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ON THE FIRST AND SECOND GJMS EIGENVALUES

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Abstract. In this paper, we define the first and the second eigenvalues of the GJMS (Graham-Jenne-Mason-Sparling) operator on compact Einstein manifold with positive scalar curvature. We show the attainability of the corresponding eigenvalues by generalized metrics.

1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, and let k be an integer such that $1 \leq k \leq \frac{n}{2}$ for n even. In 1992, Graham-Jenne-Mason-Sparling defined a family of conformally invariant differential operators (GJMS operators for short). The construction of these operators is based on the ambient metric of Fefferman-Graham (see [9]). More precisely, for any Riemannian metric g on M, there exists a differential operator

$$P_g : C^{\infty}(M) \longrightarrow C^{\infty}(M),$$

such that for all $u \in C^{\infty}(M)$, the GJMS operator P_g is given by :

$$P_g u = \Delta_g^k u + lot$$

where $\Delta_g = -div_g(\nabla)$ is the Laplace-Beltrami operator, and lot denotes differential terms of lower order. For more details, we refer to [12].

Recently, there are some existence results concerning the GJMS-operator (see [9] and [12]). The purpose of this paper is to study the first and the second GJMS eigenvalues on compact Einstein manifold with positive scalar curvature; we seek situations where these eigenvalues are attained by a generalized metric. For more details on similar work with Yamabe operator and Paneitz-Branson operator, we refer the reader to [1], [3],[4] and [5].

Our paper is organized as follows, we begin by giving some properties of the GJMS operator and we show the existence of minimizers for the first and for the second eigenvalues for generalized metrics.

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2. Some GJMS properties

In this section, we give some properties of GJMS operator. For the proof of these properties, the reader is refered to [9] and [12] and references therein.

(1) The operator P_g is elliptic, self-adjoint with respect to the inner product in $L^2(M)$ and has a discrete spectrum :
$$\lambda_1(g) < \lambda_2(g) \leq \lambda_3(g) \leq \cdots \leq \lambda_k(g) \to +\infty$$

(2) For any conformal metric, $\overline{g} = \varphi^{\frac{N-2}{k}} g$ with $n \neq 2k$, $\varphi \in C^\infty(M)$, $\varphi > 0$ and $N = \frac{2n}{n-2k}$ where the number N is the critical exponent of the Sobolev embedding $H_k^2(M) \subset L^N(M)$, the operator P_g is conformally invariant in the following sense:
for all $u \in C^\infty(M)$, we have
$$P_g(u\varphi) = \varphi^{N-1} P_{\overline{g}}(u),$$
By taking $u \equiv 1$, we get
$$\frac{n-2k}{2} Q_{\overline{g}} = P_{\widetilde{g}}(1).$$
Hence
$$P_g(\varphi) = \frac{n-2k}{2} Q_{\overline{g}} \varphi^{N-1}.$$
The quantity Q_g can be seen as the analogue of the scalar curvature for the conformal Laplacian and is called the Q-curvature. When $k = 1$, P_g it is exactly the Yamabe operator and the Q-curvature is the scalar curvature (up to a constant), and when $k = 2$, P_g is the Paneitz-Branson operator.

(3) A Riemannian manifold (M, g) is Einstein if and only if there exists a real number λ such that the Ricci tensor writes
$$Ric_g = \lambda g.$$
Here $\lambda = \frac{S_g}{n}$, where S_g is the scalar curvature and is constant in this case. On Einstein manifold, P_g expresses as an explicit product of second-order operators with constant coefficients that depend only on the scalar curvature. In otherwords, the GJSM operator of order k is given by :
$$P_g = \amalg_{l=1}^{k} (\Delta + c_l S_g),$$
where
$$c_l = \frac{(n+2l-2)(n-2l)}{4n(n-1)}.$$
Moreover, the scalar curvature S_g is positive (see also [12]):

(4) In the case of Einstein manifolds, the operator P_g is coercive.

(5) For all $u \in C^\infty(M)$ such that $P_g u > 0$, either $u > 0$ or $u \equiv 0$ and this statement is a direct consequence of k applications of the second-order comparison principle. (see [12] Proposition 4)

(6) Finally, from property (3),we have $P_g = S(\Delta_g)$ with S a polynomial with positive constant coefficients. It follows from this that the first eigenvalue of P_g is $S(0) > 0$ and P_g satisfies property (4).

