ON THE FIRST AND SECOND GJMS EIGENVALUES

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Abstract. In this paper, we define the first and the second eigenvalues of the GJMS (Graham-Jenne-Mason-Sparling) operator on compact Einstein manifold with positive scalar curvature. We show the attainability of the corresponding eigenvalues by generalized metrics.

1. Introduction

Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$, and let $k$ be an integer such that $1 \leq k \leq \frac{n}{2}$ for $n$ even. In 1992, Graham-Jenne-Mason-Sparling defined a family of conformally invariant differential operators (GJMS operators for short). The construction of these operators is based on the ambient metric of Fefferman-Graham (see [9]). More precisely, for any Riemannian metric $g$ on $M$, there exists a differential operator $P_g : C^\infty(M) \to C^\infty(M)$, such that for all $u \in C^\infty(M)$, the GJMS operator $P_g$ is given by:

$$P_g u = \Delta_g^k u + lot$$

where $\Delta_g = -\text{div}_g(\nabla)$ is the Laplace-Beltrami operator, and $lot$ denotes differential terms of lower order. For more details, we refer to [12].

Recently, there are some existence results concerning the GJMS-operator (see [9] and [12]). The purpose of this paper is to study the first and the second GJMS eigenvalues on compact Einstein manifold with positive scalar curvature; we seek situations where these eigenvalues are attained by a generalized metric. For more details on similar work with Yamabe operator and Paneitz-Branson operator, we refer the reader to [1], [3],[4] and [5].

Our paper is organized as follows, we begin by giving some properties of the GJMS operator and we show the existence of minimizers for the first and for the second eigenvalues for generalized metrics.

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2. Some GJMS properties

In this section, we give some properties of GJMS operator. For the proof of these properties, the reader is referred to [9] and [12] and references therein.

(1) The operator $P_g$ is elliptic, self-adjoint with respect to the inner product in $L^2(M)$ and has a discrete spectrum:
$$\lambda_1(g) < \lambda_2(g) \leq \cdots \leq \lambda_k(g) \to +\infty$$

(2) For any conformal metric, $g = \varphi^{\frac{4}{n-2}}g$ with $n \neq 2k$, $\varphi \in C^\infty(M)$, $\varphi > 0$ and $N = \frac{2n}{n-2k}$ where the number $N$ is the critical exponent of the Sobolev embedding $H^2_k(M) \subset L^N(M)$, the operator $P_g$ is conformally invariant in the following sense: for all $u \in C^\infty(M)$, we have
$$P_g(u\varphi) = \varphi^{N-1}P_g(u),$$
By taking $u \equiv 1$, we get
$$\frac{n-2k}{2}Q_g = P_g(1).$$
Hence
$$P_g(\varphi) = \frac{n-2k}{2}Q_g\varphi^{N-1}.$$  

The quantity $Q_g$ can be seen as the analogue of the scalar curvature for the conformal Laplacian and is called the Q-curvature. When $k = 1$, $P_g$ is exactly the Yamabe operator and the Q-curvature is the scalar curvature (up to a constant), and when $k = 2$, $P_g$ is the Paneitz-Branson operator.

(3) A Riemannian manifold $(M, g)$ is Einstein if and only if there exists a real number $\lambda$ such that the Ricci tensor writes
$$\text{Ric}_g = \lambda g.$$  

Here $\lambda = \frac{S_g}{n}$, where $S_g$ is the scalar curvature and is constant in this case. On Einstein manifold, $P_g$ expresses as an explicit product of second-order operators with constant coefficients that depend only on the scalar curvature. In otherwords, the GJSM operator of order $k$ is given by:
$$P_g = \Pi_{l=1}^k(\Delta + c_lS_g),$$
where
$$c_l = \frac{(n+2l-2)(n-2l)}{4n(n-1)}.$$  

Moreover, the scalar curvature $S_g$ is positive (see also [12]):

(4) In the case of Einstein manifolds, the operator $P_g$ is coercive.

