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GLOBAL STABILITY ANALYSIS OF AN SEIR EPIDEMIC MODEL WITH RELAPSE AND GENERAL INCIDENCE RATES

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Abstract. In this paper we propose the global dynamics of an SEIRI epidemic model with a general nonlinear incidence function. The model is based on the susceptible-exposed-infective-recovered (SEIR) compartmental structure with relapse (SEIRI). Sufficient conditions for the local and global stability of equilibria (the disease-free equilibrium and the endemic equilibrium) are obtained by means of Routh-Hurwitz criterion and Lyapunov-LaSalle theorem.

Keywords: Epidemic model, Lyapunov function, Local stability, Global stability

1. Introduction

Epidemiological modeling has long been an important tool to describe the evolution of epidemics and infectious diseases. This modeling often provides systems of ordinary differential equations or delay. The interest of these differential systems is obvious from the point of view public health (decision-making tools), but also because of the different possibilities of representation of certain characteristics related to the process spread of epidemics, namely the incidence, the phenomenon of relapse and the latency period. These different representations make this topic a popular research topic for many years (see, for example, [1, 11, 15, 20, 21, 22, 24, 27, 33, 35] and the references therein).

Recently, considerable attention has been paid to model the relapse phenomenon, i.e. the return of signs and symptoms of a disease after a remission. Hence, the recovered individual can return to the infectious class (see, [3, 7, 23, 25, 26, 28, 29, 30, 31, 34]). For the biological explanations of the relapse phenomenon, we cite two examples:

- For malaria, Bignami [4] proposed that relapses derived from persistence of small numbers of parasite in the blood. Also, it has been observed that the proportion of patients who have successive relapses is relatively constant (see [32]).
- For tuberculosis, relapse can be caused by incomplete treatment or by latent infection, being observed that HIV-positive patients are significantly more

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likely to relapse than HIV-negative patients, although it is often difficult to differentiate relapse from reinfection (see [8]).

On the other hand, the incidence function includes the following forms: The first one is the saturated incidence $\frac{\beta SI}{d+S+I}$ [2], where β and d are the positive constants. The secod one is the bilinear incidence βSI [12, 18, 36, 40, 41]. The third one is the saturated incidence $\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$ [6, 37, 39, 5, 38, 19], where α_1 and α_2 are the positive constants. The effect of the saturation factor (refer to α_1 and α_2) stems from epidemic control and the protection measures. The fourth one is the standard incidence $\frac{\beta SI}{N}$ [9, 16]. A very general form of incidence rate was considered by Hattaf and al [14]:

$$\begin{aligned} \frac{dS}{dt} &= A - \mu S - f(S,I)I, \\ \frac{dI}{dt} &= f(S(t-\tau), I(t-\tau))I(t-\tau)e^{-\mu\tau} - (\mu+\gamma+\alpha)I \\ \frac{dR}{dt} &= \gamma I - \mu R. \end{aligned} \tag{1}$$

In the present paper, we study the global dynamics of the corresponding SEIR model of system (1) with relapse effect:

$$\begin{aligned} \frac{dS}{dt} &= A - \mu S - f(S,I)I, \\ \frac{dE}{dt} &= f(S,I)I - (\mu+\sigma)E \\ \frac{dI}{dt} &= \sigma E - (\mu+\gamma)I + \delta R, \\ \frac{dR}{dt} &= \gamma I - (\mu+\delta)R. \end{aligned} \tag{2}$$

The initial condition for the above system is

$$S(0) = S_0 > 0, E(0) = E_0 > 0, I(0) = I_0 > 0, R(0) = R_0 > 0. \tag{3}$$

Here $A = \mu N$, is the recruitment rate, where $N = S + E + I + R$ is the total number of population, S is the number of susceptible individuals, I is the number of infectious individuals, E is the number of exposed individuals, R is the number of recovered individuals, μ is the natural death of the population, $f(S,I)$ is the incidence function, γ is the recovery rate of the infectious individuals, σ is the rate at which exposed individuals become infectious and δ is a constant representing the rate at which an individual in the recovered class reverts to the infective class.

In model (2) the incidence function $f(S,I)$ is a locally Lipschitz continuous differentiable function on $\mathbb{R}^+ \times \mathbb{R}^+$ satisfaying $f(0,I) = 0$ for $I \geq 0$ and the followings hold:

- **(H_1):** f is a strictly monotone increasing function of $S \geq 0$, for any fixed $I > 0$, and f is a monotone decreasing function of $I \geq 0$, for any fixed $S \geq 0$;
- **(H_2):** $\phi(S,I) = f(S,I)I$ is a monotone increasing function of $I \geq 0$, for any fixed $S \geq 0$.