(7) It is not difficult to see that the quantity :

$$\int_M uP_g u dv_g = \int_M (\left[\Delta^{\frac{k}{2}}(u)\right]^2 + \sum_{l=0}^{k-1} a_l \left|\nabla^l u\right|^2) dv_g = \|u\|^2_{H^2_k},$$

where $a_l > 0$ is some constant, is a norm on $H^2_k(M)$ wich is equivalent to the standard one :

$$\|u\|^2_{H^2_k} = \int_M \sum_{l=0}^{k} \left|\nabla^l u\right|^2 dv_g$$

Here $H^2_k(M)$ denotes the Sobolev space of functions u such that: u ,$|\nabla u|$, ..., $\left|\nabla^k u\right| \in L^2(M)$. It is well known that by the Sobolev embedding theorem [Heb97] that $H^2_k(M) \subset L^q(M)$ where $1 < q \leq N = \frac{2n}{n-2k}$ and this embedding is compact when $q < N$.

3. Variational characterization of the p^{th} eigenvalue

In this section, we quote some facts which will be used in the sequel of this paper: Grassmannians and the min-max principle.
Let $L^N_+(M) = \{u \in L^N(M), \quad u \geq 0 \quad and \quad u \neq 0\}$

Definition 1. *For all $u \in L^N_+(M)$, we define $Gr^u_p(H^2_k(M))$ as the set of all p-dimensional subspaces ($p \geq 1$) of $H^2_k(M)$ that satisfy*

$$span(v_1, ..., v_p) \in Gr^u_p(H^2_2(M))$$

if and only if $v_1, ..., v_p$ are linearly independent on $M \setminus u^{-1}(0)$.

Definition 2. *A generalized metric conformal to g is a metric of the form $\overline{g} = u^{\frac{N-2}{k}} g$ with $u \in L^N_+$.*

Definition 3. *For any generalized metric $\overline{g} = u^{\frac{N-2}{k}} g$, the p^{th} eigenvalue $\lambda_p(\overline{g})$ of P_g is characterized by (see [1]):*

$$\lambda_p(\overline{g}) = \inf_{V \in Gr^u_p(H^2_k(M))} \sup_{v \in V\setminus\{0\}} \frac{\int_M vP_g v dv_g}{\int_M u^{N-2} v^2 dv_g}, \qquad p \in \mathbb{N}^*.$$

4. The first eigenvalue for generalized metric

Lemma 4. *Let $u \in L^N_+(M)$ and let v_m be a sequence in $H^2_k(M)$ wich converges weackly to u, then*

$$\int_M u^{N-2}(\left|v_m^2 - v^2\right|) dv_g \to 0.$$

Proof. Let A be any large real number and set $u_A = inf(u, A)$. Then $(u_A)_A$ is a monotone sequence, which converges pointwise almost everywhere to u, so by the Lebesgue dominated convergence theorem, we have

$$\int_M (u^{N-2} - u_A^{N-2})^{\frac{N}{N-2}} dv_g \to 0 \text{ when } A \text{ tend to } +\infty.$$

On the other hand, we have

$$\int\limits_M u^{N-2}\left|v_m^2-v^2\right|dv_g \le \int\limits_M u_A^{N-2}\left|v_m^2-v^2\right|dv_g + (\int\limits_M (u^{N-2}-u_A^{N-2})(|v_m|+|v|)^2 dv_g$$

By using Holder inequality, we can write :

$$\int\limits_M u^{N-2}\left|v_m^2-v^2\right|dv_g \le A^{N-2}\int\limits_M \left|v_m^2-v^2\right|dv_g + (\int\limits_M (u^{N-2}-u_A^{N-2})^{\frac{N}{N-2}}dv_g)^{\frac{N-2}{N}}(\int\limits_M (|v_m|+|v|)^N dv_g)^{\frac{2}{N}}$$

Since the sequence v_m is bounded in $H_k^2(M)$ then from Sobolev embedding, the boundedness in $L^N(M)$ is assumed, and hence there exists $C>0$ such that $\int\limits_M (|v_m|+|v|)^N dv_g \le C$. By strong convergence of v_m in $L^2(M)$, we get the result. □

Theorem 5. *For any generalized metric $\overline{g}=u^{\frac{N-2}{k}}g$, there exists a non trivial function v in $H_k^2(M)$ such that in the weak sense, v satisfies :*

$$P_g(v)=\lambda_{1,\overline{g}}u^{N-2}v \tag{1}$$
$$\text{and} \quad \int\limits_M u^{N-2}v^2 dv_g = 1$$

where $\lambda_{1,\overline{g}}$ is the first eigenvalue of P_g for the metric $\overline{g}$. In other words, the first eigenvalue of P_g is attained by v.

