Electronic Journal of Mathematical Analysis and Applications
Vol. 7(2) July 2019, pp. 151-167.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

A NOTE ON SOME GROWTH PROPERTIES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS USING THEIR RELATIVE (p, q)-TH ORDERS

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Abstract. In this paper we wish to study some comparative growth properties of composite entire and meromorphic functions on the basis of relative (p, q)-th order and relative (p, q)-th lower order of entire and meromorphic function with respect to another entire function where p and q are any two positive integers.

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as $M_f(r) = \max|z| = r\,|f(z)|$. If an entire function f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s\to\infty} M_f^{-1}(s) = \infty$. In this connection we just recall the following definition which is relevant:

Definition 1. {[2]} *A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^{\sigma})$ holds. For examples of functions with or without the Property (A), one may see* [2].

When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$. The integrated counting function $N_f(r, a)$ $\left(\overline{N}_f(r, a)\right)$ of a-points (distinct a-points) of f is defined as

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + n_f(0, a) \log r$$

$$\left(\overline{N}_f(r, a) = \int_0^r \frac{\overline{n}_f(t, a) - \overline{n}_f(r, a)}{t} dt + \overline{n}_f(0, a) \log r\right),$$

Key words and phrases. Entire function, meromorphic function, index-pair, (p, q)-th order, relative (p, q)-th order, composition, growth, Property (A).
2010 Mathematics Subject Classification. 30D20, 30D30, 30D35.
Submitted Feb. 26, 2018.

where we denote by $n_f(t,a)\,(\overline{n}_f(t,a))$ the number of a-points (distinct a-points) of f in $|z|\leq t$ and an ∞ -point is a pole of f . In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively. The function $N_f(r,a)$ is called the enumerative function. On the other hand, the function $m_f(r)\equiv m_f(r,\infty)$ known as the proximity function is defined as

$$m_f(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta,$$

$$\text{where }\log^+x=\max(\log x,0)\text{ for all }x\geqslant 0$$

and an ∞ -point is a pole of f .

Analogously, $m_{\frac{1}{f-a}}(r)\equiv m_f(r,a)$ is defined when a is not an ∞-point of f. Thus the Nevanlinna's characteristic function $T_f(r)$ corresponding to f is defined as

$$T_f(r)=N_f(r)+m_f(r)\ .$$

When f is entire, $T_f(r)$ coincides with $m_f(r)$ as $N_f(r)=0$. Moreover, if f is non-constant entire then $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1}:(T_f(0),\infty)\to(0,\infty)$ exist and is such that $\lim\limits_{s\to\infty}T_f^{-1}(s)=\infty$.

However let us consider that $x\in[0,\infty)$ and $k\in\mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers. We define $\exp^{[k]}x=\exp\left(\exp^{[k-1]}x\right)$ and $\log^{[k]}x=\log\left(\log^{[k-1]}x\right)$. We also denote $\log^{[0]}x=x$, $\log^{[-1]}x=\exp x$, $\exp^{[0]}x=x$ and $\exp^{[-1]}x=\log x$. Further we assume that throughout the present paper a,b,d,m,n,p and q always denote positive integers. Now considering this, we just recall that Shen et al. [14] defined the (m,n)-φ order and (m,n)-φ lower order of entire functions f which are as follows:

Definition 2. [14] *Let $\varphi:[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function and $m\geq n$. The (m,n)-φ order $\rho^{(m,n)}(f,\varphi)$ and (m,n)-φ lower order $\lambda^{(m,n)}(f,\varphi)$ of entire functions f are defined as:*

$$\rho^{(m,n)}(f,\varphi)=\varlimsup_{r\to\infty}\frac{\log^{[m]}M_f(r)}{\log^{[n]}\varphi(r)}\ \textit{and}\ \lambda^{(m,n)}(f,\varphi)=\varliminf_{r\to\infty}\frac{\log^{[m]}M_f(r)}{\log^{[n]}\varphi(r)}.$$

If f is a meromorphic function, then

$$\rho^{(m,n)}(f,\varphi)=\varlimsup_{r\to\infty}\frac{\log^{[m-1]}T_f(r)}{\log^{[m]}\varphi(r)}\ \textit{and}\ \lambda^{(m,n)}(f,\varphi)=\varliminf_{r\to\infty}\frac{\log^{[m-1]}T_f(r)}{\log^{[n]}\varphi(r)}.$$

Further for any non-decreasing unbounded function $\varphi:[0,+\infty)\to(0,+\infty)$, if we assume $\lim\limits_{r\to+\infty}\frac{\log^{[q]}\varphi(ar)}{\log^{[q]}\varphi(r)}=1$ for all $\alpha>0$, then for any entire function f, using the inequality $T_f(r)\leq\log M_f(r)\leq 3T_f(2r)$ $\{cf.$ [8]$\}$, one can easily verify that (see [14])

$$\rho^{(m,n)}(f,\varphi)=\varlimsup_{r\to\infty}\frac{\log^{[m]}M_f(r)}{\log^{[n]}\varphi(r)}=\varlimsup_{r\to\infty}\frac{\log^{[m-1]}T_f(r)}{\log^{[n]}\varphi(r)}$$

$$\left(\lambda^{(m,n)}(f,\varphi)=\varliminf_{r\to\infty}\frac{\log^{[m]}M_f(r)}{\log^{[n]}\varphi(r)}=\varliminf_{r\to\infty}\frac{\log^{[m-1]}T_f(r)}{\log^{[n]}\varphi(r)}\right).$$

If we take $m = p$, $n = 1$ and $\varphi(r) = \log^{[q-1]} r$, then the above definition reduce to the following definition:

Definition 3. *The (p,q)-th order and (p,q)-th lower order of an entire function f are defined as:*

$$\rho^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

If f is a meromorphic function, then

$$\rho^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

Definition 3 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [9].