(5) For all $u \in C^\infty(M)$ such that $P_gu > 0$, either $u > 0$ or $u \equiv 0$ and this statement is a direct consequence of $k$ applications of the second-order comparison principle. (see [12] Proposition 4)

(6) Finally, from property (3), we have $P_g = S(\Delta_g)$ with $S$ a polynomial with positive constant coefficients. It follows from this that the first eigenvalue of $P_g$ is $S(0) > 0$ and $P_g$ satisfies property (4).
(7) It is not difficult to see that the quantity:
\[ \int_M u \mu P_g u dv_g = \int_M \left( \Delta^2 (u) \right)^k + \sum_{l=0}^{k-1} a_l \left| \nabla^l u \right|^2 dv_g = \|u\|^2_{H^2_k}, \]
where \(a_l > 0\) is some constant, is a norm on \(H^2_k(M)\) which is equivalent to the standard one:
\[ \|u\|^2_{H^2_k} = \int_M \sum_{l=0}^k \left| \nabla^l u \right|^2 dv_g. \]
Here \(H^2_k(M)\) denotes the Sobolev space of functions \(u\) such that: \(u, |\nabla u|, ..., |\nabla^k u| \in L^2(M)\). It is well known that by the Sobolev embedding theorem [Heb97] that \(H^2_k(M) \subset L^q(M)\) where \(1 < q \leq N = \frac{2n}{n-2k}\) and this embedding is compact when \(q < N\).

3. Variational characterization of the \(p^{th}\) eigenvalue

In this section, we quote some facts which will be used in the sequel of this paper: Grassmannians and the min-max principle.

Let \(L^N_+(M) = \{u \in L^N(M), \ u \geq 0 \text{ and } u \neq 0\}\)

**Definition 1.** For all \(u \in L^N_+(M)\), we define \(Gr^u_p(H^2_k(M))\) as the set of all \(p\)-dimensional subspaces \((p \geq 1)\) of \(H^2_k(M)\) that satisfy
\[ \text{span}(v_1, ..., v_p) \in Gr^u_p(H^2_k(M)) \]
if and only if \(v_1, ..., v_p\) are linearly independent on \(M \setminus u^{-1}(0)\).

**Definition 2.** A generalized metric conformal to \(g\) is a metric of the form \(\bar{g} = u^{\frac{N-2}{2}} g\) with \(u \in L^N_+\).

**Definition 3.** For any generalized metric \(\bar{g} = u^{\frac{N-2}{2}} g\), the \(p^{th}\) eigenvalue \(\lambda_p(\bar{g})\) of \(P_g\) is characterized by (see [1]):
\[ \lambda_p(\bar{g}) = \inf_{V \in Gr^u_p(H^2_k(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v P_g v dv_g}{\int_M u^{N-2} v^2 dv_g}, \quad p \in \mathbb{N}^*. \]

4. The first eigenvalue for generalized metric

**Lemma 4.** Let \(u \in L^N_+(M)\) and let \(v_m\) be a sequence in \(H^2_k(M)\) which converges weakly to \(u\), then
\[ \int_M u^{N-2} (|v_m^2 - v^2|) dv_g \to 0. \]

**Proof.** Let \(A\) be any large real number and set \(u_A = \inf(u, A)\). Then \((u_A)_A\) is a monotone sequence, which converges pointwise almost everywhere to \(u\), so by the Lebesgue dominated convergence theorem, we have
\[ \int_M (u^{N-2} - u_A^{N-2}) \frac{u^2 v^2}{u_A^2} dv_g \to 0 \quad \text{when} \ A \ \text{tend to} \ + \infty. \]
On the other hand, we have
\[
\int_M u^{N-2} |v_m^2 - v^2| \, dv_g \leq \int_M u^{N-2} |v_m^2 - v^2| \, dv_g + \left( \int_M (u^{N-2} - u_A^{N-2})(|v_m| + |v|)^2 \, dv_g \right)
\]
By using Holder inequality, we can write:
\[
\int_M u^{N-2} |v_m^2 - v^2| \, dv_g \leq A^{N-2} \int_M |v_m^2 - v^2| \, dv_g + \left( \int_M (u^{N-2} - u_A^{N-2}) \frac{N}{N-2} dv_g \right)^{\frac{N-2}{N}} \left( \int_M (|v_m| + |v|)^N \, dv_g \right)^{\frac{1}{N}}
\]
Since the sequence \(v_m\) is bounded in \(H^2(M)\) then from Sobolev embedding, the boundedness in \(L^N(M)\) is assumed, and hence there exists \(C > 0\) such that \(\int_M (|v_m| + |v|)^N \, dv_g \leq C\). By strong convergence of \(v_m\) in \(L^2(M)\), we get the result. \(\Box\)

**Theorem 5.** For any generalized metric \(\overline{g} = u^{\frac{N-2}{2}} \, g\), there exists a non trivial function \(v\) in \(H^2(M)\) such that in the weak sense, \(v\) satisfies:

\[
P_g(v) = \lambda_{1,\overline{g}} u^{N-2}v \quad \text{(1)}
\]
and \(\int_M u^{N-2}v^2 \, dv_g = 1\)

where \(\lambda_{1,\overline{g}}\) is the first eigenvalue of \(P_g\) for the metric \(\overline{g}\). In other words, the first eigenvalue of \(P_g\) is attained by \(v\).

