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NEW BOUNDS FOR HERMITE-HADAMARD TYPE INEQUALITIES WITH APPLICATIONS

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Abstract. In the present article, first we have established two integral identities attached with the right hand side of the well-known Hermite-Hadamard inequality. Second, by making use of these identities, we obtain some new Hadamard's type inequalities and these inequalities have natural applications to some special means of real numbers. At the end, some error estimations for the trapezoidal formula are also presented.

1. Introduction

The following class of functions is better known in the literature and is usually defined in the following way: a function $f : I \to \mathbb{R}$, defined on the interval I in $\mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. Also we say that f is concave, if the inequality in (1) holds in the reversed direction. The geometrical interpretation of convexity is that, if there are any three distinct points R, S and T located on the graph of function f with S lies between R and T, then the point S lies on or below the chord joining the points R and T.

A number of important inequalities have been obtained for the class of convex functions, when this idea was introduced for the first time more than a century ago. But among those the most prominent is the so called Hermite-Hadamard's inequality (or Hadamard's inequality). This double inequality is stated as (see for example [13]):

Let I be an interval in $\mathbb{R}$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on I such that $a, b \in I$ with $a < b$. Then the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

hold. If the function f is concave on I, then both the inequalities in (2) hold in the reverse direction. It gives an estimate from both sides of the mean value of a

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convex function and also ensure the integrability of convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions f. These inequalities for convex functions play a crucial role in analysis as well as in other areas of pure and applied mathematics.

For more recent results, generalizations, improvements and refinements related to Hermite-Hadamard inequality see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 20, 21, 14, 15, 16, 17, 18, 19, 22, 23, 24, 25] and the references given therein.

In 1998 Dragomir and Agarwal have proved the following important lemma:

Lemma 1.[10] Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following identity holds:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{3}$$

Here I° denotes the interior of I.
The following two results are the ultimate consequences of Lemma 1, which have been presented in [10].

Theorem 1. Under the assumptions of Lemma 1 and if $|f'|$ is convex on $[a, b]$, then we have the following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{4}$$

and

Theorem 2. Suppose the conditions of Lemma 1 are satisfied and if the new mapping $|f'|^{\frac{p}{p-1}}$ $(p > 1)$ is convex on $[a, b]$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{5}$$

In [22] Pearce and Pečarić used the above Lemma 1 and proved the following theorem.

Theorem 3. Suppose the conditions of Lemma 1 hold and if the mapping $|f'|^q$ $(q \geq 1)$ is concave on $[a, b]$, then the following inequality is valid:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{6}$$

In this paper, first we give two general integral identities for differentiable and twice differentiable functions in back to back sections (Lemma 2 and Lemma 3). Then, we apply these identities to establish our main results (Theorems 4-7) and discuss some cases of these results (Remark 1-2). Next by making use of some particular functions, we obtained new inequalities related to some special means of real numbers. Finally, we gave some applications for error estimates for the trapezoidal formula.

2. RESULTS FOR DIFFERENTIABLE FUNCTIONS

In order to prove our main results, we begin with the following lemma.

Lemma 2. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with

$a < b$. If $f' \in L[a,b]$, then the following identity holds:

$$\begin{aligned} & \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \\ = & \frac{(b-a)}{2}\left[\int_0^1 tf'(tb+(1-t)a)dt - \int_0^1 tf'(ta+(1-t)b)dt\right]. \end{aligned} \tag{7}$$

Proof. Integrating by parts

$$\begin{aligned} I_1 &= \int_0^1 tf'(ta+(1-t)b)dt \\ &= t\frac{f(ta+(1-t)b)}{a-b}\Big|_0^1 - \frac{1}{a-b}\int_0^1 f(at+(1-t)b)dt \\ &= \frac{f(a)}{a-b} + \frac{1}{b-a}.\frac{1}{b-a}\int_a^b f(x)dx \\ &= \frac{-f(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b f(x)dx, \end{aligned} \tag{8}$$

and analogously we obtain the following,

$$I_2 = \int_0^1 tf'(tb+(1-t)a)dt = \frac{f(b)}{b-a} - \frac{1}{(b-a)^2}\int_a^b f(x)dx. \tag{9}$$

The subtraction of (8) from (9) yields

$$I_2 - I_1 = \frac{f(b)+f(a)}{b-a} - \frac{2}{(b-a)^2}\int_a^b f(x)dx.$$

Thus, the multiplication of both sides by $\frac{b-a}{2}$, gives (7) and hence we have the result.

