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OSCILLATION OF THIRD-ORDER NEUTRAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, some new sufficient conditions for the oscillation of all solutions of nonlinear third order neutral dynamic equations are established aiming at extending some well known results in the literature. By using a generalized Riccati transformation and an integral averaging technique, we obtain some new results which ensure that every solution of this equation oscillates or converges to zero. Moreover, an example is given to illustrate the applicability of these results.

1. Introduction

The theory of measure chains was introduced and developed by Hilger [12]. It was created in order to unify continuous and discrete analysis, and it allows a simultaneous treatment of differential and difference equations, extending those theories to so-called dynamic equations. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers with the topology and ordering inherited from $\mathbb{R}$, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Of course many other interesting time scales exist, and they give rise to plenty of applications, for example, in the study of insect population models, neural networks, heat transfer and epidemic models. We refer the reader to the excellent introductory text by Bohner and Peterson [1] as well as the recent research monograph [2]. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various dynamic equations on time scales, e.g., see [3, 6-10, 13-17] and the references cited therein.

In [17], Zhang and Wang studied the second-order nonlinear dynamic equation

$$(r(t)((y(t)+p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+f_1(t,y(\delta_1(t))+f_2(t,y(\delta_2(t))=0$$

on a time scale $\mathbb{T}$.

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In [8], Grace et al. considered the following third-order neutral delay dynamic equation

$$(r(t)(x(t)-a(t)x(\tau(t)))^{\Delta\Delta})^{\Delta}+p(t)x^{\gamma}(\delta(t))=0.$$

Recently, some authors studied on oscillation criteria for third order neutral nonlinear dynamic equations on time scales in [4, 5, 11]. Also, in [15], Utku and Şenel considered the oscillatory behavior of all solutions of the third-order quasilinear neutral delay dynamic equation

$$[r(t)([x(t)+p(t)x(\tau_0(t))]^{\Delta\Delta})^{\gamma}]^{\Delta}+q_1(t)x^{\alpha}(\tau_1(t))+q_2(t)x^{\alpha}(\tau_2(t))=0$$

on a time scale $\mathbb{T}$.

Inspired from the above works, in this paper we consider third-order neutral dynamic equations of the following form:

$$(r_2(t)[(r_1(t)[x(t)+\sum_{i=1}^{2}p_i(t)x(\eta_i(t))]^{\Delta})^{\Delta}]^{\gamma})^{\Delta}+\sum_{i=1}^{2}f_i(t,x(\delta_i(t)))=0 \qquad (1)$$

on a time scale $\mathbb{T}$ satisfying $\inf\mathbb{T}=t_0$ and $\sup\mathbb{T}=\infty$, where $\gamma>0$ is a quotient of odd positive integers.

Throughout this paper we assume the followings:

(H_1) $\eta_i(t),\ \delta_i(t)\in C_{rd}(\mathbb{T},\mathbb{T})$ such that $\eta_1(t)\le t,\ \eta_2(t)\ge t,\ \delta_1(t)\le t,\ \delta_2(t)\ge t$ and $\lim_{t\to\infty}\eta_i(t)=\lim_{t\to\infty}\delta_i(t)=\infty,\quad i=1,2,$

(H_2) $p_i(t)\in C_{rd}(\mathbb{T},[0,1)),\ r_i(t)\in C_{rd}(\mathbb{T},(0,\infty))$ and $r_1^{\Delta}(t)<0$ such that

$$\int_{t_0}^{\infty}\frac{1}{r_1(t)}\Delta t=\infty,\quad \int_{t_0}^{\infty}\left(\frac{1}{r_2(t)}\right)^{\frac{1}{\gamma}}\Delta t=\infty,$$

(H_3) $f_i(t,u):\mathbb{T}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions such that $uf_i(t,u)>0$ for all $u\neq 0$ and there exist $q_i(t)\in C_{rd}(\mathbb{T},\ \mathbb{R}^{+})$ $(i=1,2)$ such that $|uf_i(t,u)|\ge q_i(t)|u|^{\gamma+1}$.

This paper is organized as follows. After this introduction, we introduce some basic lemmas in Section 2. In Section 3, we present the main results and give an example to illustrate the main results.

We use the following notations for convenience and for shortening the equations: $z(t):=x(t)+p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t)),\quad z^{[1]}:=(r_1z^{\Delta})^{\Delta},\quad z^{[2]}:=r_2(z^{[1]})^{\gamma}$ and $z^{[3]}:=(z^{[2]})^{\Delta}$.

For $D=\{(t,s)\in\mathbb{T}^2:t\ge s\ge 0\}$, we define the set $\mathcal{H}=\{H(t,s)\in C^1_{rd}(D,[0,\infty)):H(t,t)=0,\quad H(t,s)>0$ and $H_s^{\Delta}(t,s)\ge 0$ for $t>s\ge 0\}$, the function $\alpha\in C^1_{rd}(\mathbb{T},(0,\infty))$ is to be given Theorem 3.1 and Theorem 3.2 such that $\alpha_+^{\Delta}(t)=max\{\alpha^{\Delta}(t),0\}$ and H_s^{Δ} is the partial derivative of H with respect to second variable.

2. Some Preliminaries

To establish oscillation criteria of (1.1), we give here some useful lemmas which will play an important role in the study of the oscillation behavior for the solutions of (1.1).

Lemma 1 Assume that x is an eventually positive solution of (1.1) holds. Then, there is a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that either

$(i) \quad z(t) > 0, \quad z^{\Delta}(t) > 0, \quad z^{[1]}(t) > 0, \quad t \in [t_1, \infty)_{\mathbb{T}},$

or

$(ii) \quad z(t) > 0, \quad z^{\Delta}(t) < 0, \quad z^{[1]}(t) > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$

Proof. If $x(t)$ is an eventually positive solutions of (1.1), then there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \quad x(\eta_i(t)) > 0, \quad x(\delta_i(t)) > 0, \quad for \quad t \geq t_1, \quad i = 1, 2. \tag{2}$$

From (2.2), $z(t) > 0$ eventually. Since (1.1) and (H_3), we obtain

$$z^{[3]}(t) \leq -q_1(t)x^{\gamma}(\delta_1(t)) - q_2(t)x^{\gamma}(\delta_2(t)) < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{3}$$

which implies that $z^{[2]}(t)$ is a strictly decreasing function on $[t_1, \infty)_{\mathbb{T}}$. We claim that $z^{[2]}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not, there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $z^{[2]}(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$. Then, there exist a negative constant c and $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that

$$z^{[2]}(t) \leq c < 0, \quad t \in [t_3, \infty)_{\mathbb{T}}$$

and it follows that

$$z^{[1]}(t) \leq \left(\frac{c}{r_2(t)}\right)^{\frac{1}{\gamma}}. \tag{4}$$

Integrating (2.4) from t_3 to t and using (H_2), we obtain

$$r_1(t)z^{\Delta}(t) \leq r_1(t_3)z^{\Delta}(t_3) + c^{\frac{1}{\gamma}} \int_{t_3}^{t} \left(\frac{1}{r_2(s)}\right)^{\frac{1}{\gamma}} \Delta s,$$

which implies that $r_1(t)z^{\Delta}(t) \to -\infty$ as $t \to \infty$. Therefore, there exists a $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that

$$r_1(t)z^{\Delta}(t) \leq r_1(t_4)z^{\Delta}(t_4) < 0, \quad t \in [t_4, \infty)_{\mathbb{T}}. \tag{5}$$

Dividing both sides of (2.5) by $r_1(t)$ and integrating it from t_4 to t, we obtain

$$z(t) - z(t_4) \leq r_1(t_4)z^{\Delta}(t_4) \int_{t_4}^{t} \frac{1}{r_1(s)} \Delta s.$$

Hence, we see from (H_2) that $z(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $z(t) > 0$ and $z^{[2]}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Since $r_2(t) > 0$, $z^{[1]}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ it follows that $r_1(t)z^{\Delta}(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$ or $r_1(t)z^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. From $r_1(t) > 0$, $z^{\Delta}(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$ or $z^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$.

