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CHAOTIC SEMIFLOWS IN BANACH SPACES

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ABSTRACT. All the common notions like topological transitivity, periodic points, sensitive dependence and chaoticity are formulated in the context of semi-flows in Banach spaces. It is shown that tensor product of chaotic semiflows in Banach spaces is also chaotic.

1. INTRODUCTION

Theory of semi-flows is concerned with time evolutions of natural and iterative processes. Some are predictable whereas others are not. Even simple semi-flows may behave unpredictably - the reason for unpredictable behaviour has been called "Chaos".

Tensor products are used to describe systems consisting of multiple sub-systems. Chaoticity of the tensor product semiflows is studied, to the case of the most general possible acting monoids.

2. **Preliminaries**

First we recall the basics of tensor products of vector spaces.

A mapping A from the cartesian product $X \times Y$ of vector spaces into the vector space Z is bilinear, if it is linear in each variable, that is,

$$A\left(\alpha_1 x_1 + \alpha_2 x_2, y\right) = \alpha_1 A(x_1, y) + \alpha_2 A(x_2, y)$$

and

$$A\left(x, \beta_1 y_1 + \beta_2 y_2\right) = \beta_1 A(x, y_1) + \beta_2 A(x, y_2)$$

for all x_i, $x \in X$, y_i, $y \in Y$ and all scalars $\alpha_i, \beta_i, i = 1, 2$. We write $B(X \times Y, Z)$ for the vector space of bilinear mappings from the product $X \times Y$ into Z; when Z is the scalar field we denote the corresponding space of bilinear forms simply by $B(X \times Y)$.

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Now the tensor product, $X \otimes Y$, of the vector spaces X, Y can be constructed as a space of linear functional on $B(X \times Y)$, in the following way: for $x \in X$, $y \in Y$, we denote by $x \otimes y$ the functional given by evaluation at the point (x, y).

In other words

$$(x \otimes y)(A) = \langle A, x \otimes y \rangle = A(x, y)$$

for each bilinear form A on $X \times Y$.

The tensor product $X \otimes Y$ is the subspace of the dual $B(X \times Y)$ spanned by these elements. Thus, a typical tensor in $X \otimes Y$ has the form $u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$ where n is a natural number, $\lambda_i \in K$, $x_i \in X$ and $y_i \in Y$ [5].

We now introduce the definitions of semi-flow in a Banach space and the tensor product of semi-flows.

Definition 2.1 Let X be a Banach space and S be an abelian topological monoid whose identity element is 0. A map $\phi : S \times X \to X$ is a semi-flow, if it satisfies the following two properties
i) $\phi(0, x) = x$ for any $x \in X$
ii) $\phi(t, \phi(s, x)) = \phi((t + s), x)$ for any $t, s \in S$ and $x \in X$

Definition 2.2 Let X and Y be Banach spaces and S, T be abelian topological monoids. Tensor product of the semi-flows $\phi : S \times X \to X$ and $\psi : T \times Y \to Y$ is denoted by $\phi \otimes \psi : (S \times T) \times (X \otimes Y) \to (X \otimes Y)$ and is defined by $(\phi \otimes \psi)((s, t), x \otimes y) = \phi(s, x) \otimes \psi(t, y)$ for any $(s, t) \in S \times T$ and $x \otimes y \in X \otimes Y$.
We call $\phi : S \times X \to X$ and $\psi : T \times Y \to Y$, the factors of the tensor product semi-flows. It is easy to observe $\phi \otimes \psi$ is again a semi-flow. We assume

$$\|(x \otimes y) - (x^1 \otimes y^1)\| = max \left\{ \|x - x^1\|, \|y - y^1\| \right\}$$

where $x, x^1 \in X$ and $y, y^1 \in Y$

3. **Main results**

We introduce the common notions leading to chaoticity.