The above definition extend the generalized order $\rho^{(l)}(f)$ and generalized lower order $\lambda^{(l)}(f)$ of an entire or meromorphic function f considered in [13] for each integer $l \geq 2$ since these correspond to the particular case $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$. Clearly, $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$.

In this connection we just recall the following definition of index-pair where we will give a minor modification to the original definition (see e.g. [9]) :

Definition 4. *An entire function f is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Analogously one can easily verify that the Definition 4 of index-pair can also be applicable for a meromorphic function f.

L. Bernal [1, 2] introduced the relative order between two entire functions to avoid comparing growth just with $\exp z$. In the case of relative order, Sánchez Ruiz et al. [12] gave the definitions of relative (p,q)-th order and relative (p,q)-th lower order of an entire function with respect to another entire function and Debnath et al. [6] introduced the definitions of relative (p,q)-th order and relative (p,q)-th lower order of a meromorphic function with respect to another entire function in the light of index-pair. In order to keep accordance with Definition 2 and Definition 3, we will give a minor modification to the original definition of relative (p,q)-th order and relative (p,q)-th lower order of entire and meromorphic function (see e.g. [6, 12]).

Definition 5. *Let f and g be any two entire functions with index-pairs (m,q) and (m,p) respectively. Then the relative (p,q)-th order and relative (p,q)-th lower* *order of f with respect to g are defined as*

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

If f is a meromorphic and g is entire, then

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_g^{-1}(T_f(r))}{\log^{[q]} r}.$$

If f and g have got index-pair $(m, 1)$ and (m, k), respectively, then Definition 5 reduces to generalized relative order of f with respect to g. If f and g have the same index-pair $(p, 1)$ where p is any positive integer, we get the definition of relative order introduced by Bernal [1, 2] and Lahiri et al. [11]. Further if $g = \exp^{[m-1]} z$, then $\rho_g(f) = \rho^{(m)}(f)$ and $\rho_g^{(p,q)}(f) = \rho^{(m,q)}(f)$. Further if f have index-pair $(2, 1)$ and $g = \exp z$, then Definition 5 become the classical definitions of order and lower order.

An entire or meromorphic function f for which relative (p, q)-th order and relative (p, q)-th lower order with respect to another entire function g are the same is called a function of regular relative (p, q) growth with respect to g. Otherwise, f is said to be irregular relative (p, q) growth with respect to g.

For entire or meromorphic functions, the notions of their growth indicators such as order is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of relative orders of entire or meromorphic functions and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire and meromorphic functions in the light of their relative orders are the prime concern of this paper. Actually in this paper we establish some newly developed results related to the growth rates of composite entire and meromorphic functions on the basis of relative (p, q)-th order and relative (p, q)-th lower order where p, q are any two positive integers. We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [8, 10, 15, 16, 17].

2. **Lemmas**

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [5] *If f and g are any two entire functions then for all sufficiently large values of r,*

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{16} M_g\left(\frac{r}{2}\right)\right).$$

Lemma 2. [3] *Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leqslant \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 3. [4] *Suppose that f is a meromorphic function and g be an entire function and suppose that $0<\mu<\rho_g\leq\infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r)\geq T_f\left(\exp\left(r^{\mu}\right)\right)\ .$$

Lemma 4. [7] *Let f be an entire function which satisfies the Property (A), $\beta>0$, $\delta>1$ and $\alpha>2$. Then*

$$\beta T_f\left(r\right)<T_f\left(\alpha r^{\delta}\right).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 1. *Let f be a meromorphic function and g, h be an entire functions such that $\rho_h^{(p,q)}\left(f\right)<\infty$ and $\lambda_h^{(p,q)}\left(f\circ g\right)=\infty$. Then for every $A\left(>0\right)$,*

$$\lim_{r\to\infty}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p]}T_h^{-1}\left(T_f\left(r^A\right)\right)}=\infty\ .$$

Proof. If possible, let there exists a constant β such that for a sequence of values of r tending to infinity we have

$$\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)\leq\beta\cdot\log^{[p]}T_h^{-1}\left(T_f\left(r^A\right)\right)\ . \tag{1}$$

Again from the definition of $\rho_h^{(p,q)}\left(f\right)$, it follows for all sufficiently large values of r that

$$\log^{[p]}T_h^{-1}\left(T_f\left(r^A\right)\right)\leq\left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)\log^{[q]}r+O(1)\ . \tag{2}$$

Now combining (1) and (2) we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned}\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right) &\leq \beta\cdot\left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)\log^{[q]}r+O(1)\\ \textit{i.e., }\lambda_h^{(p,q)}\left(f\circ g\right) &\leq \beta\cdot\left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right),\end{aligned}$$

which contradicts the condition $\lambda_h^{(p,q)}\left(f\circ g\right)=\infty$. So for any positive integer q and for all sufficiently large values of r we get that

$$\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)\geq\beta\cdot\log^{[p]}T_h^{-1}T_f\left(r^A\right),$$

from which the theorem follows. □

In the line of Theorem 1, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 2. *Let f be a meromorphic function and g, h be an entire functions such that $\rho_h^{(p,q)}\left(g\right)<\infty$ and $\lambda_h^{(p,q)}\left(f\circ g\right)=\infty$. Then for every $A\left(>0\right)$,*

$$\lim_{r\to\infty}\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log^{[p]}T_h^{-1}\left(T_g\left(r^A\right)\right)}=\infty\ .$$