The proof is completed. □

Lemma 2 Assume that x is an eventually positive solution of (1.1) and (i) of Lemma 1 holds. Then, there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$z(t) \geq r_1(t, t_1)[z^{[2]}(t)]^{\frac{1}{\gamma}} \quad and \quad z^{\Delta}(t) \geq \frac{r_2(t, t_1)}{r_1(t)}[z^{[2]}(t)]^{\frac{1}{\gamma}} \tag{6}$$

for $t \in [t_1, \infty)_{\mathbb{T}}$ where $r_1(t, t_1) = \int_{t_1}^{t} \frac{r_2(s, t_1)}{r_1(s)} \Delta s$ and $r_2(t, t_1) = \int_{t_1}^{t} \frac{\Delta s}{r_2^{\frac{1}{\gamma}}(s)}$.

Proof. Since $z^{[2]}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} r_1(t)z^{\Delta}(t) &\geq r_1(t)z^{\Delta}(t) - r_1(t_1)z^{\Delta}(t_1) \\ &= \int_{t_1}^{t} \frac{[z^{[2]}(s)]^{\frac{1}{\gamma}}}{(r_2(s))^{\frac{1}{\gamma}}} \Delta s, \end{aligned}$$

it follows that

$$z^{\Delta}(t) \geq \frac{[z^{[2]}(t)]^{\frac{1}{\gamma}}}{r_1(t)} \int_{t_1}^{t} \frac{\Delta s}{(r_2(s))^{\frac{1}{\gamma}}}$$

and so

$$z^{\Delta}(t) \geq \frac{r_2(t,t_1)}{r_1(t)} [z^{[2]}(t)]^{\frac{1}{\gamma}}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{7}$$

Integrating (2.7) from t_1 to t, we obtain

$$z(t) \geq [z^{[2]}(t)]^{\frac{1}{\gamma}} \int_{t_1}^{t} \frac{r_2(s,t_1)}{r_1(s)} \Delta s$$

and so

$$z(t) \geq r_1(t,t_1)[z^{[2]}(t)]^{\frac{1}{\gamma}}, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

The proof is completed. □

Lemma 3 Let x be an eventually positive solution of (1.1). Assume that (i) of Lemma 1 and $0 \leq p_1(t) + p_2(t) \leq p < 1$ holds. If

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_{t}^{\infty} \left[\frac{1}{r_2(s)} \int_{s}^{\infty} [q_1(u) + q_2(u)] \Delta u \right]^{\frac{1}{\gamma}} \Delta s \Delta t = \infty \tag{8}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof. Since (i) of Lemma 1 is satisfied,

$$\lim_{t\to\infty} z(t) = l \geq 0.$$

We claim that $\lim_{t\to\infty} z(t) = 0$. Assume that $l > 0$. Then for any $\epsilon > 0$ we have $l < z(t) < l + \epsilon$ for a sufficiently large $t \in [t_1, \infty)$. Choose $0 < \epsilon < \frac{l(1-p)}{p}$. On the other hand, since

$$z(t) = x(t) + p_1(t)x(\eta_1(t)) + p_2(t)x(\eta_2(t)),$$

there exists a sufficiently large $t_2 \in [t_1, \infty)$, for $t \in [t_2, \infty)$ we have

$$\begin{aligned} x(t) &= z(t) - p_1(t)x(\eta_1(t)) - p_2(t)x(\eta_2(t)) \\ &\geq z(t) - p_1(t)z(\eta_1(t)) - p_2(t)z(\eta_2(t)) \\ &> l - p_1(t)(l+\epsilon) - p_2(t)(l+\epsilon) \\ &= l - (p_1(t) + p_2(t))(l+\epsilon) \\ &> l - p(l+\epsilon) \\ &= k(l+\epsilon) > kz(t), \end{aligned}$$

where $k = \frac{l-p(l+\epsilon)}{l+\epsilon} > 0$.

Then we get

$$x(t) \geq kz(t) \geq kl > 0. \tag{9}$$

Substituting (2.9) to (2.3) for $t \geq t_3$ we obtain

$$\begin{aligned} z^{[3]}(t) &\leq -q_1(t)x^{\gamma}(\delta_1(t)) - q_2(t)x^{\gamma}(\delta_2(t)) \\ &\leq -q_1(t)(kl)^{\gamma} - q_2(kl)^{\gamma} \\ &\leq -q_1(t)(kl)^{\gamma} - q_2(kl)^{\gamma} \\ &= -(kl)^{\gamma}[q_1(t) + q_2(t)]. \end{aligned} \tag{10}$$

Integrating (2.10) from t to ∞, we get

$$z^{[2]}(t) \geq (kl)^{\gamma} \int_t^{\infty} [q_1(s) + q_2(s)]\Delta s.$$

It follows that

$$z^{[1]}(t) \geq kl \left[\frac{1}{r_2(t)} \int_t^{\infty} [q_1(s) + q_2(s)]\Delta s \right]^{\frac{1}{\gamma}}. \tag{11}$$

Integrating (2.11) from t to ∞ and dividing both sides by $r_1(t)$, we obtain

$$-z^{\Delta}(t) \geq \frac{kl}{r_1(t)} \int_t^{\infty} \left[\frac{1}{r_2(u)} \int_u^{\infty} [q_1(s) + q_2(s)]\Delta s \right]^{\frac{1}{\gamma}} \Delta u. \tag{12}$$

Integrating (2.12) from t_3 to ∞, we get

$$z(t_3) \geq kl \int_{t_3}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \left[\frac{1}{r_2(s)} \int_s^{\infty} [q_1(u) + q_2(u)]\Delta u \right]^{\frac{1}{\gamma}} \Delta s \Delta t,$$

which contradicts (2.8) and therefore $l = 0$. By making use of $0 \leq x(t) \leq z(t)$ we conclude that $\lim_{t\to\infty} x(t) = 0$.