Definition 3.1 A semi-flow $\phi : S \times X \to X$ is sensitive if there exists $\epsilon > 0$ such that for any $x \in X$ and any neighbourhood U of x, there exist $x_0 \in U$ and a real number $t > 0$ such that

$$\|\phi(t, x) - \phi(t, x_0)\| > \in$$

Let X be a Banach space. For $U, V \subset X$ and a semi-flow $\phi : S \times X \to X$
we define the dewelling set

$$D_\phi(U, V) = \{t \in S / \phi(t, U) \cap V \neq \phi\}$$

Equivalently

$$D_\phi(U, V) = \left\{t \in S / \phi^{-1}(t, V) \cap U \neq \phi\right\}$$

Definition 3.2 A semi-flow $\phi : S \times X \to X$ is called topologically transitive, if for any two non-empty open sets $U, V \subset X$

$$D_\phi(U, V) \neq \phi$$

Definition 3.3 [3] A semi-flow $\phi : S \times X \to X$ is chaotic in the sense of Devaney if it is sensitive, topologically transitive and the set of all periodic points of ϕ is dense in X.

For an infinite Banach space X, and a continuous semi-flow $\phi : S \times X \to X$, topological transitivity and denseness of periodic points imply sensitivity [1].

Example 3.4 Let $S = [0, \infty)$ with addition and the standard topology and let $X = R^3$ be the Banach space. Then $\phi : S \times X \to X$ given by

$$\phi(s, (x, y, z)) = (x + s, y + 2s, z + 3s)$$

is a semiflow.
Since

$$\|(x, y, z) - (x_0, y_0, z_0)\| = \|\phi(s, (x, y, z)) - \phi(s, (x_0, y_0, z_0))\|$$

it follows that ϕ is not sensitive.

Example 3.5 Let X be a Banach space and let $f : X \to X$ be any continuous function. Further, let $S = N_0$ with discrete topology. For $n \in N$, let $\phi(n, x) = f^n(x)$. (ie) f is iterated n times at x. Then ϕ is a semiflow, called a cascade.
Cascade are commonly studied semiflows, followed by $S = R$ or $[0, \infty)$.

Example 3.6 Let $X = R$ and $f : X \to X$ be given by $f(x) = x^3$. Then the cascade generated is not sensitive, because for $(x, y) \in (-1, 1)$ we have

$$\|x - y\| > \|f(x) - f(y)\|.$$

Example 3.7 Let $X = R$ and $f(x) = x + 1$. Then the cascade ϕ generated satisfies none of the three chaos criteria.

Proof. Since f is an isometry, it cannot be sensitive. The set of periodic points is empty. Taking $U = (1, 2)$ and $V = (0, 1)$, we observe $D_\phi(U, V) = \phi$. So it is not topologically transitive also. □

Example 3.8 Let $X = R$ and $f(x) = 2x$. Then the cascade ϕ generated is sensitive. With $U = (0, 1)$ and $V = (-\infty, 0)$, we observe that $D_\phi(U, V) = \phi$. So it is not topologically transitive. The set of periodic points of ϕ is only zero and hence cannot be dense.

For any open non empty set $U \subseteq X$.

We define

$R(U, \epsilon) = t \in S/$ there exists $x, y \in U$ such that $\|\phi(t, x) - \phi(t, y)\| \geq \epsilon$.

We restate the definition of sensitivity.

Definition 3.9 Let X be a Banach space. A semiflow $\phi : S \times X \to X$ is called sensitive if there exists $\epsilon > 0$ such that for any open and non empty $U \subseteq X$, $R(U, \epsilon)$ is not empty.

Proposition 3.10 Tensor product semiflow $\phi \otimes \psi$ on $X \otimes Y$ is sensitive if and only if atleast one factor is sensitive.

Proof. We have

$R(U \otimes V, \epsilon) = \big\{(s,t) \in S \times T \mid$ there exists $x_1, x_2 \in U \subset X$ and $y_1, y_2 \in V \subset Y$ such that $\|\phi(s,x_1) \otimes \psi(t,y_1) - \phi(s,x_2) \otimes \psi(t,y_2)\| \geq \epsilon\big\}$

If $\phi \otimes \psi$ is sensitive, then $R(U \otimes V, \epsilon) \neq \phi$

ie

$R(U \otimes V, \epsilon) = (s,t) \in S \times T/$ there exists $x_1, x_2 \in U \subset X$and $y_1, y_2 \in V \subset Y$ such that $\max(\|\phi(s,x_1) - \phi(s,x_2)\|, \|\psi(t,y_1) - \psi(t,y_2)\|) \geq \epsilon \neq \phi$

Showing that atleast one of the factors is sensitive. Converse is obvious. □

Proposition 3.11 Let X and Y be a Banach spaces. $\phi : X \to X$ and $\psi : Y \to Y$ be maps. If the set of periodic points of ϕ is dense in X and the set of periodic points of ψ is dense in Y then the set of periodic points of $\phi \otimes \psi$ is dense in $X \otimes Y$, but not conversely.

