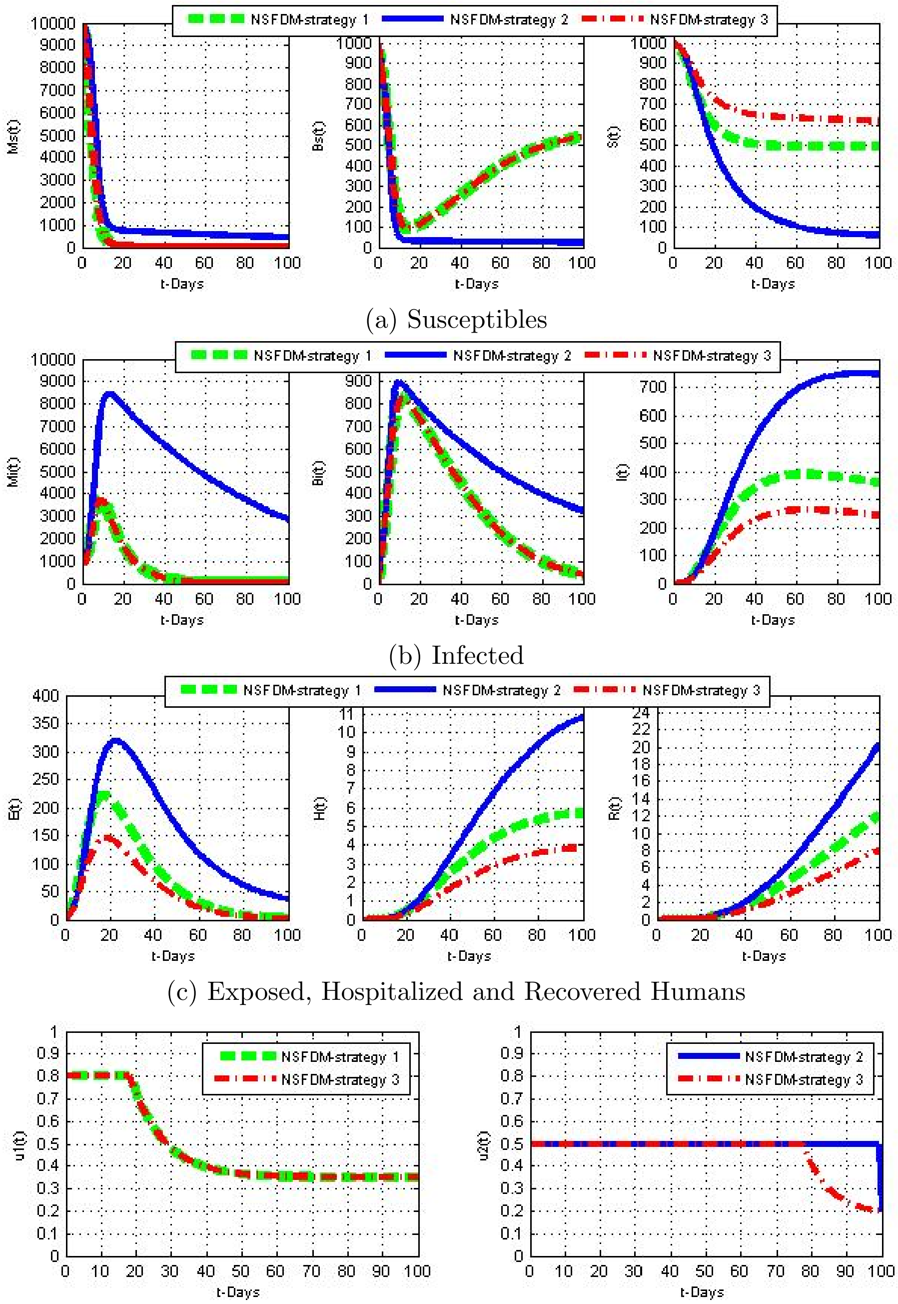


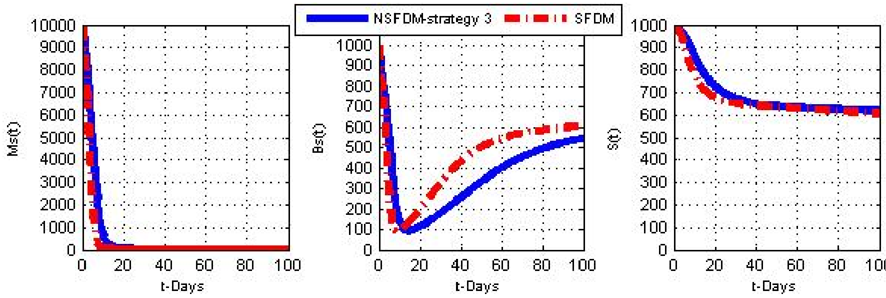
(a) Susceptibles

(b) Infected

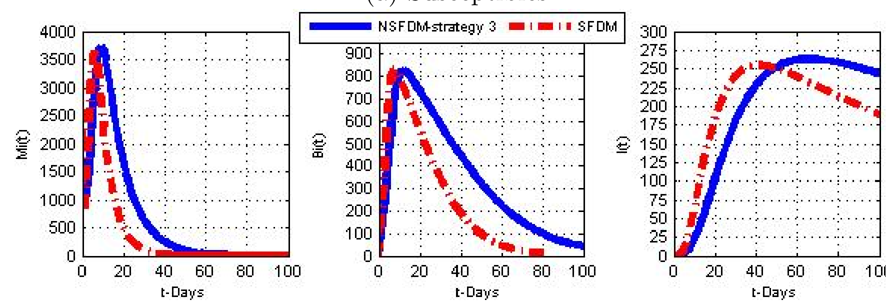
(c) Exposed, Hospitalized and Recovered Humans

(d) Control Functions

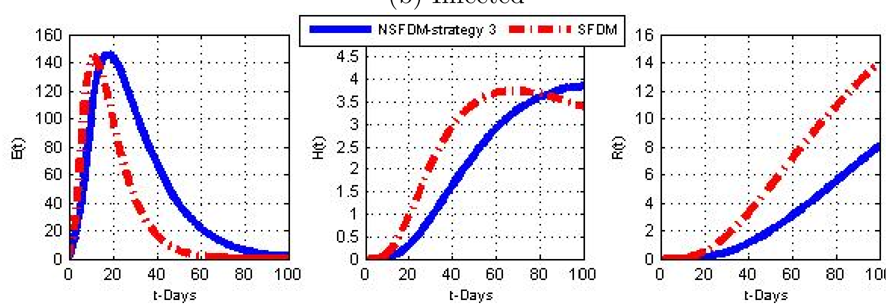
FIGURE 4. Numerical simulations of the optimality system (57) and (63) when $0 \leq u_1 \leq 0.8$, $0 \leq u_2 \leq 0.5$ with time step size $\Delta t = 1$ by using NSFD method, where parameters used are $A_1 = A_2 = 1$, $A_3 = 10^{-4}$, $B_1 = B_2 = 1$.