The proof is completed. $\square$

Lemma 4 [17] Let $g(u) = Bu - Au^{\frac{\gamma+1}{\gamma}}$, where $A > 0$ and B are constants, γ is a quotient of odd positive integers. Then g attains its maximum value on $\mathbb{R}$ $at\ \ u^* = \left(\frac{B\gamma}{A(\gamma+1)} \right)^{\gamma}$ and

$$\max_{u\in\mathbb{R}} g(u) = \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}}. \tag{13}$$

3. Main Results

In this section, we establish some oscillation criteria for (1.1). Since we are interested in asymptotic behavior of solutions we will suppose that the time scale $\mathbb{T}$ under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0, ?\infty)_{\mathbb{T}}$. Recall a solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Theorem 1 Let $\gamma > 0$. Assume that m is a positive real valued Δ-differentiable function such that

$$\frac{m(t)}{r_1(t,t_1) r_2^{\frac{1}{\gamma}}(t) r_1^{\Delta}(t)} - m^{\Delta}(t) \leq 0 \tag{14}$$

and

$$1 - p_1(t) - p_2(t) \frac{m(\eta_2(t))}{m(t)} > 0. \tag{15}$$

Furthermore

$$\int_{t_0}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \left[\frac{1}{r_2(s)} \int_s^{\infty} [q_1(u) + q_2(u)]\Delta u \right]^{\frac{1}{\gamma}} \Delta s \Delta t = \infty \tag{16}$$

and one of the following conditions satisfied for sufficiently large $t_1 \in \mathbb{T}$:

(i) there exists a function $\alpha \in C^1_{rd}(\mathbb{T},(0,\infty))$ for $\delta_1(T) > t_1$ such that

$$\lim_{t\to\infty} \sup \int_T^t \left[\alpha(s)Q(s) - \frac{\alpha_+^\Delta(s)}{r_1^\gamma(s,t_1)}\right]\Delta s = \infty, \tag{17}$$

where

$$\begin{aligned} Q(s) \quad := \quad & q_1(s)\left(\frac{m(\delta_1(s))}{m(s)}\right)^\gamma \left(1 - p_1(\delta_1(s)) - p_2(\delta_1(s))\frac{m(\eta_2(\delta_1(s)))}{m(\delta_1(s))}\right)^\gamma \\ & + q_2(s)\left(1 - p_1(\delta_2(s)) - p_2(\delta_2(s))\frac{m(\eta_2(\delta_2(s)))}{m(\delta_2(s))}\right)^\gamma, \end{aligned}$$

(ii) there exists a function $\alpha \in C^1_{rd}(\mathbb{T},(0,\infty))$ for $\delta_1(T) > t_1$ such that

$$\lim_{t\to\infty} \sup \int_T^t \left[\alpha(s)Q(s) - \frac{r_1^\gamma(s)(\alpha_+^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\alpha^\gamma(s)r_2(s,t_1)}\right]\Delta s = \infty, \tag{18}$$

(iii) there exists a function $\alpha \in C^1_{rd}(\mathbb{T},(0,\infty))$ and $H \in \mathcal{H}$ for $\delta_1(T) > t_1$ such that

$$\lim_{t\to\infty} \sup \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\alpha(s)Q(s) - \frac{r_1^\gamma(s)[C(t,s)]^{\gamma+1}}{H^\gamma(t,s)(\gamma+1)^{\gamma+1}\alpha^\gamma(s)}\right]\Delta s = \infty, \tag{19}$$

where

$$C(t,s) := H_s^\Delta(t,s)\alpha^\sigma(s) + H(t,s)\alpha^\Delta(s),$$

(iv) there exists a function $\alpha \in C^1_{rd}(\mathbb{T},(0,\infty))$ and $m \geq 1$ for $\delta_1(T) > t_1$ such that

$$\lim_{t\to\infty} \sup \frac{1}{t^m} \int_T^t \left[(t-s)^m\alpha(s)Q(s) - D(s,t_1)\frac{B^{\gamma+1}(t,s)}{(t-s)^{m\gamma}}\right]\Delta s = \infty, \tag{20}$$

where

$$D(s,t_1) := \frac{r_1^\gamma(s)(\alpha(\sigma(s)))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\alpha^\gamma(s)r_2(s,t_1)}$$

and

$$B(t,s) := (t-s)^m\frac{\alpha_+^\Delta(s)}{\alpha^\sigma(s)} - m(t-\sigma(s))^{m-1}.$$

Then every solution of (1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume the contrary and let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $t_1 \in [t_0,\infty)_\mathbb{T}$ such that

$$x(t) > 0, \quad x(\eta_i(t)) > 0, \quad x(\delta_i(t)) > 0, \quad i = 1,2 \quad for \quad t \geq t_1. \tag{21}$$

We first consider that $x(t)$ satisfies Case (i) in Lemma 1. From Lemma 1 and (H_2), we have

$$\begin{aligned} z^{\Delta\Delta}(t) \quad &= \quad \left(\frac{r_1(t)z^\Delta(t)}{r_1(t)}\right)^\Delta \\ &= \quad \frac{(r_1(t)z^\Delta(t))^\Delta r_1(t) - r_1^\Delta(t)(r_1(t)z^\Delta(t))}{r_1(t)r_1(\sigma(t))} \\ &> \quad \frac{(r_1(t)z^\Delta(t))^\Delta}{r_1(\sigma(t))} = \frac{z^{[1]}(t)}{r_1(\sigma(t))} > 0. \end{aligned} \tag{22}$$

Using Lemma 2 and (3.22), we get

$$\begin{aligned} z(t) &\geq r_1(t,t_1)[z^{[2]}(t)]^{\frac{1}{\gamma}} \\ &= r_1(t,t_1)r_2^{\frac{1}{\gamma}}(t)z^{[1]}(t) \\ &= r_1(t,t_1)r_2^{\frac{1}{\gamma}}(t)r_1^{\Delta}(t)z^{\Delta}(t)+r_1(t,t_1)r_2^{\frac{1}{\gamma}}(t)r_1^{\sigma}(t)z^{\Delta\Delta}(t) \\ &> r_1(t,t_1)r_2^{\frac{1}{\gamma}}(t)r_1^{\Delta}(t)z^{\Delta}(t). \end{aligned} \tag{23}$$

It follows from (3.23) that

$$z^{\Delta}(t) \leq \frac{z(t)}{r_1(t,t_1)r_2^{\frac{1}{\gamma}}(t)r_1^{\Delta}(t)}. \tag{24}$$

Since (3.24), we have

$$\begin{aligned} \left(\frac{z(t)}{m(t)}\right)^{\Delta} &= \frac{z^{\Delta}(t)m(t)-z(t)m^{\Delta}(t)}{m(t)m^{\sigma}(t)} \\ &\leq \frac{z(t)}{m(t)m^{\sigma}(t)}\left[\frac{m(t)}{r_1(t,t_1)r_2^{\frac{1}{\gamma}}(t)r_1^{\Delta}(t)}-m^{\Delta}(t)\right] \leq 0 \end{aligned}$$

and so $\frac{z(t)}{m(t)}$ is decreasing. Since $z^{\Delta}(t)>0$ and $\eta_1(t)\leq t$, we get

$$-z(\eta_1(t)) \geq -z(t). \tag{25}$$

Also, from $\frac{z(t)}{m(t)}$ is decreasing and $\eta_2(t)\geq t$, we get

$$-z(\eta_2(t)) \geq -\frac{m(\eta_2(t))}{m(t)}z(t). \tag{26}$$

Since $z(t)\geq x(t)$ and using (3.25)-(3.26), we obtain

$$\begin{aligned} x(t) &= z(t)-p_1(t)x(\eta_1(t))-p_2(t)x(\eta_2(t)) \\ &\geq z(t)-p_1(t)z(\eta_1(t))-p_2(t)z(\eta_2(t)) \\ &\geq \left(1-p_1(t)-p_2(t)\frac{m(\eta_2(t))}{m(t)}\right)z(t). \end{aligned} \tag{27}$$