Proof. Let (v_m) be a minimizing sequence for $\lambda_{1,\overline{g}}$, i.e a sequence $v_m\in H_k^2(M)$ such that $u^{\frac{N-2}{2}}v_m\neq 0$ and

$$\lim_m \frac{\int_M v_m P_g(v_m)dv_g}{(\int_M |u|^{N-2}v_m^2 dv_g)} = \lambda_{1,\tilde{g}}.$$

Without loss of generality, we can always normalize v_m by $\int_M u^{N-2}v_m^2 dv_g = 1$. Now for a large enough m, we have

$$\|u\|^2_{H_k^2} = \int_M v_m P_g(v_m)dv_g \le \lambda_{1,\tilde{g}}+1,$$

then the sequence (v_m) is bounded in $H_k^2(M)$, and after restriction to a subsequence we may assume that there exists v in $H_k^2(M)$ such that $v_m\to v$ weackly in $H_k^2(M)$, strongly in $H_{k-1}^2(M)$ and in $L^2(M)$ and almost everywhere in M, so that

$$\int_M vP_g(v)dv_g \le \liminf \int_M v_m P_g(v_m)dv_g = \lambda_{1,\tilde{g}}$$

and since $\lambda_{1,\tilde{g}}$ is the infimum, it follows that

$$\int_M vP_g(v)dv_g = \lambda_{1,\tilde{g}},$$

from lemma (4), we get

$$\int\limits_M u^{N-2}(v^2-v_m^2)\,dv_g \to 0 \quad \text{i.e} \quad \int\limits_M u^{N-2}v^2 dv_g = 1.$$

Consequently v is a non-trivial weak minimizer of the functional associated to $\lambda_{1,\tilde{g}}$. Writing the Euler-Lagrange equation, we find that v satisfies the equation

$$P_g(v)=\lambda_{1,\tilde{g}}u^{N-2}v.$$

Moreover, we can also obtain the sign of the first eingenvalue in this case i.e :

$$\lambda_{1,\tilde{g}} = \|v\|^2_{H^2_k} > 0.$$

□

Proposition 6. *If $u \in C^\infty_+(M)$, then the solution of equation (1) $v \in C^{2k}(M)$.*

Proof. We have $\lambda_{1,\overline{g}} u^{N-2} v \in H^2_k(M)$, $P_g(v) \in H^2_k(M)$ and by regularity theorems $v \in H^2_{3k}(M)$, it follows by successive iterations that $v \in H^2_l(M)$ where l is large enough and finally if $\frac{1}{2} < \frac{l-m}{n}$,

$$H^2_l(M) \subset C^m(M)$$

so we can take $m = 2k$ i.e

$$v \in C^{2k}(M).$$

□

Now, we are going to show that the equation (1) has a positive solution.

Proposition 7. *Let v be the solution of equation (1), there exists a non trivial positive function f in C^{2k}, such that*

$$P_g(f) = \lambda_{1,\overline{g}} u^{N-2} f \quad \textit{and} \quad \int_M u^{N-2} f^2 dv_g = 1.$$

In other word, we can say that the first eingenvalue $\lambda_{1,\overline{g}}$ is attained by a $C^{2k}(M)$ positive function.

Proof. Let v be a solution of (1) and let f be the solution of the equation

$$P_g(f) = |P_g(v)|\,,$$

we can show the existence of the function f by using the factorization of GJMS as :

$$P_g = \amalg^k_{l=1}(\Delta + c_l S_g)$$

where c_l are positive, so all operators $\Delta + c_l S_g$ are invertible and applying strong maximum principle for elliptic equations of second order for k times(see [12] Proposition 4), we show that $f > |v| > 0$ and by regularity $f \in C^{2k}(M)$. Let A be a real number such that $0 < A \leq 1$ and $\int_M (Af)^2 u^{N-2} dv_g = 1$, then