Remark 1. *Theorem 1 is also valid with "limit superior" instead of "limit" if $\lambda_h^{(p,q)}\left(f\circ g\right)=\infty$ is replaced by $\rho_h^{(p,q)}\left(f\circ g\right)=\infty$ and the other conditions remain the same.*

Remark 2. *Theorem 2 is also valid with "limit superior" instead of "limit" if* $\lambda_h^{(p,q)}(f \circ g) = \infty$ *is replaced by* $\rho_h^{(p,q)}(f \circ g) = \infty$ *and the other conditions remain the same.*

Corollary 1. *Under the assumptions of Theorem 1 and Remark 1,*

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_h^{-1}(T_f(r^A))} = \infty \text{ and } \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_h^{-1}(T_f(r^A))} = \infty$$

respectively.

Proof. By Theorem 1 we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) &\geq K \cdot \log^{[p]} T_h^{-1}\left(T_f\left(r^A\right)\right) \\ i.e.,\ \log^{[p-1]} T_h^{-1}(T_{f\circ g}(r)) &\geq \left\{\log^{[p-1]} T_h^{-1}\left(T_f\left(r^A\right)\right)\right\}^K, \end{aligned}$$

from which the first part of the corollary follows.

Similarly using Remark 1, we obtain the second part of the corollary. □

Corollary 2. *Under the assumptions of Theorem 2 and Remark 2,*

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_h^{-1}(T_g(r^A))} = \infty \text{ and } \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_h^{-1}(T_g(r^A))} = \infty$$

respectively.

In the line of Corollary 1, one can easily verify Corollary 2 with the help of Theorem 2 and Remark 2 respectively and therefore its proof is omitted.

Theorem 3. *Let f be a meromorphic function and g, h be any two entire functions such that* $\rho^{(m,n)}(g) < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$. *If h satisfies the Property (A), then*

$$(i)\ \lim_{r\to\infty} \frac{\log^{[p]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_h^{-1} T_f\left(\exp^{[q-1]} r\right)} = 0 \text{ if } q \geqslant m$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{[p+m-q-1]} T_h^{-1}(T_{f\circ g}(r))}{\log^{[p-1]} T_h^{-1} T_f\left(\exp^{[q-1]} r\right)} = 0 \text{ if } q < m.$$

Proof. Let us suppose that $\alpha > 2$ and $\delta \to 1^+$ in Lemma 4. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2, Lemma 4 and the inequality $T_g(r) \leq \log M_g(r)$ {cf. [8] } for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1}(T_{f\circ g}(r)) &\leqslant T_h^{-1}(\{1 + o(1)\} T_f(M_g(r))) \\ i.e.,\ T_h^{-1}(T_{f\circ g}(r)) &\leqslant \beta\left[T_h^{-1}(T_f(M_g(r)))\right]^\delta \\ i.e.,\ \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) &\leqslant \log^{[p]} T_h^{-1}(T_f(M_g(r))) + O(1) \end{aligned}$$

$$i.e.,\ \log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} M_g(r) + O(1)\ . \tag{3}$$

Now the following cases may arise :

Case I. Let $q \geqslant m$. Then we have from (3) for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1}(T_{f\circ g}(r)) \leqslant \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[m-1]} M_g(r) + O(1)\ . \tag{4}$$

Now from the definition of (m,n)-th order of g in terms of maximum modulus, we get for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[m]} M_g(r) \leqslant \left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]} r$$

$$i.e.,\ \log^{[m]} M_g(r) \leqslant \left(\rho^{(m,n)}(g)+\varepsilon\right)\log r\ . \tag{5}$$

So for all sufficiently large values of r it follows from (5) that

$$\log^{[m-1]} M_g(r) \leqslant r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}\ . \tag{6}$$

Therefore from (4) and (6) it follows for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \left(\rho_h^{(p,q)}(f)+\varepsilon\right) r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)} + O(1)\ . \tag{7}$$

Case II. Let $q<m$. Then we get from(3) for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q]}\log^{[m]} M_g(r)+O(1). \tag{8}$$

Also we obtain from (5) for all sufficiently large values of r that

$$\exp^{[m-q]}\log^{[m]} M_g(r) \leqslant \exp^{[m-q]}\log r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}$$

$$i.e.,\ \exp^{[m-q]}\log^{[m]} M_g(r) \leqslant \exp^{[m-q-1]} r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}. \tag{9}$$

Now from (8) and (9) we obtain for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q-1]} r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)$$

$$i.e.,\ \log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)\ . \tag{10}$$

From the definition of relative (p,q)-th order, we obtain for all sufficiently large values of r that

$$\log^{[p]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right) \geqslant \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}\exp^{[q-1]} r$$

$$i.e.,\ \log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right) \geqslant r^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}\ . \tag{11}$$

As $\rho^{(m,n)}(g)<\lambda_h^{(p,q)}(f)$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho^{(m,n)}(g)+\varepsilon<\lambda_h^{(p,q)}(f)-\varepsilon\ . \tag{12}$$

Now if $q\geqslant m$, combining $(7), (11)$ and in view of (12) we have for all sufficiently large values of r that

$$\frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)} \leqslant \frac{\left(\rho_h^{(p,q)}(f)+\varepsilon\right) r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)}{r^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}}$$

$$i.e.,\ \lim_{r\to\infty}\frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)}=0\ .$$

This proves the first part of the theorem.