In the present paper, we extend the local and global stability to the SEIRI epidemic model with general incidence function (2) satisfies the hypothesis (H_1)

and (H_2), we apply Lyapunov-LaSalle invariance principle to prove the global stability of the disease-free equilibrium and we apply Routh-Hurwitz criterion and Lyapunov-LaSalle invariance principle to prove the local and the global stability of endemic equilibrium of the SEIR model with a relapse rate (see (2)). The rest of the paper is organized as follows: In Section 2, we offer a basic result. In Section 3, we apply Lyapunov-LaSalle invariance principle to prove the global stability of the disease-free equilibrium and we apply Routh-Hurwitz criterion and Lyapunov-LaSalle invariance principle to prove the local and the global stability of endemic equilibrium. In Section 4, we present some concluding remarks.

2. Preliminary

In this section, we prove the following basic result, which guarantees the existence and uniqueness of the solution $(S(t), E(t), I(t), R(t))$ for system (2) satisfying initial conditions (3).

Lemma 2.1. *The solution* $(S(t), E(t), I(t), R(t))$ *of system* (2) *with initial conditions* (3) *uniquely exists and is positive for all* $t \geq 0$. *Furthermore, it holds that*

$$\lim_{t\to+\infty}(S(t)+E(t)+I(t)+R(t)) = \frac{A}{\mu}. \tag{4}$$

Proof. We notice that the right hand side of system (2) is completely continuous and locally Lipschitzian on $\mathbb{C}$. Then it follows that the solution of system (2) exists and is unique on $[0, \alpha)$ for some $\alpha > 0$. It is easy to prove that $S(t) > 0$ for all $t \in [0, \alpha)$. Indeed, this follows from that $\frac{dS}{dt} = A > 0$ for any $t \in [0, \alpha)$ when $S(t) = 0$. Let us now show that $I(t) > 0$ for all $t \in [0, \alpha)$. Suppose on the contrary that there exists som $t_1 \in [0, \alpha)$ such that $I(t_1) = 0$ and $I(t) > 0$ for $t \in [0, t_1)$. By the third equation of system (2) we have $I'(t_1) = \sigma E(t_1) + \delta R(t_1)$. Solving $E(t)$ and $R(t)$ in the second and fourth equation of system (2), we have

$$E(t_1) = E(0)e^{-(\mu+\sigma)t_1} + \int_0^{t_1} \phi(S(\theta), I(\theta))e^{-(\mu+\sigma)(t_1-\theta)}d\theta > 0$$

and

$$R(t_1) = R(0)e^{-(\mu+\delta)t_1} + \int_0^{t_1} \gamma I(\theta)e^{-(\mu+\delta)(t_1-\theta)}d\theta > 0.$$

It follows that $I'(t_1) > 0$, and hence the $I(t)$ are nonnegative for all $t \in [0, \alpha)$. And for all $t \in [0, \alpha)$.

$$E(t) = E(0)e^{-(\mu+\sigma)t} + \int_0^{t} \phi(S(\theta), I(\theta))e^{-(\mu+\sigma)(t-\theta)}d\theta \geq 0$$

and

$$R(t) = R(0)e^{-(\mu+\delta)t} + \int_0^{t} \gamma I(\theta)e^{-(\mu+\delta)(t-\theta)}d\theta \geq 0.$$

Furthermore, for $t \in [0, \alpha)$, we obtain

$$\frac{dN}{dt} = A - \mu(S(t)+E(t)+I(t)+R(t)) = A - \mu N \tag{5}$$

which implies that $(S(t), E(t), I(t), R(t))$ is uniformly bounded on $[0, \alpha)$. It follows that $(S(t), E(t), I(t), R(t))$ exists and is unique and positive for all $t \geq 0$. From (5), we immediately have (4), which completes the proof. □

3. Stability Analysis of SEIRI Model

3.1. **Local stability of the endemic equilibrium.** In this section, we discuss the local asymptotic stability of the endemic equilibrium of the generalized SEIRI epidemic model (2). Note that the system (2) always has a disease-free equilibrium $P_0 = (N, 0, 0, 0)$. On the other hand, The existence of endemic equilibrium is determined by the following proposition:

Proposition 3.1. *Under the hypothesis* (H_1)*, if* $R_0 > 1$*, then system* (2) *admits a unique endemic equilibrium* $P^* = (S^*, E^*, I^*, R^*)$*, with*

$$I^* = \frac{A - \mu S^*}{\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}},\ E^* = I^* \frac{\mu(\mu+\delta+\gamma)}{\sigma(\mu+\delta)},\ R^* = \frac{\gamma I^*}{\mu+\delta},$$

and S^* *is the unique solution of the following equation:*

$$f(S, \frac{A - \mu S}{\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}}) = \frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}. \tag{6}$$

with

$$R_0 = \frac{\sigma}{(\mu+\sigma)} \frac{(\mu+\delta) f(N, 0)}{\mu(\mu+\delta+\gamma)} \tag{7}$$

Proof. We prove the existence and the uniqueness of the endemic equilibrium P^*. At a fixed point (S, E, I, R) of system (2), the following equations hold.

$$\begin{aligned} A - \mu S - f(S, I)I &= 0, \\ f(S, I)I - (\mu+\sigma)E &= 0, \\ \sigma E - (\mu+\gamma)I + \delta R &= 0, \\ \gamma I - (\mu+\delta)R &= 0. \end{aligned} \tag{8}$$

A simple calculation gives the following system:

$$\begin{aligned} f(S, I)I - \frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)} I &= 0, \\ E &= I \frac{\mu(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}, \\ I &= \frac{A - \mu S}{\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}}, \\ R &= \frac{\gamma I}{\mu+\delta}. \end{aligned} \tag{9}$$

From the first equation of (9) we get $I = 0$ or $f(S, I) = \frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}$.
If I = 0, we obtain the disease-free equilibrium point $P_0 = (\frac{A}{\mu}, 0, 0, 0)$.
If $I \neq 0$, then using the (9) we get the following equation

$$f(S, \frac{A - \mu S}{\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}}) = \frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)} \tag{10}$$

we have $I = \frac{A-\mu S}{\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}} \geq 0$ implies that $S \leq \frac{A}{\mu}$. Hence, there is no positive equilibrium point if $S > \frac{A}{\mu}$. Now, we consider the following function g defined on the interval $[0, \frac{A}{\mu}]$

$$g(S) := f(S, \frac{A-\mu S}{\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}}) - \frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}$$

Since,

$$\begin{aligned} g(\frac{A}{\mu}) &= f(\frac{A}{\mu},0) - \frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)} \\ &= \frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}(\frac{f(\frac{A}{\mu},0)}{\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}} - 1) \\ &= \frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}(R_0 - 1) > 0 \ for \ R_0 > 1 \end{aligned}$$

and

$$g(0) = -\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)} < 0$$

Further

$$g^{'}(S) = \frac{\partial f}{\partial S} - \frac{\mu}{\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}}\frac{\partial f}{\partial I}$$

bay the hypothesis (H_1), we have $g^{'}(S) > 0$.
Hence, there exists a unique endemic equilibrium $P^* = (S^*, E^*, I^*, R^*)$ with $S^* \in]0, \frac{A}{\mu}[$ and $E^* > 0$, $I^* > 0$, $R^* > 0$ satisfies the equations $I = \frac{A-\mu S}{\frac{\mu(\mu+\sigma)(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}}$, $E = I\frac{\mu(\mu+\delta+\gamma)}{\sigma(\mu+\delta)}$ and $R = \frac{\gamma I}{\mu+\delta}$. Hence, we conclude the existence and uniqueness of the endemic equilibrium P^*. □

The total population size N satisfies the equation $N = S + E + I + R$, which reduces the system (2) to the following system:

$$\begin{aligned} \frac{dS}{dt} &= A - \mu S - f(S,I)I, \\ \frac{dE}{dt} &= f(S,I)I - (\mu+\sigma)E \\ \frac{dI}{dt} &= \sigma E - (\mu+\gamma)I + \delta N - \delta S - \delta E - \delta I. \end{aligned} \tag{11}$$

In the next, we will study the local stability of the positive equilibrium $P_1^* = (S^*, E^*, I^*)$ of the system (11), with S^*, E^* *et* I^* are defined in Proposition 3.1.