Proof. Let $W \subset X \otimes Y$ be any non empty open set. Then there exists open sets $U \subset X$ and $V \subset Y$ with $U \otimes V \subset W$

By assumption, there exists a point $x \in U$ such that $\phi(s,x) = x$ and there exists $y \in V$ such that $\psi(t,y) = y$ with $s, t > 0$. Now

$$(\phi \otimes \psi)((s,t), x \otimes y) = \phi(s,x) \otimes \psi(t,y) = x \otimes y$$

showing that W contains a periodic point $x \otimes y$. Since tensor product of dense sets is dense, the result follows.

Conversely, let $x \otimes y$ be a periodic point of $\phi \otimes \psi$. Then there exists $s, t > 0$ such that $(\phi \otimes \psi)((s,t), x \otimes y) = x \otimes y$. So that $\phi(s,x) \otimes \psi(t,y) = x \otimes y$. This need not imply $\phi(s,x) = x$, $\psi(t,y) = y$. In fact, there exists a non zero scalar β such that $\phi(s,x) = \beta x$, $\psi(t,y) = \frac{1}{\beta} y$. [Lemma 4.3 of [4]]. The next example 3.12 illustrates this point. □

Example 3.12 Let $X = Y$ be the Banach space of all 2×2 real matrices. Define $\phi : X \to X$, $\psi : Y \to Y$ by

$\phi = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

and $\psi = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$.

Then $\phi \otimes \psi = I_4$, the identity matrix of order 4.

Also let $x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$

and $y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$.

It is easy to observe that $(\phi \otimes \psi)(x \otimes y) = x \otimes y$ but $\phi(x) = 4x$ and $\psi(y) = \frac{1}{4} y$.

It is known that product of topologically transitive semi-flows is not topologically transitive [2]. In the tensor product case we have the semi-flows.

Proposition 3.13 $\phi : S \times X \to X$ and $\psi : T \times Y \to Y$ are topologically transitive if and only if their tensor product semi-flow $\phi \otimes \psi : (S \times T) \times (X \otimes Y) \to X \otimes Y$ is topologically transitive.

Proof. For any non-empty open sets $U, V \subset X$ and for $P, Q \subset Y$ we have

$$D_\phi(U, V) = \{s \in S/\phi(s, U) \cap V \neq \phi\}$$

$$D_\phi(P, Q) = \{t \in T/\psi(t, P) \cap Q \neq \phi\}$$

and

$$\begin{aligned} D_{\phi\otimes\psi}(U \otimes P, V \otimes Q) &= \{(s,t) \in S \times T/(\phi \otimes \psi)((s,t), (U \otimes P)) \cap (V \otimes Q)\} \neq \phi \\ &= \{(s,t) \in S \times T/(\phi(s,U) \otimes \psi(t,P)) \cap (V \otimes Q)\} \neq \phi \\ &= \{(s,t) \in S \times T/(\phi(s,U) \cap V) \otimes (\psi(t,P) \cap Q)\} \neq \phi \end{aligned}$$

Clearly $D_{\phi\otimes\psi}(U \otimes P, V \otimes Q)$ is non empty iff both $D_\phi(U, V)$ and $D_\psi(P, Q)$ are non empty. By definition of topological transitivity, the result follows. □

Examples are there to show that product of chaotic maps on metric spaces need not be chaotic [2]. But in the tensor product case, we have

Theorem 3.14 The tensor product of two chaotic semi-flows is again chaotic.

Proof. By proposition 3.11 and 3.13 the result follows. □

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