When $q<m$, combining (10) and (11) it follows for all sufficiently large values of r that

$$\frac{\log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)} \leqslant \frac{r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)}{r^{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)}}\ .$$

Since $\rho^{(m,n)}(g) < \lambda_h^{(p,q)}(f)$ and $\varepsilon\,(>0)$ is arbitrary, we get from above

$$\lim_{r\to\infty} \frac{\log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)} = 0,$$

which is the second part of the theorem. □

Theorem 4. *Let f be a meromorphic function and g, h be any two entire functions such that $\lambda_g(m,n) < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$. If h satisfies the Property (A), then*

$$(i)\ \ \underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)} = 0 \text{ if } q \geqslant m$$

and

$$(ii)\ \ \underline{\lim}_{r\to\infty} \frac{\log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)} = 0 \text{ if } q < m.$$

Proof of Theorem 4 is omitted as it can be carried out in the line of Theorem 3.

Theorem 5. *Let f be a meromorphic function and h be an entire function such that $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$. Also let g be an entire function with finite (m,q)-th order where $q < m$. If h satisfies the Property (A), then*

$$\overline{\lim}_{r\to\infty} \frac{\log^{[p+m-q]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p]} T_h^{-1}\left(T_f(r)\right)} \le \frac{\rho^{(m,q)}(g)}{\lambda_h^{(p,q)}(f)},$$

Proof. Since $q < m$, we get from(8) for all sufficiently large values of r that

$$\log^{[p+m-q]} T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \log^{[m]} M_g(r) + O(1)$$

$$i.e.,\ \frac{\log^{[p+m-q]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p]} T_h^{-1}\left(T_f(r)\right)} \le \frac{\log^{[m]} M_g(r) + O(1)}{\log^{[q]} r} \cdot \frac{\log^{[q]} r}{\log^{[p]} T_h^{-1}\left(T_f(r)\right)}$$

$$i.e.,\ \overline{\lim}_{r\to\infty} \frac{\log^{[p+m-q]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p]} T_h^{-1}\left(T_f(r)\right)} \le \overline{\lim}_{r\to\infty} \frac{\log^{[m]} M_g(r) + O(1)}{\log^{[q]} r} \cdot \overline{\lim}_{r\to\infty} \frac{\log^{[q]} r}{\log^{[p]} T_h^{-1}\left(T_f(r)\right)}$$

$$i.e.,\ \overline{\lim}_{r\to\infty} \frac{\log^{[p+m-q]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p]} T_h^{-1}\left(T_f(r)\right)} \le \rho^{(m,q)}(g) \cdot \frac{1}{\lambda_h^{(p,q)}(f)} = \frac{\rho^{(m,q)}(g)}{\lambda_h^{(p,q)}(f)}\ .$$

This proves the theorem. □

In the line of Theorem 5 we may state the following theorem without proof.

Theorem 6. *Let f be a meromorphic function and g, h be an two entire functions satisfying (i) $\rho_h^{(p,q)}(f) < \infty$, (ii) $\lambda_h^{(p,n)}(g) > 0$ and (iii) $\rho^{(m,n)}(g) < \infty$ where $q < m$. If h satisfies the Property (A), then*

$$\overline{\lim}_{r\to\infty} \frac{\log^{[p+m-q]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log^{[p]} T_h^{-1}\left(T_g(r)\right)} \le \frac{\rho^{(m,n)}(g)}{\lambda_h^{(p,n)}(g)}.$$

Theorem 7. *Let f be a meromorphic function and h be an entire function such that $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$ and g be an entire function with finite (m,n)-th lower order. If h satisfies the Property (A), then*

$$(i)\quad \overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]}r\right)\right)}=\infty \text{ if } q\geq m \text{ and } A>1,$$

$$(ii)\quad \overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right)}=\infty \text{ if } q\geq m$$

$$\text{or } q=m\,(\neq 1)-1 \text{ and } \lambda^{(m,n)}(g)<A$$

and

$$(iii)\quad \overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p+m-q-1]}T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]}r)\right)}=\infty \text{ if } m>q+1 \text{ and } A>\lambda^{(m,n)}(g)\ .$$

Proof. From the definition of $\lambda_h^{(p,q)}(f)$, we obtain for arbitrary positive $\varepsilon\,(>0)$ and for all sufficiently large values of r that

$$\log^{[p]}T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)r^A\ . \tag{13}$$

Also from the definition of (m,n)-th lower order of g, we get for a sequence of values of r tending to infinity that

$$\log^{[m]}M_g\left(\exp^{[n-1]}r\right)\leq(\lambda^{(m,n)}(g)+\varepsilon)\log^{[n]}(\exp^{[n-1]}r)$$

$$i.e.,\ \log^{[m]}M_g\left(\exp^{[n-1]}r\right)\leq\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log r$$

$$i.e.,\ \log^{[m]}M_g\left(\exp^{[n-1]}r\right)\leq\log r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)} \tag{14}$$

$$i.e.,\ \log^{[m-1]}M_g\left(\exp^{[n-1]}r\right)\leq r^{(\lambda^{(m,n)}(g)+\varepsilon)}\ . \tag{15}$$