Since $\frac{z(t)}{m(t)}$ is decreasing, $\delta_1(t)\leq t$ and $\gamma>0$, we have

$$-z^{\gamma}(\delta_1(t)) \leq -\left(\frac{m(\delta_1(t))}{m(t)}\right)^{\gamma}z^{\gamma}(t). \tag{28}$$

Also, from $z^{\Delta}(t)>0$, $\delta_2(t)\geq t$ and $\gamma>0$, we get

$$-z^{\gamma}(\delta_2(t)) \leq -z^{\gamma}(t). \tag{29}$$

By virtue of (3.27)-(3.29), (1.1) and (H_3), we obtain

$$\begin{aligned}
z^{[3]}(t) &= -f_1(t, x(\delta_1(t)) - f_2(t, x(\delta_2(t)) \\
&\leq -q_1(t)x^\gamma(\delta_1(t)) - q_2(t)x^\gamma(\delta_2(t)) \\
&\leq -q_1(t)\left(1 - p_1(\delta_1(t)) - p_2(\delta_1(t))\frac{m(\eta_2(\delta_1(t)))}{m(\delta_1(t))}\right)^\gamma z^\gamma(\delta_1(t)) \\
&\quad -q_2(t)\left(1 - p_1(\delta_2(t)) - p_2(\delta_2(t))\frac{m(\eta_2(\delta_2(t)))}{m(\delta_2(t))}\right)^\gamma z^\gamma(\delta_2(t)) \\
&\leq -\Bigg[q_1(t)\left(1 - p_1(\delta_1(t)) - p_2(\delta_1(t))\frac{m(\eta_2(\delta_1(t)))}{m(\delta_1(t))}\right)^\gamma \left(\frac{m(\delta_1(t))}{m(t)}\right)^\gamma \\
&\quad +q_2(t)\left(1 - p_1(\delta_2(t)) - p_2(\delta_2(t))\frac{m(\eta_2(\delta_2(t)))}{m(\delta_2(t))}\right)^\gamma\Bigg] z^\gamma(t) \\
&= -Q(t)z^\gamma(t).
\end{aligned}$$

It follows that

$$z^{[3]}(t) + Q(t)z^\alpha(t) \leq 0. \tag{30}$$

(i) Let we define

$$w(t) := \alpha(t)\frac{z^{[2]}(t)}{z^\gamma(t)}, \quad for \quad t \geq t_1. \tag{31}$$

Then $w(t) > 0$ for $t \geq t_1$. From (3.31), we have

$$\begin{aligned}
w^\Delta(t) &= (z^{[2]}(t))^\Delta\left(\frac{\alpha(t)}{z^\gamma(t)}\right) + (z^{[2]}(t))^\sigma\left(\frac{\alpha^\Delta(t)z^\gamma(t) - \alpha(t)(z^\gamma(t))^\Delta}{z^\gamma(t)(z^\gamma(t))^\sigma}\right) \\
&\leq -\alpha(t)Q(t) + \frac{(z^{[2]}(t))^\sigma}{z^\gamma(\sigma(t))}\alpha_+^\Delta(t) - \frac{(z^{[2]}(t))^\sigma\alpha(t)(z^\gamma(t))^\Delta}{z^\gamma(t)(z^\gamma(t))^\sigma}.
\end{aligned} \tag{32}$$

By the P?tzsche chain rule, if $z^\Delta(t) > 0$ and $\gamma \geq 1$, then

$$\begin{aligned}
(z^\gamma(t))^\Delta &= \gamma\int_0^1 [z(t) + \mu(t)hz^\Delta(t)]^{\gamma-1}z^\Delta(t)dh \\
&= \gamma\int_0^1 [(1-h)z(t) + hz^\sigma(t)]^{\gamma-1}z^\Delta(t)dh \\
&\geq \gamma\int_0^1 (z(t))^{\gamma-1}z^\Delta(t)dh \\
&= \gamma(z(t))^{\gamma-1}z^\Delta(t).
\end{aligned} \tag{33}$$

Also again by the P?tzsche chain rule, if $z^\Delta(t) > 0$ and $0 < \gamma < 1$, then

$$\begin{aligned}
(z^\gamma(t))^\Delta &= \gamma\int_0^1 [z(t) + \mu(t)hz^\Delta(t)]^{\gamma-1}z^\Delta(t)dh \\
&= \gamma\int_0^1 [(1-h)z(t) + hz^\sigma(t)]^{\gamma-1}z^\Delta(t)dh \\
&\geq \gamma\int_0^1 (z^\sigma(t))^{\gamma-1}z^\Delta(t)dh \\
&= \gamma(z^\sigma(t))^{\gamma-1}z^\Delta(t).
\end{aligned} \tag{34}$$

Since $z^{[2]}(t) > 0$ and using (3.33)-(3.34), we get

$$\frac{(z^{[2]}(t))^\sigma \alpha(t)(z^\gamma(t))^\Delta}{z^\gamma(t)(z^\gamma(t))^\sigma} \geq 0.$$

In view of $z^{[2]}(t)$ is decreasing and $t \leq \sigma(t)$, we have

$$(z^{[2]}(t))^\sigma \leq z^{[2]}(t). \tag{35}$$

Therefore, from (3.35), Lemma 2 and $z^\Delta(t) > 0$, we obtain

$$w^\Delta(t) \leq -\alpha(t)Q(t) + \frac{\alpha_+^\Delta(t)}{r_1^\gamma(t, t_1)}.$$

Integrating the above inequality from T to t for $t \geq T$, we get

$$\int_T^t \left[\alpha(s)Q(s) - \frac{\alpha_+^\Delta(s)}{r_1^\gamma(s, t_1)}\right]\Delta s \leq w(t_1) - w(t) < w(t_1). \tag{36}$$

Taking lim sup on both sides as $t \to \infty$, we obtain a contradiction with (3.17). Therefore, every solution $x(t)$ of (1.1) is oscillatory. When (ii) of Lemma 1 holds, we can conclude from Lemma 3 that $\lim_{t\to\infty} x(t) = 0$.

(ii) From (3.32), we have

$$w^\Delta(t) \leq -\alpha(t)Q(t) + w^\sigma(t)\frac{\alpha^\Delta(t)}{\alpha^\sigma(t)} - \frac{(z^{[2]}(t))^\sigma \alpha(t)(z^\gamma(t))^\Delta}{z^\gamma(t)(z^\gamma(t))^\sigma}.$$

Since $z(t)$ is increasing, $z^{[2]}(t)$ is decreasing and using Lemma 2, for $\gamma > 1$, we get

$$\begin{aligned}
\frac{(z^{[2]}(t))^\sigma \alpha(t)(z^\gamma(t))^\Delta}{z^\gamma(t)(z^\gamma(t))^\sigma} &\geq \frac{\gamma(z(t))^{\gamma-1}z^\Delta(t)\alpha(t)(z^{[2]}(t))^\sigma}{z^\gamma(t)(z^\gamma(t))^\sigma} \\
&\geq \frac{\gamma z^\Delta(t)\alpha(t)(z^{[2]}(t))^\sigma}{z^{\gamma+1}(\sigma(t))} \\
&\geq \frac{\gamma r_2(t, t_1)[z^{[2]}(t)]^{\frac{1}{\gamma}}\alpha(t)(z^{[2]}(t))^\sigma}{r_1(t)z^{\gamma+1}(\sigma(t))} \\
&\geq \gamma\frac{r_2(t, t_1)\alpha(t)}{r_1(t)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(t))}(w^\sigma(t))^{\frac{\gamma+1}{\gamma}}.
\end{aligned} \tag{37}$$