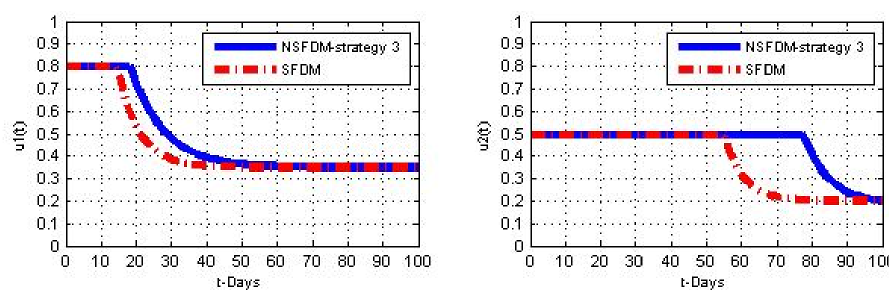
(a) Susceptibles



(b) Infected

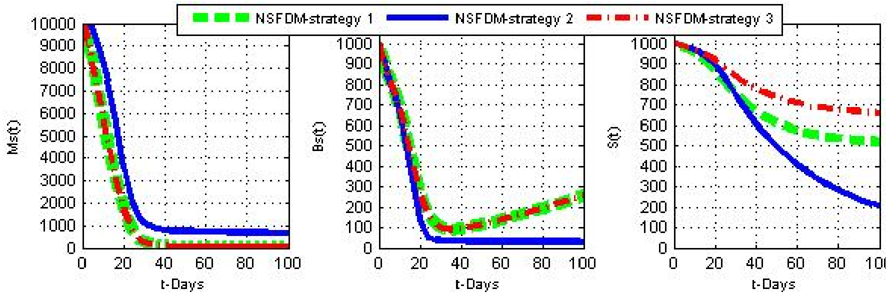


(c) Exposed, Hospitalized and Recovered Humans

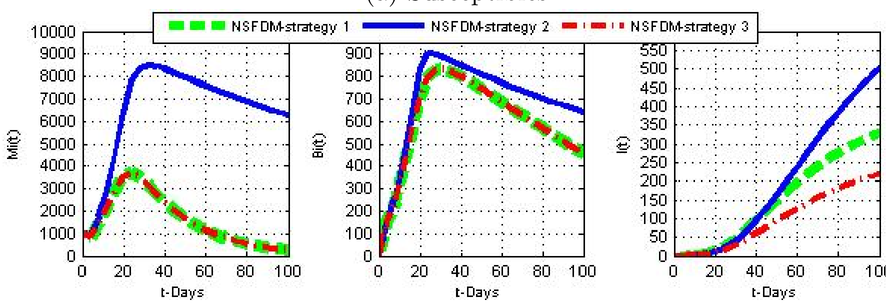


(d) Control Functions

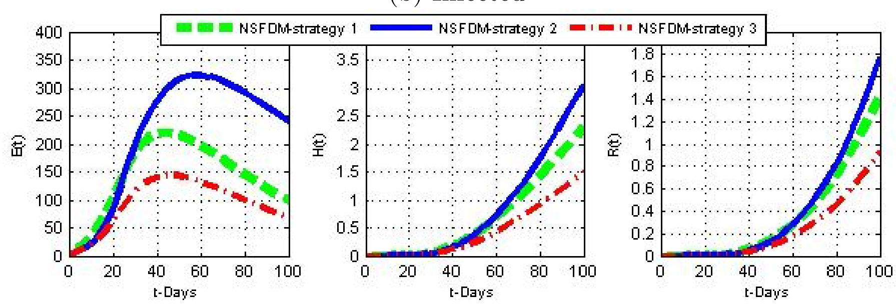
FIGURE 5. Numerical simulations of the optimality system (57) and (63) when $0 \le u_1 \le 0.8$, $0 \le u_2 \le 0.5$ with time step size $\Delta t = 1$ by using NSFD and SFD methods, where parameters used are $A_1 = A_2 = 1$, $A_3 = 10^{-4}$, $B_1 = B_2 = 1$.