Case I. Let $q\geq m$. Then it follows from (3) for a sequence of values of r tending to infinity that

$$\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]}r\right)\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]}M_g(\exp^{[n]}r)+O(1)$$

$$i.e.,\ \log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]}r\right)\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[m]}M_g(\exp^{[n]}r)+O(1)$$

$$i.e.,\ \log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]}r\right)\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\lambda^{(m,n)}(g)+\varepsilon\right)r+O(1)\ . \tag{16}$$

Case II. Also let $q\geq m$ or $q=m\,(\neq 1)-1$. Then also we obtain from (15) and (3) for a sequence of values of r tending to infinity that

$$i.e.,\ \log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]}M_g(\exp^{[n-1]}r)+O(1)$$

$$i.e.,\ \log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[m-1]}\mu_g(\exp^{[n-1]}r)+O(1)$$

$$i.e.,\ \log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^{(\lambda^{(m,n)}(g)+\varepsilon)}+O(1)\ . \tag{17}$$

Case III. Again let $m > q+1$. Then we get from (14) and (3) for a sequence of values of r tending to infinity that

$$\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right) \leq \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]} M_g(\exp^{[n-1]} r)+O(1)$$

$$i.e.,\ \log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]} r\right)\right) \leq \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q-m]}\log^{[m]} M_g(\exp^{[n-1]} r)+O(1)$$

$$i.e.,\ \log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]} r\right)\right) \leq \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q]}\log^{[m]} M_g(\exp^{[n-1]} r)+O(1)$$

$$i.e.,\ \log^{[p]} T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]} r)\right) \leq \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q]}\log r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)}+O(1)$$

$$i.e.,\ \log^{[p]} T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]} r)\right) \leq \left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q-1]} r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)}+O(1)$$

$$i.e.,\ \log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]} r)\right) \leq r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)}+O(1)\ . \tag{18}$$

Now if $q \geq m$ and $\mu > 1$, we get from (13) and (16) of Case I for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} T_h^{-1}\left(T_f(\exp^{[q]}(r^A))\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]} r\right)\right)} \geq \frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right) r^A}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\lambda^{(m,n)}(g)+\varepsilon\right) r+O(1)},$$

from which the first part of the theorem follows.

Again combining (13) and (17) of Case II we obtain for a sequence of values of r tending to infinity when $q \geq m$ or $q = m\,(\neq 1)-1$

$$\frac{\log^{[p]} T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)} \geq \frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right) r^A}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right) r^{(\lambda^{(m,n)}(g)+\varepsilon)}+O(1)}\ . \tag{19}$$

As $A > \lambda^{(m,n)}(g)$ we can choose $\varepsilon\,(>0)$ in such a way that

$$\lambda^{(m,n)}(g)+\varepsilon < A\ . \tag{20}$$

Thus from (19) and (20) we get that

$$\varlimsup_{r\to\infty}\frac{\log^{[p]} T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)} = \infty\ .$$

This establishes the second part of the theorem .

When $m > q+1$, it follows from (13) and (18) of Case III for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]} r)\right)} \geq \frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right) r^A}{r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)}+O(1)}\ . \tag{21}$$

Now from (20) and (21) we obtain that

$$\varlimsup_{r\to\infty}\frac{\log^{[p]} T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]} r)\right)} = \infty\ .$$

This proves the third part of the theorem.

Thus the theorem follows . $\square$

In the line of Theorem 7 we may state the following theorem without proof.

Theorem 8. *Let f be a meromorphic function and h be an entire function such that $\rho_h^{(p,q)}(f)$ is finite and g be a entire function with (m,n)-th lower order and non zero (p,n)-th relative lower order with respect to h. If h satisfies the Property (A), then*

$$
\begin{aligned}
(i) \quad \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_g(\exp^{[n]}\left(r^A\right))\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]} r\right)\right)} &= \infty \text{ if } q \geq m \text{ and } A > 1, \\
(ii) \quad \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_g(\exp^{[n]}\left(r^A\right))\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)} &= \infty \text{ if } q \geq m \\
& \text{or } q = m\,(\neq 1) - 1 \text{ and } \lambda^{(m,n)}(g) < A
\end{aligned}
$$

and

$$
\begin{aligned}
(iii) \quad \overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1}\left(T_g(\exp^{[n]}\left(r^A\right))\right)}{\log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]} r)\right)} &= \infty \text{ if } m > q+1 \text{ and} \\
A &> \lambda^{(m,n)}(g) \ .
\end{aligned}
$$

Theorem 9. *Let f be a meromorphic function and h be an entire function such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and g be an entire function with finite (m,n)-th order. If h satisfies the Property (A), then*

$$
\begin{aligned}
(i) \quad \lim_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]} r\right)\right)} &= \infty \text{ if } q \geq m \text{ and } A > 1, \\
(ii) \quad \lim_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)} &= \infty \text{ if } q \geq m \\
& \text{or } q = m\,(\neq 1) - 1 \text{ and } \rho^{(m,n)}(g) < A
\end{aligned}
$$

and

$$
\begin{aligned}
(iii) \quad \lim_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_f(\exp^{[q]}\left(r^A\right))\right)}{\log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]} r)\right)} &= \infty \text{ if } m > q+1 \text{ and} \\
A &> \rho^{(m,n)}(g) \ .
\end{aligned}
$$