**Proof.** Let \((v_m)\) be a minimizing sequence for \(\lambda_{1,\overline{g}}\), i.e a sequence \(v_m \in H^2(M)\) such that \(u^{\frac{N-2}{2}} v_m \neq 0\) and

\[
\lim_m \int_M v_m P_g(v_m) \, dv_g = \lambda_{1,\overline{g}}.
\]
Without loss of generality, we can always normalize \(v_m\) by \(\int_M u^{N-2}v_m^2 \, dv_g = 1\).

Now for a large enough \(m\), we have

\[
\|u\|_{H^2}^2 = \int_M v_m P_g(v_m) \, dv_g \leq \lambda_{1,\overline{g}} + 1,
\]
then the sequence \((v_m)\) is bounded in \(H^2(M)\), and after restriction to a subsequence we may assume that there exists \(v\) in \(H^2(M)\) such that \(v_m \rightarrow v\) weakly in \(H^2(M)\), strongly in \(H^2_{k-1}(M)\) and in \(L^2(M)\) and almost everywhere in \(M\), so that

\[
\int_M v P_g(v) \, dv_g \leq \lim inf \int_M v_m P_g(v_m) \, dv_g = \lambda_{1,\overline{g}}
\]
and since \(\lambda_{1,\overline{g}}\) is the infimum, it follows that

\[
\int_M v P_g(v) \, dv_g = \lambda_{1,\overline{g}},
\]
from lemma (4), we get

\[
\int_M u^{N-2}(v^2 - v_m^2) \, dv_g \rightarrow 0 \quad \text{i.e.} \quad \int_M u^{N-2}v^2 \, dv_g = 1.
\]
Consequently \(v\) is a non-trivial weak minimizer of the functional associated to \(\lambda_{1,\overline{g}}\). Writing the Euler-Lagrange equation, we find that \(v\) satisfies the equation

\[
P_g(v) = \lambda_{1,\overline{g}} u^{N-2}v.
\]
Moreover, we can also obtain the sign of the first eigenvalue in this case i.e:

\[ \lambda_{1, \tilde{g}} = \| v \|_{H^2_k}^2 > 0. \]

\[ \square \]

**Proposition 6.** If \( u \in C^{\infty}_c(M) \), then the solution of equation (1) \( v \in C^{2k}(M) \).

**Proof.** We have \( \lambda_{1, \tilde{g}} u^{N-2} \in H^2_k(M) \), \( P_g(v) \in H^2_k(M) \) and by regularity theorems \( v \in H^2_{3k}(M) \), it follows by successive iterations that \( v \in H^2_l(M) \) where \( l \) is large enough and finally if \( \frac{1}{2} < \frac{l-m}{n} \),

\[ H^2_l(M) \subset C^m(M) \]

so we can take \( m = 2k \) i.e

\[ v \in C^{2k}(M). \]

\[ \square \]

Now, we are going to show that the equation (1) has a positive solution.

**Proposition 7.** Let \( v \) be the solution of equation (1), there exists a non-trivial positive function \( f \) in \( C^{2k}(M) \), such that

\[ P_g(f) = \lambda_{1, \tilde{g}} u^{N-2} f \quad \text{and} \quad \int_M u^{N-2} f^2 \, dv_g = 1. \]

In other words, we can say that the first eigenvalue \( \lambda_{1, \tilde{g}} \) is attained by a \( C^{2k}(M) \) positive function.

