CHAOTIC SEMIFLOWS IN BANACH SPACES

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Abstract. All the common notions like topological transitivity, periodic points, sensitive dependence and chaoticity are formulated in the context of semi-flows in Banach spaces. It is shown that tensor product of chaotic semiflows in Banach spaces is also chaotic.

1. Introduction

Theory of semi-flows is concerned with time evolutions of natural and iterative processes. Some are predictable whereas others are not. Even simple semi-flows may behave unpredictably - the reason for unpredictable behaviour has been called "Chaos".

Tensor products are used to describe systems consisting of multiple sub-systems. Chaoticity of the tensor product semiflows is studied, to the case of the most general possible acting monoids.

2. Preliminaries

First we recall the basics of tensor products of vector spaces.

A mapping $A$ from the cartesian product $X \times Y$ of vector spaces into the vector space $Z$ is bilinear, if it is linear in each variable, that is,

$$A(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 A(x_1, y) + \alpha_2 A(x_2, y)$$

and

$$A(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 A(x, y_1) + \beta_2 A(x, y_2)$$

for all $x_i, x \in X, y_i, y \in Y$ and all scalars $\alpha_i, \beta_i, i = 1, 2$. We write $B(X \times Y, Z)$ for the vector space of bilinear mappings from the product $X \times Y$ into $Z$; when $Z$ is the scalar field we denote the corresponding space of bilinear forms simply by $B(X \times Y)$. 

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Now the tensor product, $X \otimes Y$, of the vector spaces $X,Y$ can be constructed as a space of linear functional on $B(X \times Y)$, in the following way: for $x \in X$, $y \in Y$, we denote by $x \otimes y$ the functional given by evaluation at the point $(x,y)$.

In other words

$$(x \otimes y)(A) = \langle A, x \otimes y \rangle = A(x,y)$$

for each bilinear form $A$ on $X \times Y$.

The tensor product $X \otimes Y$ is the subspace of the dual $B(X \times Y)$ spanned by these elements. Thus, a typical tensor in $X \otimes Y$ has the form $u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$ where $n$ is a natural number, $\lambda_i \in K$, $x_i \in X$ and $y_i \in Y$ [5].

We now introduce the definitions of semi-flow in a Banach space and the tensor product of semi-flows.

**Definition 2.1** Let $X$ be a Banach space and $S$ be an abelian topological monoid whose identity element is 0. A map $\phi : S \times X \rightarrow X$ is a semi-flow, if it satisfies the following two properties

1) $\phi(0,x) = x$ for any $x \in X$
2) $\phi(t,\phi(s,x)) = \phi(t+s,x)$ for any $t,s \in S$ and $x \in X$

**Definition 2.2** Let $X$ and $Y$ be Banach spaces and $S,T$ be abelian topological monoids. Tensor product of the semi-flows $\phi : S \times X \rightarrow X$ and $\psi : T \times Y \rightarrow Y$ is denoted by $\phi \otimes \psi : (S \times T) \times (X \otimes Y) \rightarrow (X \otimes Y)$ and is defined by $(\phi \otimes \psi)((s,t),x \otimes y) = \phi(s,x) \otimes \psi(t,y)$ for any $(s,t) \in S \times T$ and $x \otimes y \in X \otimes Y$.

We call $\phi : S \times X \rightarrow X$ and $\psi : T \times Y \rightarrow Y$, the factors of the tensor product semi-flows. It is easy to observe $\phi \otimes \psi$ is again a semi-flow. We assume

$$\|(x \otimes y) - (x^1 \otimes y^1)\| = \max \{\|x - x^1\|, \|y - y^1\|\}$$

where $x,x^1 \in X$ and $y,y^1 \in Y$.

3. **Main results**

We introduce the common notions leading to chaoticity.