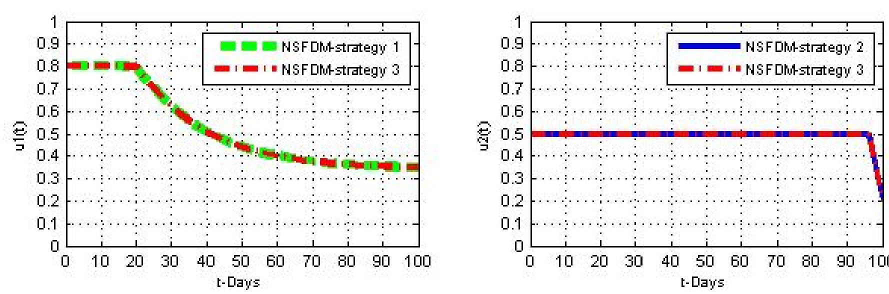
(a) Susceptibles



(b) Infected

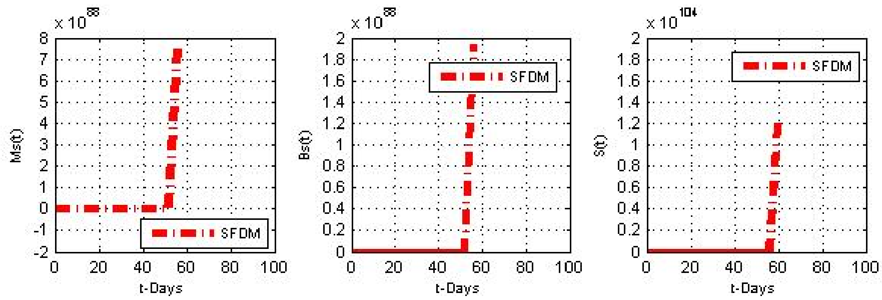


(c) Exposed, Hospitalized and Recovered Humans

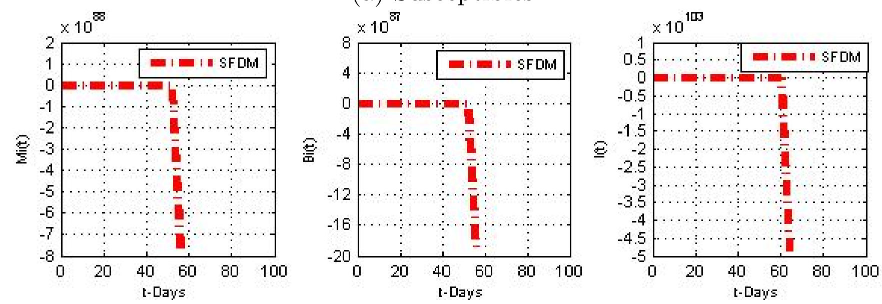


(d) Control Functions

FIGURE 6. Numerical simulations of the optimality system (57) and (63) when $0 \le u_1 \le 0.8$, $0 \le u_2 \le 0.5$ with time step size $\Delta t = 4$ by using NSFD method, where parameters used are $A_1 = A_2 = 1$, $A_3 = 10^{-4}$, $B_1 = B_2 = 1$.



(a) Susceptibles



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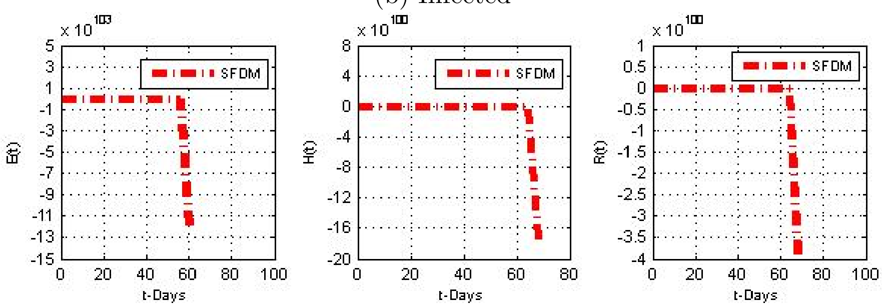

(c) Exposed, Hospitalized and Recovered Humans

FIGURE 7. Numerical simulations of the optimality system (57) and (63) when $0 \le u_1 \le 0.8$, $0 \le u_2 \le 0.5$ with time step size $\Delta t = 4$ by using SFD method, where parameters used are $A_1 = A_2 = 1$, $A_3 = 10^{-4}$, $B_1 = B_2 = 1$.

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