Now using Lemma 2 we proceed to prove the following interesting results.

Theorem 4. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If for $q \geq 1$ the mapping $|f'|^q$ is concave on $[a,b]$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\left|f'\left(\frac{a+2b}{3}\right)\right| + \left|f'\left(\frac{2a+b}{3}\right)\right|\right]. \tag{10}$$

Proof. By power mean inequality, we have

$$\begin{aligned} \left(t|f'(a)| + (1-t)|f'(b)|\right)^q &\leq t|f'(a)|^q + (1-t)|f'(b)|^q \\ &\leq |f'(ta+(1-t)b)|^q, \text{ (by concavity of } |f'|^q \text{)} \end{aligned}$$

that is

$$|f'(ta+(1-t)b)| \geq t|f'(a)| + (1-t)|f'(b)|,$$

this shows that $|f'|$ is concave. Now using Lemma 2 we have

$$\begin{aligned} & \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq & \frac{(b-a)}{2}\int_0^1 \Big[t\,|f'(tb+(1-t)a)| + t\,|f'(ta+(1-t)b)|\Big]dt. \end{aligned} \tag{11}$$

Accordingly by Jensen's integral inequality we have

$$\begin{aligned}\int_0^1 t\left|f'(tb+(1-t)a)\right|dt &\leq \left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1 t(tb+(1-t)a)dt}{\int_0^1 tdt}\right)\right| \\ &= \frac{1}{2}\left|f'\left(\frac{a+2b}{3}\right)\right|\end{aligned}$$

and equivalently , we have

$$\begin{aligned}\int_0^1 t\left|f'(ta+(1-t)b)\right|dt &\leq \left(\int_0^1 tdt\right)\left|f'\left(\frac{\int_0^1 t(ta+(1-t)b)dt}{\int_0^1 tdt}\right)\right| \\ &= \frac{1}{2}\left|f'\left(\frac{2a+b}{3}\right)\right|.\end{aligned}$$

This completes the proof.

Theorem 5. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If for $q \geq 1$ the mapping $|f'|^q$ is convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{2|f'(b)|^q+|f'(a)|^q}{3}\right)^{\frac{1}{q}}\right. \\ &\quad + \left.\left(\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right]. \qquad (12)\end{aligned}$$

Proof. Likewise in Theorem 4, again here we consider (11), that is

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq \frac{(b-a)}{2}\int_0^1 \Big[t\left|f'(tb+(1-t)a)\right| + t\left|f'(ta+(1-t)b)\right|\Big]dt\end{aligned}$$

and by power mean integral inequality, we can write

$$\int_0^1 t\left|f'(tb+(1-t)a)\right|dt \leq \left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

and

$$\int_0^1 t\left|f'(ta+(1-t)b)\right|dt \leq \left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since, $|f'|^q$ is convex, so we have

$$\begin{aligned}\int_0^1 t\left|f'(tb+(1-t)a)\right|^q dt &\leq \int_0^1 t[t|f'(b)|^q+(1-t)|f'(a)|^q]dt \\ &\leq \frac{2|f'(b)|^q+|f'(a)|^q}{3}\end{aligned}$$

and equivalently, we have

$$\int_0^1 t\left|f'(ta+(1-t)b)\right|^q dt \leq \frac{2|f'(a)|^q+|f'(b)|^q}{3}.$$

Also $\int_0^1 tdt = \frac{1}{2}$, therefore all the above inequalities and the facts enable us to get the required result.

Remark 1. Further more by using the fact: $\sum_{k=1}^{n}(\alpha_k+\beta_k)^s \leq \sum_{k=1}^{n}(\alpha_k)^s + \sum_{k=1}^{n}(\beta_k)^s$ for $(0 \leq s \leq 1)$, $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n \geq 0$; $\beta_1, \beta_2, \beta_3, ..., \beta_n \geq 0$ with $0 < \frac{q-1}{q} < 1$, for $q > 1$, on the right hand side of the inequality (12), we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{2|f'(b)|^q+|f'(a)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{2|f'(a)|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left(\frac{1}{3}\right)^{\frac{1}{q}} + \left(\frac{2}{3}\right)^{\frac{1}{q}}\right)\Big[|f'(a)|+|f'(b)|\Big]. \qquad (13)
\end{aligned}$$

3. RESULTS FOR TWICE DIFFERENTIABLE FUNCTIONS

In order to prove our main results for twice differentiable functions, first we prove the following lemma.