Theorem 10. *Let f be a meromorphic function and h be an entire function such that $\rho_h^{(p,q)}(f)$ is finite and g be a entire function with (m,n)-th order and non zero (p,n)-th relative lower order with respect to h. If h satisfies the Property (A), then*

$$
\begin{aligned}
(i) \quad \lim_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_g(\exp^{[n]}\left(r^A\right))\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n]} r\right)\right)} &= \infty \text{ if } q \geq m \text{ and } A > 1, \\
(ii) \quad \lim_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_g(\exp^{[n]}\left(r^A\right))\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)} &= \infty \text{ if } q \geq m \\
& \text{or } q = m\,(\neq 1) - 1 \text{ and } \rho^{(m,n)}(g) < A
\end{aligned}
$$

and

$$
\begin{aligned}
(iii) \quad \lim_{r\to\infty} \frac{\log^{[p]} T_h^{-1}\left(T_g(\exp^{[n]}\left(r^A\right))\right)}{\log^{[p+m-q-1]} T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]} r)\right)} &= \infty \text{ if } m > q+1 \text{ and} \\
A &> \rho^{(m,n)}(g) \ .
\end{aligned}
$$

We omit the proof of Theorem 9 and Theorem 10 as those can be carried out in the line of Theorem 7 and Theorem 8 respectively.

Theorem 11. *Let f be a meromorphic and h be an entire function such that $0<\lambda_h^{(p,q)}(f)\le\rho_h^{(p,q)}(f)<\infty$. Also let g be an entire function with non zero finite order. Then for every positive constant A and real number α,*

$$\overline{\lim_{r\to\infty}}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(\exp^{[n-1]}r)}{\left\{\log^{[p]}T_h^{-1}T_f(r^A)\right\}^{1+\alpha}}=\infty\ .$$

Proof. If α be such that $1+\alpha\le 0$ then the theorem is trivial. So we suppose that $1+\alpha>0$. Now from the definition of $\rho_h^{(p,q)}(f)$, it follows for all sufficiently large values of r that

$$\log^{[p]}T_h^{-1}\left(T_f(r^A)\right)\le\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r+O(1)$$

$$\log^{[p]}T_h^{-1}\left(T_f(r^A)\right)\le\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r\left(1+\frac{O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}\right)$$

i.e., $\left\{\log^{[p]}T_h^{-1}\left(T_f(r^A)\right)\right\}^{1+\alpha}\le$

$$\left(\rho_h^{(p,q)}(f)+\varepsilon\right)^{1+\alpha}\left(\log^{[q]}r\right)^{1+\alpha}\left(1+\frac{O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}\right)^{1+\alpha}.\qquad(22)$$

Now from Lemma 3 we get a sequence of values of r tending to infinity that

$$\log^{[p]}T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]}r)\right)\ge\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q-1]}\left(\exp^{[n-1]}r\right)^{\mu}.\qquad(23)$$

Now from (22) and (23) we have for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}T_h^{-1}\left(T_{f\circ g}(\exp^{[n-1]}r)\right)}{\left\{\log^{[p]}T_h^{-1}\left(T_f(r^A)\right)\right\}^{1+\alpha}}\ge$$

$$\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q-1]}\left(\exp^{[n-1]}r\right)^{\mu}}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)^{1+\alpha}\left(\log^{[q]}r\right)^{1+\alpha}\left(1+\frac{O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}\right)^{1+\alpha}}.$$

Since $\frac{\log^{[q-1]}\left(\exp^{[n-1]}r\right)^{\mu}}{\left(\log^{[q]}r\right)^{1+\alpha}}\to\infty$ as $r\to\infty$, then the theorem follows from above. □

Theorem 12. *Let f be a meromorphic function and l, h be any two entire functions such that $\lambda_h^{(p,d)}(l)>0$ and $\rho_h^{(p,q)}(f)<\infty$ .Also let g and k are two entire function with $\rho^{(m,n)}(g)<\lambda^{(a,b)}(k)$. If h satisfies the Property (A), then*

$$(i)\ \lim_{r\to\infty}\frac{\log^{[p-1]}T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]}r\right)\right)}{\log^{[p-1]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right)\cdot\log^{[p-1]}T_h^{-1}\left(T_f\left(\exp^{[q-1]}r\right)\right)}=\infty,$$

if $d\le a-1$ and $q\ge m-1$,

$$(ii)\ \lim_{r\to\infty}\frac{\log^{[p-1]}T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]}r\right)\right)}{\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right)\cdot\log^{[p-1]}T_h^{-1}\left(T_f\left(\exp^{[q-1]}r\right)\right)}=\infty,$$

if $d\leq a-1$ *and* $m-q=2$,

and

$$(iii)\ \lim_{r\to\infty}\frac{\log^{[p-1]}T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]}r\right)\right)}{\log^{[p+m-q-2]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right)\cdot\log^{[p-1]}T_h^{-1}\left(T_f\left(\exp^{[q-1]}r\right)\right)}=\infty,$$

if $d\leq a-1$ *and* $m-q>2$.