**Definition 3.1** A semi-flow $\phi : S \times X \rightarrow X$ is sensitive if there exists $\epsilon > 0$ such that for any $x \in X$ and any neighbourhood $U$ of $x$, there exist $x_0 \in U$ and a real number $t > 0$ such that

$$\|\phi(t,x) - \phi(t,x_0)\| > \epsilon$$

Let $X$ be a Banach space. For $U,V \subset X$ and a semi-flow $\phi : S \times X \rightarrow X$ we define the dewelling set

$$D_\phi(U,V) = \{t \in S/\phi(t,U) \cap V \neq \phi\}$$

Equivalently

$$D_\phi(U,V) = \{t \in S/\phi^{-1}(t,V) \cap U \neq \phi\}$$
Definition 3.2 A semi-flow $\phi : S \times X \to X$ is called topologically transitive, if for any two non-empty open sets $U, V \subset X$

$$D_{\phi}(U, V) \neq \phi$$

Definition 3.3 A semi-flow $\phi : S \times X \to X$ is chaotic in the sense of Devaney if it is sensitive, topologically transitive and the set of all periodic points of $\phi$ is dense in $X$.

For an infinite Banach space $X$, and a continuous semi-flow $\phi : S \times X \to X$, topological transitivity and denseness of periodic points imply sensitivity [1].

Example 3.4 Let $S = [0, \infty)$ with addition and the standard topology and let $X = R^3$ be the Banach space. Then $\phi : S \times X \to X$ given by

$$\phi(s, (x, y, z)) = (x + s, y + 2s, z + 3s)$$

is a semiflow.

Since

$$\|(x, y, z) - (x_0, y_0, z_0)\| = \|\phi(s, (x, y, z)) - \phi(s, (x_0, y_0, z_0))\|$$

it follows that $\phi$ is not sensitive.

Example 3.5 Let $X$ be a Banach space and let $f : X \to X$ be any continuous function. Further, let $S = N_0$ with discrete topology. For $n \in N$, let $\phi(n, x) = f^n(x)$. (ie) $f$ is iterated $n$ times at $x$. Then $\phi$ is a semiflow, called a cascade.

Cascade are commonly studied semiflows, followed by $S = R$ or $[0, \infty)$.

Example 3.6 Let $X = R$ and $f : X \to X$ be given by $f(x) = x^3$. Then the cascade generated is not sensitive, because for $(x, y) \in (-1, 1)$ we have

$$\|x - y\| > \|f(x) - f(y)\|.$$

Example 3.7 Let $X = R$ and $f(x) = x + 1$. Then the cascade $\phi$ generated satisfies none of the three chaos criteria.

Proof. Since $f$ is an isometry, it cannot be sensitive. The set of periodic points is empty. Taking $U = (1, 2)$ and $V = (0, 1)$, we observe $D_{\phi}(U, V) = \phi$. So it is not topologically transitive also.

Example 3.8 Let $X = R$ and $f(x) = 2x$. Then the cascade $\phi$ generated is sensitive. With $U = (0, 1)$ and $V = (-\infty, 0)$, we observe that $D_{\phi}(U, V) = \phi$. So it is not topologically transitive. The set of periodic points of $\phi$ is only zero and hence cannot be dense.

For any open non empty set $U \subseteq X$.

We define

$$R(U, \epsilon) = t \in S/ \text{there exists } x, y \in U \text{ such that } \|\phi(t, x) - \phi(t, y)\| \geq \epsilon.$$
**Definition 3.9** Let $X$ be a Banach space. A semiflow $\phi : S \times X \to X$ is called sensitive if there exists $\epsilon > 0$ such that for any open and non empty $U \subseteq X$, $R(U, \epsilon)$ is not empty.

**Proposition 3.10** Tensor product semiflow $\phi \otimes \psi$ on $X \otimes Y$ is sensitive if and only if at least one factor is sensitive.

*Proof.* We have

$$R(U \otimes V, \epsilon) = \{(s, t) \in S \times T \mid \text{there exists } x_1, x_2 \in U \subseteq X \text{ and } y_1, y_2 \in V \subseteq Y \text{ such that } \|\phi(s, x_1) \otimes \psi(t, y_1) - \phi(s, x_2) \otimes \psi(t, y_2)\| \geq \epsilon\}$$

If $\phi \otimes \psi$ is sensitive, then $R(U \otimes V, \epsilon) \neq \phi$.