Lemma 3. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L[a,b]$, then the following identity holds:

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \\
=\ & \frac{(b-a)^2}{4}\left[\int_0^1 (1-t^2)f''(ta+(1-t)b)dt - \int_0^1 t^2 f''(tb+(1-t)a)dt\right]. \qquad (14)
\end{aligned}$$

Proof. Integrating by parts

$$\begin{aligned}
I_1 &= \int_0^1 (1-t^2)f''(ta+(1-t)b)dt \\
&= (1-t^2)\frac{f'(ta+(1-t)b)}{a-b}\Bigg|_0^1 + \frac{2}{a-b}\int_0^1 tf'(at+(1-t)b)dt \\
&= \frac{-f'(b)}{a-b} - \frac{2}{b-a}\left[\frac{f(a)}{a-b} + \frac{1}{b-a}.\frac{1}{b-a}\int_a^b f(x)dx\right] \\
&= \frac{f'(b)}{b-a} + \frac{2f(a)}{(b-a)^2} - \frac{2}{(b-a)^3}\int_a^b f(x)dx \qquad (15)
\end{aligned}$$

and in similar way we get,

$$I_2 = \int_0^1 t^2 f''(tb+(1-t)a)dt = \frac{f'(b)}{b-a} - \frac{2f(b)}{(b-a)^2} + \frac{2}{(b-a)^3}\int_a^b f(x)dx. \qquad (16)$$

Subtracting (16) from (15), we have

$$I_1 - I_2 = \frac{2(f(b)+f(a))}{(b-a)^2} - \frac{4}{(b-a)^3}\int_a^b f(x)dx.$$

Thus, by multiplying both sides by $\frac{(b-a)^2}{4}$, we arrive at (14).

Using this lemma, we are going to obtain our first main result of the section in the next theorem.

Theorem 6. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If for $q \geq 1$ the function $|f''|^q$ is concave on $[a, b]$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{4}\left[\left|f''\left(\frac{a+2b}{3}\right)\right|\right]. \tag{17}$$

Proof. Since from Theorem 4 one can see that the concavity of $|f''|^q$ implies the following inequality

$$|f''(ta+(1-t)b)| \geq t|f''(a)| + (1-t)|f''(b)|$$

and so $|f''|$ is concave. Now using Lemma 3 we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{4}\int_0^1 \left[(1-t^2)\left|f''(ta+(1-t)b)\right| + t^2\left|f''(tb+(1-t)a)\right|\right]dt \\
\leq\ & \frac{(b-a)^2}{4}\int_0^1 \left[\left|f''\left((1-t^2)(ta+(1-t)b)+t^2(tb+(1-t)a)\right)\right|\right]dt \\
=\ & \frac{(b-a)^2}{4}\int_0^1 \left[\left|f''\left((t+t^2-2t^3)a+(1-t-t^2+2t^3)b\right)\right|\right]dt \\
=\ & \frac{(b-a)^2}{4}\int_0^1 \left[\left|f''\left(g(t)\right)\right|\right]dt
\end{aligned}$$

where, $g(t) = (t+t^2-2t^3)a + (1-t-t^2+2t^3)b$.

Using Jensen's integral inequality for the concave function $|f''|$, we have

$$\int_0^1 \left|f''\left(g(t)\right)\right|dt \leq \left|f''\left(\int_0^1 g(t)dt\right)\right|. \tag{18}$$

Further, since

$$\int_0^1 g(t)dt = \int_0^1 ((t+t^2-2t^3)a + (1-t-t^2+2t^3)b)dt = \frac{a}{3} + \frac{2b}{3} \tag{19}$$

a combination of (18)-(19) immediately gives the required inequality (17).