Proof. From the definition of (m,n)-th order of g, we get for arbitrary positive ε and for all sufficiently large values of r that

$$\begin{aligned}\log^{[m]}M_g\left(\exp^{[n-1]}r\right)&\leq\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}\exp^{[n-1]}r\\ i.e.,\ \log^{[m]}M_g\left(\exp^{[n-1]}r\right)&\leq\left(\rho^{(m,n)}(g)+\varepsilon\right)\log r\\ i.e.,\ \log^{[m]}M_g\left(\exp^{[n-1]}r\right)&\leq\log r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}\qquad(24)\\ i.e.,\ \log^{[m-1]}M_g\left(\exp^{[n-1]}r\right)&\leq r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}.\qquad(25)\end{aligned}$$

Also from the definition of (a,b)-th lower order of k, we get for all sufficiently large values of r that

$$\begin{aligned}\log^{[a]}M_k\left(\frac{\exp^{[b-1]}r}{2}\right)&\geqslant\left(\lambda^{(a,b)}(k)-\varepsilon\right)\log^{[b]}\left(\frac{\exp^{[b-1]}r}{2}\right)\\ i.e.,\ \log^{[a]}M_k\left(\frac{\exp^{[b-1]}r}{2}\right)&\geqslant\log r^{\left(\lambda^{(a,b)}(k)-\varepsilon\right)}+O(1)\qquad(26)\\ i.e.,\ \log^{[a-1]}M_k\left(\frac{\exp^{[b-1]}r}{2}\right)&\geqslant r^{\left(\lambda^{(a,b)}(k)-\varepsilon\right)}+O(1).\qquad(27)\end{aligned}$$

Again from the definition of (p,q)-th relative order of f with respect to h, we have for all sufficiently large values of r that

$$\begin{aligned}\log^{[p]}T_h^{-1}\left(T_f\left(\exp^{[q-1]}r\right)\right)&\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log r\\ i.e.,\ \log^{[p-1]}T_h^{-1}\left(T_f\left(\exp^{[q-1]}r\right)\right)&\leq r^{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)}.\qquad(28)\end{aligned}$$

Now in view of Lemma 1, and the inequality $T_f(r)\leq\log M_f(r)\leq 3T_f(2r)$ {cf. [8]}for any entire f , we get for all sufficiently large values of r that

$$\begin{aligned}M_{l\circ k}(r)&\geqslant M_l\left(\frac{1}{16}M_k\left(\frac{r}{2}\right)\right)\\ i.e.,\ 3T_{l\circ k}(r)&\geqslant T_l\left(\frac{1}{32}M_k\left(\frac{r}{2}\right)\right).\end{aligned}$$

Since $T_h^{-1}(r)$ is an increasing function of r, we obtain from above and in view of Lemma 4, for any $\beta>2$, $\delta\to 1^+$ and for all sufficiently large values of r

that

$$\begin{aligned} T_h^{-1}\left(3T_{l\circ k}\left(r\right)\right) &\geqslant T_h^{-1}\left(T_l\left(\frac{1}{32}M_k\left(\frac{r}{2}\right)\right)\right) \\ i.e.,\ T_h^{-1}\left(T_{l\circ k}\left(r\right)\right) &\geqslant \frac{1}{\beta}\left[T_h^{-1}\left(T_l\left(\frac{1}{32}M_k\left(\frac{r}{2}\right)\right)\right)\right]^{\frac{1}{\delta}} \\ i.e.,\ \log^{[p]}T_h^{-1}\left(T_{l\circ k}\left(r\right)\right) &\geqslant \log^{[p]}T_h^{-1}\left(T_l\left(\frac{1}{32}M_k\left(\frac{r}{2}\right)\right)\right)+O(1) \\ i.e.,\ \log^{[p]}T_h^{-1}\left(T_{l\circ k}\left(r\right)\right) &\geqslant \left(\lambda_h^{(p,d)}\left(l\right)-\varepsilon\right)\log^{[d]}M_k\left(\frac{r}{2}\right)+O(1)\ . \end{aligned} \tag{29}$$

Case I. Let $d \leq a-1$. Then we get from (27) and (29) for all sufficiently large values of r that

$$\log^{[p]}T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]}r\right)\right) \geqslant \left(\lambda_h^{(p,d)}\left(l\right)-\varepsilon\right)\log^{[a-1]}M_k\left(\frac{\exp^{[b-1]}r}{2}\right)+O(1)$$

$$i.e.,\ \log^{[p]}T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]}r\right)\right) \geqslant \left(\lambda_h^{(p,d)}\left(l\right)-\varepsilon\right)r^{\left(\lambda^{(a,b)}(k)-\varepsilon\right)}+O(1)$$

$$i.e.,\ \log^{[p-1]}T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]}r\right)\right) \geqslant \exp\left\{\left(\lambda_h^{(p,d)}\left(l\right)-\varepsilon\right)r^{\left(\lambda^{(a,b)}(k)-\varepsilon\right)}+O(1)\right\}\ . \tag{30}$$

Case II. Again let $q \geq m-1$. Then we obtain from (3) and (25) for all sufficiently large values of r that

$$\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right) \leq \left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)\log^{[m-1]}M_g\left(\exp^{[n-1]}r\right)+O(1)$$

$$i.e.,\ \log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right) \leq \left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)$$

$$i.e.,\ \log^{[p-1]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right) \leq \exp\left\{\left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)\right\}\ . \tag{31}$$

Case III. Also let $q < m$. Then it follows from (3) and (24) for all sufficiently large values of r that

$$\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right) \leq \left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)\exp^{[m-q]}\log^{[m]}M_g(\exp^{[n-1]}r)+O(1)$$

$$i.e.,\ \log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right) \leq \left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)\exp^{[m-q-1]}r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)\ . \tag{32}$$

Now if $m-q=2$, then we get from (32) for all sufficiently large values of r that

$$\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right) \leq \left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)\exp r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)\ . \tag{33}$$

Also if $m-q>2$, then we get from (32) for all sufficiently large values of r that

$$\begin{aligned} &\log^{[m-q-2]}\left[\log^{[p]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right)\right] \\ \leq\ &\log^{[m-q-2]}\left[\left(\rho_h^{(p,q)}\left(f\right)+\varepsilon\right)\exp^{[m-q-1]}r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)\right] \\ i.e.,\ &\log^{[p+m-q-2]}T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]}r\right)\right) \leq \exp r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}+O(1)\ . \end{aligned} \tag{34}$$