$$\int_M (Af) P_g(Af) dv_g - \lambda_{1,\tilde{g}} = A^2 \int_M (f)\,|P_g(v)|\,dv_g - \lambda_{1,\tilde{g}}$$

$$= A^2\,|\lambda_{1,\tilde{g}}| \int_M (f) u^{N-2}\,|v|\,dv_g - \lambda_{1,\tilde{g}} = A\,|\lambda_{1,\tilde{g}}| \int_M (Af) u^{\frac{N-2}{2}} u^{\frac{N-2}{2}}\,|v|\,dv_g - \lambda_{1,\tilde{g}}$$

$$\leq A\,|\lambda_{1,\tilde{g}}| \left[\int_M (Af)^2 u^{N-2} dv_g\right]^{\frac{1}{2}} \left[\int_M (v)^2 u^{N-2} dv_g\right]^{\frac{1}{2}} - \lambda_{1,\tilde{g}}$$

$$\leq (A-1)\lambda_{1,\tilde{g}} \quad \text{as} \quad \lambda_{1,\tilde{g}} > 0.$$

Thus,

$$\int_M (Af)P_g(Af)dv_g \le \lambda_{1,\tilde{g}}$$

and as $\lambda_{1,\tilde{g}}$ is the infimum, we get equality i.e Af is a positive solution of $P_g(v) = \lambda_{1,\tilde{g}} u^{N-2} v$. $\square$

5. The second eigenvalue for generalized metric

Proposition 8. *Let v be the solution of equation (1). Then the set :*

$$E = \{ \quad w \in H_k^2(M) \quad \textit{such that } u^{\frac{N-2}{2}} w \neq 0 \,, \int_M u^{N-2} w^2 dv_g = 1 \textit{ and } \int_M u^{N-2} wv dv_g = 0\},$$

is not empty.

Proof. Let $v, s \in H_k^2(M)$ non-collinear functions, by multiplying if necessary v and s by certain constants, we assume that : $\int_M u^{N-2} v^2 dv_g = \int_M u^{N-2} s^2 dv_g = 1$, and thus $u^{\frac{N-2}{2}} v \neq 0$ and $u^{\frac{N-2}{2}} s \neq 0$. We set

$$w = \alpha v + \beta s$$

where α, β are real numbers.
Now we are going to find α, β such that $w \in E$. We begin by multiplying w by $u^{N-2} v$ and we integrate :

$$\int_M u^{N-2} wv dv_g = \alpha + \beta \int_M u^{N-2} sv dv_g = 0$$

i.e

$$\beta = -\frac{\alpha}{\int_M u^{N-2} sv dv_g}.$$

If $\int_M u^{N-2} sv dv_g = 0$, then $s \in E$ and E is not empty and if $\int_M u^{N-2} sv dv_g \neq 0$, hence β is well defined.

By the equality $\int_M u^{N-2} w^2 dv_g = 1$, we obtain: $\int_M u^{N-2} (\alpha v + \beta s)^2 dv_g = 1$ i.e

$$\alpha^2 + \beta^2 + 2\alpha\beta \int_M u^{N-2} sv dv_g = 1,$$

therefore

$$\alpha = \pm \frac{\int_M u^{N-2} sv dv_g}{(1 - [\int_M u^{N-2} sv dv_g]^2)^{\frac{1}{2}}}$$

then the number α is also well defined because $\int_M u^{N-2} sv dv_g < 1$ is always true. In fact, if $\int_M u^{N-2} sv dv_g \ge 1$, Holder inequality implies that

$$1 \le \int_M u^{N-2} sv dv_g = \int_M u^{\frac{N-2}{2}} s \; u^{\frac{N-2}{2}} v dv_g \le [\int_M u^{N-2} v^2 dv_g]^{\frac{1}{2}} [\int_M u^{N-2} s^2 dv_g]^{\frac{1}{2}} \le 1,$$

then there is equality in Holder inequality, and this is possible if and only if there is a real constant c such that $v = cs$, hence v and s are colinear and we get a contradiction. $\square$