Theorem 7. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If for $q \geq 1$ the function $|f''|^q$ is concave on $[a, b]$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \qquad (20)\\
\leq\ & \frac{(b-a)^2}{12}\left[2\left|f''\left(\frac{3a+5b}{8}\right)\right| + \left|f''\left(\frac{3a+b}{4}\right)\right|\right].
\end{aligned}$$

Proof. Since, from above we know that if $|f''|^q$ is concave for $q \geq 1$, then so is $|f''|$ and again from Lemma 3 we have

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ \leq\; & \frac{(b-a)^2}{4}\int_0^1 \Big[(1-t^2)\left|f''(ta+(1-t)b)\right| + t^2\left|f''(tb+(1-t)a)\right| \Big] dt. \end{aligned}$$

Now by making use of Jensen's integral inequality we have

$$\begin{aligned} \int_0^1 (1-t^2)\left|f''(ta+(1-t)b)\right| dt &\leq \left(\int_0^1 (1-t^2)dt\right) \left| f''\left(\frac{\int_0^1 (1-t^2)(ta+(1-t)b)dt}{\int_0^1 (1-t^2)dt} \right) \right| \\ &= \frac{2}{3}\left| f''\left(\frac{3a+5b}{8}\right)\right| \end{aligned}$$

and similarly, we have

$$\begin{aligned} \int_0^1 t^2\left|f''(tb+(1-t)a)\right| dt &\leq \left(\int_0^1 t^2 dt\right) \left| f''\left(\frac{\int_0^1 t^2(tb+(1-t)a)dt}{\int_0^1 t^2 dt} \right) \right| \\ &= \frac{1}{3}\left| f''\left(\frac{a+3b}{4}\right)\right|. \end{aligned}$$

Hence, from the above inequalities the result in (20) is obvious.

Remark 2. It is also important to note that since $|f''|$ is concave on the interval $[a,b]$, therefore we have

$$\begin{aligned} & \frac{(b-a)^2}{12}\left[2\left|f''\left(\frac{3a+5b}{8}\right)\right| + \left|f''\left(\frac{3a+b}{4}\right)\right| \right] \\ =\; & \frac{(b-a)^2}{4}\left[\frac{2}{3}\left|f''\left(\frac{3a+5b}{8}\right)\right| + \frac{1}{3}\left|f''\left(\frac{3a+b}{4}\right)\right| \right] \\ \leq\; & \frac{(b-a)^2}{4}\left|f''\left(\frac{a+b}{2}\right)\right|. \end{aligned}$$

4. APPLICATION TO SPECIAL MEANS

As in [10], we will consider the following particular means for any $\alpha, \beta \in \mathbb{R}$, $\alpha \neq \beta$ which are well known in the literature:

$$\begin{aligned} A(\alpha,\beta) &= \frac{\alpha+\beta}{2}, && \alpha,\beta \in \mathbb{R}, \\ \bar{L}(\alpha,\beta) &= \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|}, && \alpha,\beta \in \mathbb{R}\setminus\{0\}, \\ L_n(\alpha,\beta) &= \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, && n \in N,\ n \geq 1,\ \alpha,\beta \in \mathbb{R},\ \alpha < \beta. \end{aligned}$$

Proposition 1. Let $0 < a < b$ and $n \in N$, $n \geq 2$, then for all $q \geq 1$ the following inequality holds:

$$\begin{aligned} &\left|A(a^n,b^n) - L_n^n(a,b)\right| \qquad (21) \\ \leq\; & \frac{n(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[\left(\frac{2|b|^{(n-1)q} + |a|^{(n-1)q}}{3}\right)^{\frac{1}{q}} + \left(\frac{2|a|^{(n-1)q} + |b|^{(n-1)q}}{3}\right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. Using the convex function $f(x) = x^n$, $x > 0$ in Theorem 5, one can easily obtained the result.

Proposition 2. Let $0 < a < b$, then for all $q \geq 1$ the following inequality holds:

$$\begin{aligned} & \left|A(a^{-1}, b^{-1}) - L^{-1}(a,b)\right| \\ \leq \quad & \frac{n(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{2|b|^{-2q}+|a|^{-2q}}{3}\right)^{\frac{1}{q}} + \left(\frac{2|a|^{-2q}+|b|^{-2q}}{3}\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{22}$$

Proof. The result follows directly from Theorem 5 under the utility of convex function $f(x) = \frac{1}{x}$, $x > 0$.