Now as $\rho^{(m,n)}(g) < \lambda^{(a,b)}(k)$, we can choose $\varepsilon\,(>0)$ in such a manner that

$$\rho^{(m,n)}(g) + \varepsilon < \lambda^{(a,b)}(k) - \varepsilon \,. \tag{35}$$

Therefore combining (28), (30) of Case I and (31) of Case II it follows for all sufficiently large values of r that

$$\frac{\log^{[p-1]} T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]} r\right)\right)}{\log^{[p-1]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)\cdot \log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)}$$

$$\geqslant \frac{\exp\left\{\left(\lambda_h^{(p,d)}(l) - \varepsilon\right) r^{\left(\lambda^{(a,b)}(k)-\varepsilon\right)} + O(1)\right\}}{r^{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)}\cdot \exp\left\{\left(\rho_h^{(p,q)}(f) + \varepsilon\right) r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)} + O(1)\right\}}\,.$$

Thus in view of (35) first part of the theorem follows from above.

Again combining (28), (30) of Case I, (33) of Case III and (35) we obtain for all sufficiently large values of r that

$$\frac{\log^{[p-1]} T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]} r\right)\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)\cdot \log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)}$$

$$\geqslant \frac{\exp\left\{\left(\lambda_h^{(p,d)}(l) - \varepsilon\right) r^{\left(\lambda^{(a,b)}(k)-\varepsilon\right)} + O(1)\right\}}{r^{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)}\cdot \left[\left(\rho_h^{(p,q)}(f) + \varepsilon\right) \exp r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)} + O(1)\right]}$$

$$i.e.,\ \lim_{r\to\infty}\frac{\log^{[p-1]} T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]} r\right)\right)}{\log^{[p]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)\cdot \log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)} = \infty,$$

which is the second part of the theorem.

Similarly combining (28), (30) of Case I, (34) of Case III and (35) we get for all sufficiently large values of r that

$$\frac{\log^{[p-1]} T_h^{-1}\left(T_{l\circ k}\left(\exp^{[b-1]} r\right)\right)}{\log^{[p+m-q-2]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)\cdot \log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)}$$

$$\geqslant \frac{\exp\left\{\left(\lambda_h^{(p,d)}(l) - \varepsilon\right) r^{\left(\lambda^{(a,b)}(k)-\varepsilon\right)} + O(1)\right\}}{r^{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)}\cdot \left[\exp r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)} + O(1)\right]}$$

$$i.e.,\ \lim_{r\to\infty}\frac{\log^{[p-1]} T\left({}_h^{-1}T_{l\circ k}\left(\exp^{[b-1]} r\right)\right)}{\log^{[p+m-q-2]} T_h^{-1}\left(T_{f\circ g}\left(\exp^{[n-1]} r\right)\right)\cdot \log^{[p-1]} T_h^{-1}\left(T_f\left(\exp^{[q-1]} r\right)\right)} = \infty\,.$$

This proves the third part of the theorem.

Thus the theorem follows . □

Remark 3. *If we consider $\rho^{(m,n)}(g) < \rho^{(a,b)}(k)$ instead of $\rho^{(m,n)}(g) < \lambda^{(a,b)}(k)$ and the other conditions remain the same, the conclusion of Theorem 12 remains valid with "limit superior " replaced by " limit "*

In the line of Theorem 12, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 13. *Let f be a meromorphic function and l, h be any two entire functions such that $\lambda_h^{(p,d)}(l) > 0$, $\rho_h^{(p,q)}(f) < \infty$. Also let g and k are two entire function with $\rho^{(m,n)}(g) < \lambda^{(a,b)}(k)$ and $\rho_h^{(p,n)}(g) < \infty$. If h satisfies the Property (A), then*

$$(i)\ \lim_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1} T_{l\circ k}\left(\exp^{[b-1]} r\right)}{\log^{[p-1]} T_h^{-1} T_{f\circ g}\left(\exp^{[n-1]} r\right) \cdot \log^{[p-1]} T_h^{-1} T_g\left(\exp^{[n-1]} r\right)} = \infty,$$

if $d \le a-1$ and $q \ge m-1$,

$$(ii)\ \lim_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1} T_{l\circ k}\left(\exp^{[b-1]} r\right)}{\log^{[p]} T_h^{-1} T_{f\circ g}\left(\exp^{[n-1]} r\right) \cdot \log^{[p-1]} T_h^{-1} T_g\left(\exp^{[n-1]} r\right)} = \infty,$$

if $d \le a-1$ and $m-q = 2$,

and

$$(iii)\ \lim_{r\to\infty} \frac{\log^{[p-1]} T_h^{-1} T_{l\circ k}\left(\exp^{[b-1]} r\right)}{\log^{[p+m-q-2]} T_h^{-1} T_{f\circ g}\left(\exp^{[n-1]} r\right) \cdot \log^{[p-1]} T_h^{-1} T_g\left(\exp^{[n-1]} r\right)} = \infty,$$

if $d \le a-1$ and $m-q > 2$.

Remark 4. *If we consider $\rho^{(m,n)}(g) < \rho^{(a,b)}(k)$ instead of $\rho\rho^{(m,n)}(g) < \lambda^{(a,b)}(k)$ and the other conditions remain the same, the conclusion of Theorem 13 remains valid with "limit superior " replaced by " limit "*

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