From $z(t)$ is increasing, $z^{[2]}(t)$ is decreasing and Lemma 2, for $0 < \gamma \leq 1$, we get

$$\begin{aligned}
\frac{(z^{[2]}(t))^\sigma \alpha(t)(z^\gamma(t))^\Delta}{z^\gamma(t)(z^\gamma(t))^\sigma} &\geq \frac{\gamma(z^\sigma(t))^{\gamma-1}z^\Delta(t)\alpha(t)(z^{[2]}(t))^\sigma}{z^\gamma(t)(z^\gamma(t))^\sigma} \\
&\geq \frac{\gamma z^\Delta(t)\alpha(t)(z^{[2]}(t))^\sigma}{z^{\gamma+1}(\sigma(t))} \\
&\geq \frac{\gamma r_2(t, t_1)[z^{[2]}(t)]^{\frac{1}{\gamma}}\alpha(t)(z^{[2]}(t))^\sigma}{r_1(t)z^{\gamma+1}(\sigma(t))} \\
&\geq \gamma\frac{r_2(t, t_1)\alpha(t)}{r_1(t)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(t))}(w^\sigma(t))^{\frac{\gamma+1}{\gamma}}.
\end{aligned} \tag{38}$$

From (3.37) and (3.38), we obtain

$$w^\Delta(t) \leq -\alpha(t)Q(t) + w^\sigma(t)\frac{\alpha^\Delta(t)}{\alpha^\sigma(t)} - \gamma\frac{r_2(t, t_1)\alpha(t)}{r_1(t)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(t))}(w^\sigma(t))^{\frac{\gamma+1}{\gamma}}. \tag{39}$$

Setting

$$B=\frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}(t)},\quad A=\gamma\frac{r_2(t,t_1)\alpha(t)}{r_1(t)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(t))},\quad u=w^{\sigma}(t)$$

and using Lemma 4, we obtain

$$w^{\Delta}(t)\leq-\left[\alpha(t)Q(t)+\frac{r_1^{\gamma}(s)(\alpha_+^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\alpha^{\gamma}(s)r_2(t,t_1)}\right].$$

Integrating the above inequality from T to t for $t\geq T$, we obtain

$$\int_T^t\left[\alpha(s)Q(s)-\frac{r_1^{\gamma}(s)(\alpha_+^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\alpha^{\gamma}(s)r_2(s,t_1)}\right]\Delta s\leq w(t_1)-w(t)<w(t_1). \tag{40}$$

Taking lim sup on both sides as $t\to\infty$, we obtain a contradiction with (3.18). Therefore, every solution $x(t)$ of (1.1) is oscillatory. When Case (ii) of Lemma 1 holds, we can conclude from Lemma 3 that $\lim_{t\to\infty}x(t)=0$.

(iii) Since (3.39), we have that for $H\in\mathcal{H}$ and $t\geq T$

$$\begin{aligned}\int_T^t H(t,s)\alpha(s)Q(s)\Delta s\quad\leq\quad&-\int_T^t H(t,s)w^{\Delta}(s)\Delta s+\int_T^t H(t,s)w^{\sigma}(s)\frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)}\Delta s\\&-\int_T^t\gamma\frac{H(t,s)r_2(s,t_1)\alpha(s)}{r_1(s)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(s))}(w^{\sigma}(s))^{\frac{\gamma+1}{\gamma}}\Delta s.\end{aligned}$$

By integration by parts we obtain

$$-\int_T^t H(t,s)w^{\Delta}(s)\Delta s=H(t,T)w(T)+\int_T^t H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s.$$

It follows that

$$\begin{aligned}\int_T^t H(t,s)\alpha(s)Q(s)\Delta s\quad\leq\quad&H(t,t_1)w(t_1)+\int_T^t\left[H_s^{\Delta}(t,s)+H(t,s)\frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)}\right]w^{\sigma}(s)\Delta s\\&-\int_T^t\gamma\frac{H(t,s)r_2(s,t_1)\alpha(s)}{r_1(s)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(s))}(w^{\sigma}(s))^{\frac{\gamma+1}{\gamma}}\Delta s.\end{aligned}$$

Let

$$B=\frac{C(t,s)}{\alpha^{\sigma}(s)},\quad A=\frac{\gamma H(t,s)r_2(s,t_1)\alpha(s)}{r_1(s)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(s))},\quad u=w^{\sigma}(s).$$

From Lemma 4, we obtain that for all $t\geq T$,

$$\int_T^t H(t,s)\alpha(s)Q(s)\Delta s\quad\leq\quad H(t,T)w(T)+\int_T^t\frac{[C(t,s)]^{\gamma+1}r_1^{\gamma}(s)}{H^{\gamma}(t,s)(\gamma+1)^{\gamma+1}\alpha^{\gamma}(s)}\Delta s.$$

That is,

$$\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\alpha(s)Q(s)-\frac{r_1^{\gamma}(s)[C(t,s)]^{\gamma+1}}{H^{\gamma}(t,s)(\gamma+1)^{\gamma+1}\alpha^{\gamma}(s)}\right]\Delta s\leq w(T).$$

Taking lim sup on both sides as $t\to\infty$, we obtain a contradiction with (3.19). Therefore, every solution $x(t)$ of (1.1) is oscillatory. When (ii) of Lemma 1 holds, we can conclude from Lemma 3 that $\lim_{t\to\infty}x(t)=0$.

(iv) Multiplying (3.39) by $(t-s)^m$ and integrating from T to t, we have

$$\begin{aligned}\int_T^t (t-s)^m \alpha(s)Q(s)\Delta s &\leq -\int_T^t (t-s)^m w^{\Delta}(s)\Delta s + \int_T^t (t-s)^m w^{\sigma}(s)\frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)}\Delta s \\ &\quad - \int_T^t \gamma \frac{(t-s)^m r_2(s,t_1)\alpha(s)}{r_1(s)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(s))}(w^{\sigma}(s))^{\frac{\gamma+1}{\gamma}}\Delta s. \qquad (41)\end{aligned}$$

By integration by parts we obtain

$$-\int_T^t (t-s)^m w^{\Delta}(s)\Delta s = (t-T)^m w(T) + \int_T^t ((t-s)^m)^{\Delta_s} w^{\sigma}(s)\Delta s. \qquad (42)$$

Next, we show that if $t \geq \sigma(s)$ and $m \geq 1$, then

$$((t-s)^m)^{\Delta_s} \leq -m(t-\sigma(s))^{m-1}. \qquad (43)$$

If $\mu(s) = 0$ then we have

$$((t-s)^m)^{\Delta_s} = -m(t-s)^{m-1}.$$

If $\mu(s) \neq 0$ then we have

$$\begin{aligned}((t-s)^m)^{\Delta_s} &= \frac{1}{\mu(s)}[(t-\sigma(s))^m - (t-s)^m] \\ &= -\frac{1}{\sigma(s)-s}[(t-s)^m - (t-\sigma(s))^m].\end{aligned}$$

Using inequality

$$A^{\gamma} - B^{\gamma} \geq \gamma B^{\gamma-1}(A-B),$$

where A and B are nonnegative constants and $\gamma \geq 1$, we have

$$[(t-s)^m - (t-\sigma(s))^m] \geq m(t-\sigma(s))^{m-1}(\sigma(s)-s)$$

so we see that (3.43) holds.