**Proof.** Let \( v \) be a solution of (1) and let \( f \) be the solution of the equation

\[ P_g(f) = \| P_g(v) \|, \]

we can show the existence of the function \( f \) by using the factorization of GJMS as:

\[ P_g = \Pi_{l=1}^k (\Delta + c_l S_g) \]

where \( c_l \) are positive, so all operators \( \Delta + c_l S_g \) are invertible and applying strong maximum principle for elliptic equations of second order for \( k \) times (see [12] Proposition 4), we show that \( f > |v| > 0 \) and by regularity \( f \in C^{2k}(M) \). Let \( A \) be a real number such that \( 0 < A \leq 1 \) and \( \int_M (Af)^2 u^{N-2} dv_g = 1 \), then

\[ \int_M (Af) P_g(Af) \, dv_g - \lambda_{1, \tilde{g}} = A^2 \int_M (f) |P_g(v)| \, dv_g - \lambda_{1, \tilde{g}} \]

\[ = A^2 |\lambda_{1, \tilde{g}}| \int_M (f) u^{N-2} |v| \, dv_g - \lambda_{1, \tilde{g}} = A |\lambda_{1, \tilde{g}}| \int_M (Af) u^{\frac{N-2}{2}} u^{\frac{N-2}{2}} |v| \, dv_g - \lambda_{1, \tilde{g}} \]

\[ \leq A |\lambda_{1, \tilde{g}}| \left[ \int_M (Af)^2 u^{N-2} \, dv_g \right]^{\frac{1}{2}} \left[ \int_M (v)^2 u^{N-2} \, dv_g \right]^{\frac{1}{2}} - \lambda_{1, \tilde{g}} \]

\[ \leq (A - 1)\lambda_{1, \tilde{g}} \quad \text{as} \quad \lambda_{1, \tilde{g}} > 0. \]
Thus,
\[ \int_M (Af)P_\varrho(Af) dv_g \leq \lambda_1, \]
and as \( \lambda_1, \varrho \) is the infimum, we get equality i.e \( Af \) is a positive solution of \( P_\varrho(v) = \lambda_1, \varrho u^{-2}v \).

5. The second eigenvalue for generalized metric

**Proposition 8.** Let \( v \) be the solution of equation (1). Then the set :

\[ E = \{ w \in H^2_k(M) \text{ such that } u^{N-2}w \neq 0, \int_M u^{N-2}w^2 dv_g = 1 \text{ and } \int_M u^{N-2}wvdv_g = 0 \}, \]

is not empty.

**Proof.** Let \( v, s \in H^2_k(M) \) non-collinear functions, by multiplying if necessary \( v \) and \( s \) by certain constants, we assume that :

\[ \int_M u^{N-2}v^2 dv_g = 1, \]

and thus \( u^{N-2}v \neq 0 \) and \( u^{N-2}s \neq 0 \). We set

\[ w = \alpha v + \beta s \]

where \( \alpha, \beta \) are real numbers.

Now we are going to find \( \alpha, \beta \) such that \( w \in E \). We begin by multiplying \( w \) by \( u^{N-2}v \) and we integrate :

\[ \int_M u^{N-2}wvdv_g = \alpha \int_M u^{N-2}svdvg = 0 \]

i.e

\[ \beta = -\frac{\alpha}{\int_M u^{N-2}svdvg}. \]

If \( \int_M u^{N-2}svdvg = 0 \), then \( s \in E \) and \( E \) is not empty and if \( \int_M u^{N-2}svdvg \neq 0 \), hence \( \beta \) is well defined.

By the equality \( \int_M u^{N-2}w^2 dv_g = 1 \), we obtain: \( \int_M u^{N-2}(\alpha v + \beta s)^2 dv_g = 1 \) i.e

\[ \alpha^2 + \beta^2 + 2\alpha\beta \int_M u^{N-2}svdvg = 1, \]

therefore

\[ \alpha = \pm \frac{\int_M u^{N-2}svdvg}{(1 - [\int_M u^{N-2}svdvg]^2)^{\frac{1}{2}}} \]

then the number \( \alpha \) is also well defined because \( \int_M u^{N-2}svdvg < 1 \) is always true. In fact, if \( \int_M u^{N-2}svdvg \geq 1 \), Holder inequality implies that

\[ 1 \leq \int_M u^{N-2}svdvg = \int_M u^{\frac{N-2}{2}}s u^{\frac{N-2}{2}}vdvg \leq [\int_M u^{N-2}v^2 dv_g]^{\frac{1}{2}}[\int_M u^{N-2}s^2 dv_g]^{\frac{1}{2}} \leq 1, \]
then there is equality in Holder inequality, and this is possible if and only if there is a real constant $c$ such that $v = cs$, hence $v$ and $s$ are colinear and we get a contradiction.