5. APPLICATIONS TO TRAPEZOIDAL FORMULA

Let $p = \{x_1, x_2, .., x_n\}$ be a partition of the points $x_i \in [a,b]$, with $a = x_0$, $x_n = b$ and $x_i < x_{i+1}$ for $i = \overline{1,n}$. Then the well known Trapezoidal formula for the partition p is given by:

$$\int_a^b f(x)dx = \tau(f,p) + e(f,p),$$

where

$$\tau(f,p) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)$$

denotes the trapezoidal formula and $e(f,p)$ represents the error approximation associated to it.

Proposition 3. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If for $q \geq 1$ the function $|f'|^q$ is concave on $[a,b]$, then for every partition p of $[a,b]$, the following inequality holds:

$$\begin{aligned} & |e(f,d))| \\ \leq \quad & \frac{1}{4}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\left|f'\left(\frac{x_i+2x_{i+1}}{3}\right)\right| + \left|f'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|\right\}. \end{aligned} \tag{23}$$

Proof. Utilizing Theorem 4 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ of the partition p, we get

$$\begin{aligned} & \left|\frac{f(x_{i+1}) + f(x_i)}{2}(x_{i+1} - x_i) - \int_{x_i}^{x_{i+1}} f(x)dx\right| \\ \leq \quad & \frac{(x_{i+1}-x_i)^2}{4}\left\{\left|f'\left(\frac{x_i+2x_{i+1}}{3}\right)\right| + \left|f'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|\right\}. \end{aligned}$$

Summing over i from 0 to $n-1$ and by the triangle inequality, we have

$$\begin{aligned} & \left|\tau(f,d)) - \int_a^b f(x)dx\right| \\ \leq \quad & \frac{1}{4}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left\{\left|f'\left(\frac{x_i+2x_{i+1}}{3}\right)\right| + \left|f'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|\right\}. \end{aligned}$$

Proposition 4. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If for $q \geq 1$ the function $|f'|^q$ is convex on $[a, b]$, then for every partition p of $[a, b]$, the following inequality is valid:

$$\begin{aligned} & |e(f,d)| \\ \leq & \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \frac{1}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[\left(\frac{2|f'(x_{i+1})|^q + |f'(x_i)|^q}{3} \right)^{\frac{1}{q}} \right. \\ + & \left. \left(\frac{2|f'(x_i)|^q + |f'(x_{i+1})|^q}{3} \right)^{\frac{1}{q}} \right]. \end{aligned} \tag{24}$$

Proof. The proof utilizes Theorem 5 and is parallel to that of Proposition 3 but here we consider the convexity of function instead of concavity.

Proposition 5. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If for $q \geq 1$ the function $|f''|^q$ is concave on $[a, b]$, then for every partition p of $[a, b]$, we have the following inequality :

$$|e(f,d))| \leq \frac{1}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left[\left| f'' \left(\frac{x_i + 2x_{i+1}}{3} \right) \right| \right]. \tag{25}$$

Proof. Applying Theorem 6 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ of the partition p, we get

$$\begin{aligned} & \left| \frac{f(x_{i+1}) + f(x_i)}{2} (x_{i+1} - x_i) - \int_{x_i}^{x_{i+1}} f(x)dx \right| \\ \leq & \frac{(x_{i+1} - x_i)^3}{4} \left[\left| f'' \left(\frac{x_i + 2x_{i+1}}{3} \right) \right| \right]. \end{aligned}$$

Summing over i from 0 to $n-1$ and by the triangle inequality, we have

$$\begin{aligned} & \left| \tau(f,d)) - \int_a^b f(x)dx \right| \\ \leq & \frac{1}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left[\left| f'' \left(\frac{x_i + 2x_{i+1}}{3} \right) \right| \right]. \end{aligned}$$

Proposition 6. Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If for $q \geq 1$ the function $|f''|^q$ is concave on $[a, b]$, then for every partition p of $[a, b]$, the following inequality holds:

$$\begin{aligned} & |e(f,d)| \\ \leq & \frac{1}{12} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left[2 \left| f'' \left(\frac{3x_i + 5x_{i+1}}{8} \right) \right| + \left| f'' \left(\frac{3x_i + x_{i+1}}{4} \right) \right| \right]. \end{aligned} \tag{26}$$

Proof. The proof uses Theorem 7 and is similar to that of Proposition 5.

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