From (3.41)-(3.43), we obtain

$$\begin{aligned}\int_T^t (t-s)^m \alpha(s)Q(s)\Delta s &\leq (t-T)w(T) \\ &\quad + \int_T^t \left[(t-s)^m \frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)} - m(t-\sigma(s))^{m-1}\right] w^{\sigma}(s)\Delta s \\ &\quad - \int_T^t \gamma \frac{(t-s)^m r_2(s,t_1)\alpha(s)}{r_1(s)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(s))}(w^{\sigma}(s))^{\frac{\gamma+1}{\gamma}}\Delta s.\end{aligned}$$

Let

$$u := w^{\sigma}(s), \quad B := (t-s)^m \frac{\alpha^{\Delta}(s)}{\alpha^{\sigma}(s)} - m(t-\sigma(s))^{m-1}, \quad A := \gamma \frac{(t-s)^m r_2(s,t_1)\alpha(s)}{r_1(s)\alpha^{\frac{\gamma+1}{\gamma}}(\sigma(s))}.$$

From Lemma 4, we obtain that for all $t \geq T$,

$$\begin{aligned}\int_T^t (t-s)^m \alpha(s)Q(s)\Delta s &\leq (t-T)^m w(T) \\ &\quad + \int_T^t \frac{[B(t,s)]^{\gamma+1} r_1^{\gamma}(s)}{(t-s)^{m\gamma}(\gamma+1)^{\gamma+1}\alpha^{\gamma}(s) r_2(s,t_1)}\Delta s,\end{aligned}$$

that is,

$$\frac{1}{t^m}\int_T^t \left[(t-s)^m \alpha(s)Q(s) - D(s,t_1)\frac{B^{\gamma+1}(t,s)}{(t-s)^{m\gamma}}\right]\Delta s \le w(T). \tag{44}$$

Taking $\limsup$ on both sides as $t\to\infty$, we obtain a contradiction with (3.20). Therefore, every solution $x(t)$ of (1.1) is oscillatory. When Case (ii) of Lemma 1 holds, we can conclude from Lemma 3 that $\lim_{t\to\infty} x(t) = 0$.

The proof is completed. $\square$

Theorem 2 Let $\gamma \ge 1$. Assume that m is a positive real valued Δ-differentiable function such that

$$\frac{m(t)}{r_1(t,t_1)r_2^{\frac{1}{\gamma}}(t)r_1^{\Delta}(t)} - m^{\Delta}(t) \le 0 \tag{45}$$

and

$$1 - p_1(t) - p_2(t)\frac{m(\eta_2(t))}{m(t)} > 0. \tag{46}$$

Furthermore

$$\int_{t_0}^{\infty}\frac{1}{r_1(t)}\int_t^{\infty}\left[\frac{1}{r_2(s)}\int_s^{\infty}[q_1(u)+q_2(u)]\Delta u\right]^{\frac{1}{\gamma}}\Delta s\Delta t = \infty \tag{47}$$

and one of the following conditions satisfied for sufficiently large $t_1 \in \mathbb{T}$:

(i) there exists a function $\alpha \in C^1_{rd}(\mathbb{T},(0,\infty))$ for $\delta_1(T) > t_1$ such that

$$\lim_{t\to\infty}\sup\int_T^t\left[\alpha(s)Q(s) - \frac{r_1^{\gamma}(s)(\alpha_+^{\Delta}(s))^2}{2^{3-\gamma}(\mu(s))^{\gamma-1}\alpha(s)r_2^{\gamma}(s,t_1)}\right]\Delta s = \infty, \tag{48}$$

where

$$\begin{aligned} Q(s) \quad := \quad & q_1(s)\left(\frac{m(\delta_1(s))}{m(s)}\right)^{\gamma}\left(1 - p_1(\delta_1(s)) - p_2(\delta_1(s))\frac{m(\eta_2(\delta_1(s)))}{m(\delta_1(s))}\right)^{\gamma} \\ & + q_2(s)\left(1 - p_1(\delta_2(s)) - p_2(\delta_2(s))\frac{m(\eta_2(\delta_2(s)))}{m(\delta_2(s))}\right)^{\gamma}, \end{aligned}$$

(ii) there exists a function $\alpha \in C^1_{rd}(\mathbb{T},(0,\infty))$ for $\delta_1(T) > t_1$ such that

$$\lim_{t\to\infty}\sup\frac{1}{H(t,T)}\int_T^t\left[D(t,s) - \frac{[C(t,s)]^2 r_1^{\gamma}(s)}{H(t,s)2^{3-\gamma}\alpha(s)(\mu(s))^{\gamma-1}r_2^{\gamma}(s,t_1)}\right]\Delta s = \infty, \tag{49}$$

where

$$C(t,s) := H_s^{\Delta}(t,s)\alpha^{\sigma}(s) + H(t,s)\alpha^{\Delta}(s), \quad D(t,s) := H(t,s)\alpha(s)Q(s),$$

(iii) there exists a function $\alpha \in C^1_{rd}(\mathbb{T},(0,\infty))$ and $H \in \mathcal{H}$ for $\delta_1(T) > t_1$ such that

$$\lim_{t\to\infty}\sup\frac{1}{t^m}\int_T^t\left[(t-s)^m\alpha(s)Q(s) - E(s,t_1)\frac{[F(t,s)]^2}{(t-s)^m}\right]\Delta s = \infty, \tag{50}$$

where

$$E(s,t_1) := \frac{r_1^{\gamma}(s)}{2^{3-\gamma}\alpha(s)(\mu(s))^{\gamma-1}r_2^{\gamma}(s,t_1)}$$

and

$$F(t,s) := (t-s)^m\alpha^{\Delta}(s) - m(t-\sigma(s))^{m-1}\alpha^{\sigma}(s).$$

Then every solution of (1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume the contrary and let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \quad x(\eta_i(t)) > 0, \quad x(\delta_i(t)) > 0, \quad i = 1, 2 \quad for \quad t \geq t_1.$$

We first consider that $x(t)$ satisfies Case (i) in Lemma 1.