ie

$$R(U \otimes V, \epsilon) = \{(s, t) \in S \times T \mid \text{there exists } x_1, x_2 \in U \subseteq X \text{ and } y_1, y_2 \in V \subseteq Y \text{ such that } \max\{\|\phi(s, x_1) - \phi(s, x_2)\|, \|\psi(t, y_1) - \psi(t, y_2)\|\} \geq \epsilon \neq \phi$$

Showing that at least one of the factors is sensitive. Converse is obvious. \[ \square \]

**Proposition 3.11** Let $X$ and $Y$ be a Banach spaces. $\phi : X \to X$ and $\psi : Y \to Y$ be maps. If the set of periodic points of $\phi$ is dense in $X$ and the set of periodic points of $\psi$ is dense in $Y$ then the set of periodic points of $\phi \otimes \psi$ is dense in $X \otimes Y$, but not conversely.

*Proof.* Let $W \subseteq X \otimes Y$ be any non empty open set. Then there exists open sets $U \subseteq X$ and $V \subseteq Y$ with $U \otimes V \subseteq W$.

By assumption, there exists a point $x \in U$ such that $\phi(s, x) = x$ and there exists $y \in V$ such that $\psi(t, y) = y$ for some $s, t > 0$. Now

$$(\phi \otimes \psi)((s, t), x \otimes y) = \phi(s, x) \otimes \psi(t, y) = x \otimes y$$

showing that $W$ contains a periodic point $x \otimes y$. Since tensor product of dense sets is dense, the result follows.

Conversely, let $x \otimes y$ be a periodic point of $\phi \otimes \psi$. Then there exists $s, t > 0$ such that $(\phi \otimes \psi)((s, t), x \otimes y) = x \otimes y$. So that $\phi(s, x) \otimes \psi(t, y) = x \otimes y$. This need not imply $\phi(s, x) = x, \psi(t, y) = y$. In fact, there exists a non zero scalar $\beta$ such that $\phi(s, x) = \beta x, \psi(t, y) = \frac{1}{\beta} y$. [Lemma 4.3 of \[2\]]. The next example 3.12 illustrates this point. \[ \square \]

**Example 3.12** Let $X = Y$ be the Banach space of all $2 \times 2$ real matrices. Define $\phi : X \to X$, $\psi : Y \to Y$ by

$$\phi = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

and $\psi = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$.

Then $\phi \otimes \psi = I_4$, the identity matrix of order 4.

Also let $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

and $y = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}$.

It is easy to observe that $(\phi \otimes \psi)(x \otimes y) = x \otimes y$ but $\phi(x) = 4x$ and $\psi(y) = \frac{1}{2} y$. 

It is known that product of topologically transitive semi-flows is not topologically transitive [2]. In the tensor product case we have the semi-flows.

**Proposition 3.13** $\phi: S \times X \to X$ and $\psi: T \times Y \to Y$ are topologically transitive if and only if their tensor product semi-flow $\phi \otimes \psi: (S \times T) \times (X \otimes Y) \to X \otimes Y$ is topologically transitive.

**Proof.** For any non-empty open sets $U, V \subset X$ and for $P, Q \subset Y$ we have

$$D_\phi(U, V) = \{s \in S/\phi(s, U) \cap V \neq \phi\}$$

and

$$D_\phi(P, Q) = \{t \in T/\psi(t, P) \cap Q \neq \phi\}$$

and

$$D_{\phi \otimes \psi}(U \otimes P, V \otimes Q) = \{(s, t) \in S \times T/(\phi \otimes \psi)((s, t), (U \otimes P)) \cap (V \otimes Q)\} \neq \phi$$

$$= \{(s, t) \in S \times T/\phi(s, U) \otimes \psi(t, P) \cap (V \otimes Q)\} \neq \phi$$

Clearly $D_{\phi \otimes \psi}(U \otimes P, V \otimes Q)$ is non empty if both $D_\phi(U, V)$ and $D_\psi(P, Q)$ are non empty. By definition of topological transitivity, the result follows. □

Examples are there to show that product of chaotic maps on metric spaces need not be chaotic [2]. But in the tensor product case, we have

**Theorem 3.14** The tensor product of two chaotic semi-flows is again chaotic.

**Proof.** By proposition 3.11 and 3.13 the result follows. □

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**References**


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