**Proposition 9.** Let $u$ and $v$ two functions as in the theorem (5), then there exists a function $w$ in $H^2_k(M)$ solution in the weak sense of the equation

$$P_g(w) = \lambda'_{2,g} u^{N-2}w,$$

such that $\int_M u^{N-2}w^2dv_g = 1$ and $\int_M u^{N-2}vwdv_g = 0$ where

$$\lambda'_{2,g} = \inf_E \frac{\int_M \nabla P_g(v)dv_g}{\int_M u^{N-2}v^2dv_g}.$$

**Proof.** Let $(w_m)$ be a minimizing sequence for $\lambda'_{2,g}$ i.e $w_m \in E$ is such that:

$$\lim_m \int_M w_mP_g(w_m)dv_g = \lambda'_{2,g},$$

with the same proof of theorem (5), we can find $w \in C^2(M)$ solution of $P_g(w) = \lambda'_{2,g} u^{N-2}w$ such that $\int_M u^{N-2}w^2 = 1$.

Now writing

$$\int_M u^{N-2}wvdv_g = \int_M u^{N-2}w_m v - u^{N-2}w_m v + u^{N-2}wvdv_g$$

$$= \int_M u^{N-2}v(w - w_m)dv_g + \int_M u^{N-2}w_mvdv_g = 0.$$

As the sequence $w_m \in E$, $\int_M u^{N-2}w_mvdv_g = 0$, and using the weak convergence of $w_m$ to $w$ in $L^N(M)$, we get

$$\int_M u^{N-2}v(w - w_m)dv_g \to 0 \quad (u^{N-2}v \in L^{\frac{N}{N-2}} \text{ dual space of } L^N(M)).$$

**Proposition 10.** We have

$$\lambda'_{2,g} = \lambda_{2,\tilde{g}}.$$

**Proof.** Since $w \in E$, the functions $u^{\frac{N-2}{2}}v$, $u^{\frac{N-2}{2}}w$ are linearly independent, then the space

$$V_0 = \text{span}(v, w) \in \text{Gr}_2^k(H^2_k(M)).$$

Putting $f = \alpha v + \beta w$ with $\alpha, \beta$ are non-zero real numbers, we evaluate

$$s = \frac{\int_M fP_g(f)dv_g}{\int_M u^{N-2}f^2dv_g} \quad \text{over} \quad V_0$$

we find

$$s = \frac{\alpha^2 \int_M vL_g(v)dv_g + \beta^2 \int_M wP_g(w)dv_g}{\alpha^2 + \beta^2}$$

$$s = \frac{\alpha^2}{\alpha^2 + \beta^2} \lambda_{1,\tilde{g}} + \frac{\beta^2}{\alpha^2 + \beta^2} \lambda'_{2,\tilde{g}}.$$
and as \( \frac{\alpha^2}{\alpha^2 + \beta^2} + \frac{\beta^2}{\alpha^2 + \beta^2} = 1 \), so for \( \theta \in \mathbb{R} \), we can get:

\[
s = \lambda_{1,\tilde{g}} \cos^2 \theta + \lambda'_{2,\tilde{g}} \sin^2 \theta.
\]

On the other hand

\[
\frac{ds}{d\theta} = (\lambda'_{2,\tilde{g}} - \lambda_{1,\tilde{g}}) \sin 2\theta,
\]

and taking into account that:

\[
\lambda_{1,\tilde{g}} = \inf_{H^2_0} \leq \lambda'_{2,\tilde{g}} = \inf_E \text{ because } (E \subset H^2_k(M)).
\]

We get that \( \lambda_{1,\tilde{g}} \) is a minimum of \( s(\theta) \) and \( \lambda'_{2,\tilde{g}}(\theta) \) is a maximum of \( s \),

\[
\lambda'_{2,\tilde{g}} = \sup_{f \in V_0} \int_M fP_\tilde{g}(f)dv_{\tilde{g}},
\]

and as the infimum of the quantity \( \sup_{f \in V_0 \setminus \{0\}} \int_M \frac{fP_\tilde{g}(f)}{u^{N-2}f^2}dv_{\tilde{g}} \) on all elements of \( Gr^0_2(H^2_k(M)) \) is attained over \( V_0 \), it follows

\[
\lambda'_{2,\tilde{g}} = \inf_{V \in Gr^0_2(H^2_k(M))} \sup_{v \in V \setminus \{0\}} \int_M \frac{vP_\tilde{g}vdv_{\tilde{g}}}{u^{N-2}vdv_{\tilde{g}}} = \lambda_{2,\tilde{g}}.
\]

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\square
\]

**References**


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