(i) Using the inequality

$$x^\gamma - y^\gamma \geq 2^{1-\gamma}(x-y)^\gamma, \quad \gamma \geq 1$$

we have

$$\begin{aligned}
(z^\gamma(t))^\Delta &= \frac{z^\gamma(\sigma(t)) - z^\gamma(t)}{\mu(t)} \geq 2^{1-\gamma}\frac{1}{\mu(t)}(z^\gamma(\sigma(t)) - z^\gamma(t))^\gamma \\
&= 2^{1-\gamma}(\mu(t))^{\gamma-1}\left(\frac{z^\gamma(\sigma(t)) - z^\gamma(t)}{\mu(t)}\right)^\gamma = 2^{1-\gamma}(\mu(t))^{\gamma-1}(z^\Delta(t))^\gamma.
\end{aligned}$$

From (3.33), (3.39), $z^\Delta(t) > 0$, Lemma 2 and using the fact that $u - mu^2 \leq \frac{1}{4m}$ for every u, we obtain

$$\begin{aligned}
w^\Delta(t) &\leq -\alpha(t)Q(t) + w^\sigma(t)\frac{\alpha^\Delta(t)}{\alpha^\sigma(t)} - \frac{(z^{[2]}(t))^\sigma \alpha(t) 2^{1-\gamma}(\mu(t))^{\gamma-1}(z^\Delta(t))^\gamma}{z^\gamma(t)(z^\gamma(t))^\sigma} \\
&\leq -\alpha(t)Q(t) + w^\sigma(t)\frac{\alpha^\Delta(t)}{\alpha^\sigma(t)} - \frac{2^{1-\gamma}(\mu(t))^{\gamma-1}(z^\Delta(t))^\gamma}{(z^{[2]}(t))^\sigma(\alpha^\sigma(t))^2}(w^\sigma(t))^2 \\
&\leq -\alpha(t)Q(t) + w^\sigma(t)\frac{\alpha^\Delta(t)}{\alpha^\sigma(t)} - \frac{\alpha(t)2^{1-\gamma}(\mu(t))^{\gamma-1}r_2^\gamma(t,t_1)}{(\alpha^\sigma(t))^2 r_1^\gamma(t)}(w^\sigma(t))^2 \\
&\leq -\alpha(t)Q(t) + \frac{\alpha^\Delta(t)}{\alpha^\sigma(t)}\left[w^\sigma(t) - \frac{\alpha(t)2^{1-\gamma}(\mu(t))^{\gamma-1}r_2^\gamma(t,t_1)}{(\alpha^\sigma(t)r_1^\gamma(t)\alpha^\Delta(t)}(w^\sigma(t))^2\right] \\
&\leq -\alpha(t)Q(t) + \frac{r_1^\gamma(t)(\alpha^\Delta(t))^2}{2^{3-\gamma}(\mu(t))^{\gamma-1}\alpha(t)r_2^\gamma(t,t_1)}.
\end{aligned}$$

Integrating the last inequality from T to t, we obtain

$$\int_T^t \left[\alpha(s)Q(s) - \frac{r_1^\gamma(s)(\alpha_+^\Delta(s))^2}{2^{3-\gamma}(\mu(s))^{\gamma-1}\alpha(s)r_2^\gamma(s,t_1)}\right]\Delta s \leq w(t_1).$$

Taking $\limsup$ on both sides as $t \to \infty$, we obtain a contradiction with (3.48). Therefore, every solution $x(t)$ of (1.1) is oscillatory. When Case (ii) of Lemma 1 holds, we can conclude from Lemma 3 that $\lim_{t\to\infty} x(t) = 0$.

(ii) From (3.39), we have that for $H \in \mathcal{H}$ and $t \geq t_1$

$$\begin{aligned}
\int_T^t H(t,s)\alpha(s)Q(s)\Delta s &\leq -\int_T^t H(t,s)w^\Delta(s)\Delta s + \int_T^t H(t,s)\frac{\alpha^\Delta(s)}{\alpha^\sigma(s)}w^\sigma(s) \\
&\quad - \int_T^t \frac{H(t,s)\alpha(s)2^{1-\gamma}(\mu(s))^{\gamma-1}r_2^\gamma(s,t_1)}{(\alpha^\sigma(s))^2 r_1^\gamma(s)}(w^\sigma(s))^2\Delta s.
\end{aligned}$$

By integration by parts we obtain

$$-\int_T^t H(t,s)w^\Delta(s)\Delta s = H(t,T)w(T) + \int_T^t H_s^\Delta(t,s)w^\sigma(s)\Delta s.$$

Using the fact that $u - mu^2 \leq \frac{1}{4m}$, we obtain

$$\begin{aligned}\int_T^t H(t,s)\alpha(s)Q(s)\Delta s &\leq H(t,T)w(T) + \int_T^t [H_s^\Delta(t,s) + H(t,s)\frac{\alpha^\Delta(s)}{\alpha^\sigma(s)}]w^\sigma(s)\Delta s \\ &\quad - \int_T^t \frac{H(t,s)\alpha(s)2^{1-\gamma}(\mu(s))^{\gamma-1}r_2^\gamma(s,t_1)}{(\alpha^\sigma(s))^2 r_1^\gamma(s)}(w^\sigma(s))^2\Delta s \\ &\leq H(t,T)w(T) + \int_T^t \frac{[C(t,s)]^2 r_1^\gamma(s)}{H(t,s)2^{3-\gamma}\alpha(s)(\mu(s))^{\gamma-1}r_2^\gamma(s,t_1)}\Delta s.\end{aligned}$$

It follows that

$$\frac{1}{H(t,T)}\int_T^t \left[H(t,s)\alpha(s)Q(s) - \frac{[C(t,s)]^2 r_1^\gamma(s)}{H(t,s)2^{3-\gamma}\alpha(s)(\mu(s))^{\gamma-1}r_2^\gamma(s,t_1)}\right]\Delta s \leq w(T).$$

Taking lim sup on both sides as $t \to \infty$, we obtain a contradiction with (3.49). Therefore, every solution $x(t)$ of (1.1) is oscillatory. When Case (ii) of Lemma 1 holds, we can conclude from Lemma 3 that $\lim_{t\to\infty} x(t) = 0$.

(iii) Multiplying (3.39) by $(t-s)^m$ and integrating from T to t, we have

$$\begin{aligned}\int_T^t (t-s)^m\alpha(s)Q(s)\Delta s &\leq -\int_T^t (t-s)^m w^\Delta(s)\Delta s + \int_T^t (t-s)^m\frac{\alpha^\Delta(s)}{\alpha^\sigma(s)}w^\sigma(s) \\ &\quad - \int_T^t \frac{(t-s)^m\alpha(s)2^{1-\gamma}(\mu(s))^{\gamma-1}r_2^\gamma(s,t_1)}{(\alpha^\sigma(s))^2 r_1^\gamma(s)}(w^\sigma(s))^2\Delta s.\end{aligned}$$

Since (3.42), (3.43) and the fact that $u - mu^2 \leq \frac{1}{4m}$, we obtain

$$\begin{aligned}\int_T^t (t-s)^m\alpha(s)Q(s)\Delta s &\leq (t-T)^m w(T) \\ &\quad + \int_T^t [(t-s)^m\frac{\alpha^\Delta(s)}{\alpha^\sigma(s)} - m(t-\sigma(s))^{m-1}]w^\sigma(s)\Delta s \\ &\quad - \int_T^t \frac{(t-s)^m\alpha(s)2^{1-\gamma}(\mu(s))^{\gamma-1}r_2^\gamma(s,t_1)}{(\alpha^\sigma(s))^2 r_1^\gamma(s)}(w^\sigma(s))^2\Delta s \\ &\leq H(t,T)w(T) + \int_T^t \frac{[B(t,s)]^2 r_1^\gamma(s)}{(t-s)^m 2^{3-\gamma}\alpha(s)(\mu(s))^{\gamma-1}r_2^\gamma(s,t_1)}\Delta s.\end{aligned}$$

That is,

$$\frac{1}{t^m}\int_T^t \left[(t-s)^m\alpha(s)Q(s) - E(s,t_1)\frac{[F(t,s)]^2}{(t-s)^m}\right]\Delta s \leq w(T).$$

Taking lim sup on both sides as $t \to \infty$, we obtain a contradiction with (3.50). Therefore, every solution $x(t)$ of (1.1) is oscillatory. When Case (ii) of Lemma 1 holds, we can conclude from Lemma 3 that $\lim_{t\to\infty} x(t) = 0$.