Proposition 9. *Let u and v two functions as in the theorem (5), then there exists a fonction w in $H_k^2(M)$ solution in the weak sense of the equation*

$$P_g(w) = \lambda'_{2,g} u^{N-2} w,$$

such that $\int\limits_M u^{N-2} w^2 dv_g = 1$ *and* $\int\limits_M u^{N-2} wv dv_g = 0$ *where*

$$\lambda'_{2,g} = \inf_E \frac{\int_M vP_g v dv_g}{\int_M u^{N-2} v^2 dv_g}.$$

Proof. let (w_m) be a minimizing sequence for $\lambda'_{2,g}$ i.e $w_m \in E$ is such that :

$$\lim_m \frac{\int_M w_m P_g(w_m) dv_g}{(\int_M u^{N-2} w_m^2 dv_g)} = \lambda'_{2,g},$$

with the same proof of theorem (5), we can find $w \in C^{2k}(M)$ solution of $P_g(w) = \lambda'_{2,g} u^{N-2} w$ such that $\int\limits_M u^{N-2} w^2 = 1$.

Now writing

$$\begin{aligned}\int\limits_M u^{N-2} wv dv_g &= \int\limits_M u^{N-2} w_m v - u^{N-2} w_m v + u^{N-2} wv dv_g \\ &= \int\limits_M u^{N-2} v(w - w_m) dv_g + \int\limits_M u^{N-2} w_m v dv_g = 0.\end{aligned}$$

As the sequence $w_m \in E$, $\int\limits_M u^{N-2} w_m v dv_g = 0$, and using the weak convergence of w_m to w in $L^N(M)$, we get

$$\int\limits_M u^{N-2} v(w - w_m) dv_g \to 0 \quad (u^{N-2} v \in L^{\frac{N}{N-1}} \text{ dual space of } L^N(M)).$$

$\square$

Proposition 10. *We have*

$$\lambda'_{2,g} = \lambda_{2,\tilde{g}}$$

Proof. Since $w \in E$, the functions $u^{\frac{N-2}{2}} v$, $u^{\frac{N-2}{2}} w$ are linearly independent, then the space

$$V_0 = span(v, w) \in Gr_2^u(H_k^2(M)).$$

Putting $f = \alpha v + \beta w$ with α, β are non-zero real numbers, we evaluate

$$s = \frac{\int_M f P_g(f) dv_g}{\int_M u^{N-2} f^2 dv_g} \quad over \quad V_0$$

we find

$$s = \frac{\alpha^2 \int_M v L_g(v) dv_g + \beta^2 \int_M w P_g(w) dv_g}{\alpha^2 + \beta^2}$$

$$s = \frac{\alpha^2}{\alpha^2 + \beta^2} \lambda_{1,\tilde{g}} + \frac{\beta^2}{\alpha^2 + \beta^2} \lambda'_{2,\tilde{g}}$$

and as $\frac{\alpha^2}{\alpha^2+\beta^2}+\frac{\beta^2}{\alpha^2+\beta^2}=1$, so for $\theta\in\mathbb{R}$, we can get :

$$s=\lambda_{1,\tilde{g}}\cos^2\theta+\lambda'_{2,\tilde{g}}\sin^2\theta.$$

On the other hand

$$\frac{ds}{d\theta}=(\lambda'_{2,\tilde{g}}-\lambda_{1,\tilde{g}})\sin 2\theta,$$

and taking into account that:

$$\lambda_{1,\tilde{g}}=\inf_{H_2^2}\ \leq\lambda'_{2,\tilde{g}}=\inf_{E}\qquad\text{because } (E\subset H_k^2(M)).$$

We get that $\lambda_{1,\tilde{g}}$ is a minimum of $s(\theta)$ and $\lambda'_{2,\tilde{g}}(\theta)$ is a maximum of s,

$$\lambda'_{2,\tilde{g}}\quad=\quad\sup_{f\in V_0}\frac{\int_M fP_g(f)dv_g}{\int_M u^{N-2}f^2dv_g},$$

and as the infimum of the quantity $\sup\limits_{f\in V_0}\frac{\int_M fP_g(f)dv_g}{\int_M u^{N-2}f^2dv_g}$ on all elements of $Gr_2^u(H_k^2(M))$ is attained over V_0, it folows

$$\lambda'_{2,\tilde{g}}=\inf_{V\in Gr_2^u(H_k^2(M))}\quad\sup_{v\in V\setminus\{0\}}\frac{\int\limits_M vP_gvdv_g}{\int\limits_M u^{N-2}v^2dv_g}=\lambda_{2,\tilde{g}}.$$

□

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