The proof is completed. $\square$

Example 1 Let $\mathbb{T} = \mathbb{Z}$ and we consider the following third order neutral dynamic equation:

$$\left(\left(\left(\frac{1}{t}(x(t)+\frac{1}{2}x(t-1)+\frac{t}{3(t+1)}x(t+1)\right)^\Delta\right)^\Delta\right)^3\right)^\Delta + \frac{8}{t^{\frac{1}{3}}}x^3\left(\frac{t}{2}\right) + \frac{2}{t^{\frac{1}{3}}}x^3(2t) = 0,$$

$t \in [2,\infty)_{\mathbb{T}}$.

(H_1) $\eta_1(t) = t-1 \le t, \quad \eta_1(t) = t+1 \ge t, \quad \delta_1(t) = \frac{t}{2} \le t, \quad \delta_2(t) = 2t \ge t,$ $\lim_{t\to\infty} \eta_i(t) = \lim_{t\to\infty} \delta_i(t) = \infty,$

(H_2) $r_1(t) = \frac{1}{t} \in C_{rd}(\mathbb{T}, \mathbb{R}^+), \quad r_1^{\Delta}(t) = -\frac{1}{t(t+1)} < 0, \quad r_2(t) = 1 \in C_{rd}(\mathbb{T}, \mathbb{R}^+),$ $\int_2^\infty \left(\frac{1}{r_2(t)}\right)^{\frac{1}{\gamma}} \Delta t = \lim_{a\to\infty} \int_2^a \Delta t = \lim_{a\to\infty} a - 2 = \infty, \quad \int_2^\infty \frac{1}{\frac{1}{t}} \Delta t = \int_2^\infty t\Delta t = \lim_{a\to\infty} \sum_2^a k = \lim_{a\to\infty} \frac{a(a+1)}{2} - 1 = \infty, \quad p_1(t) = \frac{1}{2} \in C_{rd}(\mathbb{T}, [0,1)), \quad p_2(t) = \frac{t}{3(t+1)} \in C_{rd}(\mathbb{T}, [0,1)), \quad 0 < p_1(t) + p_2(t) < \frac{1}{2} + \frac{1}{3} < 1,$

(H_3) $q_1(t) = \frac{8}{t^{\frac{1}{3}}} \in C_{rd}(\mathbb{T}, \mathbb{R}^+), q_2(t) = \frac{2}{t^{\frac{1}{3}}} \in C_{rd}(\mathbb{T}, \mathbb{R}^+),$ $r_2(t, t_1) = \int_{t_1}^t \Delta s = t - t_1,$

$$\begin{aligned} r_1(t,t_1) &= \int_{t_1}^t \frac{s-t_1}{\frac{1}{s}} \Delta s = \int_{t_1}^t (s^2 - t_1 s)\Delta s = \sum_1^{t-1} s^2 - t_1 s - \sum_1^{t_1-1} s^2 - t_1 s \\ &= \frac{(t-1)t(2t-1)}{6} - \frac{t_1(t-1)t}{2} - \frac{(t_1-1)t_1(2t_1-1)}{6} + \frac{t_1(t_1-1)t_1}{2} \\ &= \frac{t(t-1)(2t-1-3t_1) + t_1(t_1-1)(t_1+1)}{6} \end{aligned}$$

and so

$$r_1(t,t_1) r_2^{\frac{1}{3}}(t) r_1^{\Delta}(t) = \left(\frac{t(t-1)(2t-1-3t_1) + t_1(t_1-1)(t_1+1)}{6}\right)\left(-\frac{1}{t(t+1)}\right).$$

Let $m(t) = t$, thus

$$1 - p_1(t) - p_2(t)\frac{m(\eta_2(t))}{m(t)} = 1 - \frac{1}{2} - \frac{t}{3(t+1)}\frac{t+1}{t} = \frac{1}{6} > 0.$$

Let we define,

$$h(t) = t(t-1)(2t-1-3t_1) + t_1(t_1-1)(t_1+1).$$

For $t > t_1$, we get

$$\begin{aligned} h^{\Delta}(t) &= (t-1)(2t-1-3t_1) + (t+1)[1.(2t-1-3t_1) + t.2] \\ &= (t-1)(2t-1-3t_1) + (t+1)[4t-1-3t_1] \\ &= 2t^2 - t - 3tt_1 - 2t + 1 + 3t_1 + 4t^2 - 3tt_1 - t + 4t - 3t_1 - 1 \\ &= 6t^2 - 6tt_1 = 6t(t-t_1) > 0, \end{aligned}$$

so the function $h(t)$ is increasing.
Since

$$\begin{aligned} h(t_1) &= t_1(t_1-1)(2t_1-1-3t_1) + t_1(t_1-1)(t_1+1) \\ &= t_1(t_1-1)(2t_1-1-3t_1+t_1+1) = 0, \end{aligned}$$

we have

$$h(t) > h(t_1) = 0.$$

It follows that

$$-\frac{6t^2(t+1)}{t(t-1)(2t-1-3t_1) + t_1(t_1-1)(t_1+1)} \le 0.$$

So we have

$$\frac{m(t)}{r_1(t,t_1)r_2^{\frac{1}{3}}(t)r_1^{\Delta}(t)} - m^{\Delta}(t) = -\frac{6t^2(t+1)}{t(t-1)(2t-1-3t_1)+t_1(t_1-1)(t_1+1)} - 1 \leq 0$$

and

$$\begin{aligned} Q(s) &:= q_1(s)\left(\frac{m(\delta_1(s))}{m(s)}\right)^{\gamma}\left(1-p_1(\delta_1(s))-p_2(\delta_1(s))\frac{m(\eta_2(\delta_1(s)))}{m(\delta_1(s))}\right)^{\gamma} \\ &\quad +q_2(s)\left(1-p_1(\delta_2(s))-p_2(\delta_2(s))\frac{m(\eta_2(\delta_2(s)))}{m(\delta_2(s))}\right)^{\gamma} \\ &= \frac{8}{s^{\frac{1}{3}}}\left(\frac{\frac{s}{2}}{s}\right)^3\left(1-\frac{1}{2}-\frac{\frac{t}{2}}{3(\frac{t}{2}+1)}\frac{\frac{t}{2}+1}{\frac{t}{2}}\right)^3+\frac{2}{s^{\frac{1}{3}}}\left(1-\frac{1}{2}-\frac{2t}{3(2t+1)}\frac{2t+1}{2t}\right)^3 \\ &= \frac{1}{108s^{\frac{1}{3}}}. \end{aligned}$$

Let $\alpha(s)=1$ in (ii) of Theorem 3.1, $\delta_1(T)=\frac{T}{2}>t_1$ and we obtain

$$\limsup_{t\to\infty}\int_T^t Q(s)\Delta s = \limsup_{t\to\infty}\int_T^t\left[\frac{1}{108s^{\frac{1}{3}}}\right]\Delta s = \frac{1}{108}\limsup_{t\to\infty}\sum_T^{t-1}\frac{1}{k^{\frac{1}{3}}} = \infty.$$

Since Theorem 1, every solution of the neutral dynamic equation is oscillatory or $\lim_{t\to\infty} x(t)=0$.

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