Theorem 3.1. *Suppose the hypothesis* (H_1) *hold.*
If $R_0 > 1$, *then the endemic equilibrium* P_1^* *of system* (11) *is locally asymptotically stable.*

Proof. Let $x = S - S^*$, $y = E - E^*$ and $z = I - I^*$. Then by linearizing system (11) around P_1^*, we have

$$\begin{aligned} \frac{dx}{dt} &= (-\mu - \frac{\partial f(S^*,I^*)}{\partial S}I^*)x(t) - (\frac{\partial f(S^*,I^*)}{\partial I}I^* + f(S^*,I^*))z(t), \\ \frac{dy}{dt} &= \frac{\partial f(S^*,I^*)}{\partial S}I^*x(t) + (\frac{\partial f(S^*,I^*)}{\partial I}I^* + f(S^*,I^*))z(t) - (\mu+\sigma)y(t), \\ \frac{dz}{dt} &= -\delta x(t) + (\sigma - \delta)y(t) - (\mu+\gamma+\delta)z(t). \end{aligned} \tag{12}$$

The Jacobian matrix $M(\lambda)$ of equation (12) is defined as follows:

$$\mathcal{M}(\lambda) = \begin{pmatrix} \lambda + (\mu + \frac{\partial f(S^*,I^*)}{\partial S} I^*) & 0 & (\frac{\partial f(S^*,I^*)}{\partial I} I^* + f(S^*,I^*)) \\ -\frac{\partial f(S^*,I^*)}{\partial S} I^* & \lambda + (\mu+\sigma) & -(\frac{\partial f(S^*,I^*)}{\partial I} I^* + f(S^*,I^*)) \\ \delta & -(\sigma - \delta) & \lambda + (\mu + \gamma + \delta) \end{pmatrix}$$

The characteristic equation associated to system (12) is given by

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \tag{13}$$

where

$$A = (\mu+\gamma+\delta) + (\mu+\sigma) + (\mu + \frac{\partial f(S^*,I^*)}{\partial S} I^*),$$

$$B = (\mu+\sigma)(\mu+\gamma+\delta) - \sigma(\frac{\partial f(S^*,I^*)}{\partial I} I^* + f(S^*,I^*)) + (\mu + \frac{\partial f(S^*,I^*)}{\partial S} I^*)((\mu+\gamma+\delta)+(\mu+\sigma)),$$

and

$$C = (\mu+\sigma)(\mu+\gamma+\delta)(\mu + \frac{\partial f(S^*,I^*)}{\partial S} I^*) - \sigma(\mu+\delta)(\frac{\partial f(S^*,I^*)}{\partial I} I^* + f(S^*,I^*)).$$

Firstly, form hypothesis (H_1), we have $\frac{\partial f(S^*,I^*)}{\partial S} I^* \geq 0$, which implies that $A > 0$. Secondly, by using the first and the second and the third equation in system (11), we find that

$$f(S^*,I^*) = \frac{(\mu+\sigma)(\mu+\gamma+\delta)}{(\mu+\delta)} \frac{\mu}{\sigma}. \tag{14}$$

Hence, the hypothesis (H_1) and equation (14) we have:

$$\begin{aligned} C &= (\mu+\sigma)(\mu+\gamma+\delta)\mu + (\mu+\sigma)(\mu+\gamma+\delta)\frac{\partial f(S^*,I^*)}{\partial S} I^* \\ &- \sigma(\mu+\delta) f(S^*,I^*) - \sigma(\mu+\delta)\frac{\partial f(S^*,I^*)}{\partial I} I^* \\ &= \sigma(\mu+\delta)\Big(\frac{(\mu+\sigma)(\mu+\gamma+\delta)}{(\mu+\delta)}\frac{\mu}{\sigma} - f(S^*,I^*)\Big) \\ &+ (\mu+\sigma)(\mu+\gamma+\delta)\frac{\partial f(S^*,I^*)}{\partial S} I^* - \sigma(\mu+\delta)\frac{\partial f(S^*,I^*)}{\partial I} I^* \\ &= (\mu+\sigma)(\mu+\gamma+\delta)\frac{\partial f(S^*,I^*)}{\partial S} I^* - \sigma(\mu+\delta)\frac{\partial f(S^*,I^*)}{\partial I} I^* \geq 0. \end{aligned}$$

and

$$
\begin{aligned}
AB-C &= \Big((\mu+\gamma+\delta)+(\mu+\sigma)+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\Big)\times \\
&\Big((\mu+\sigma)(\mu+\gamma+\delta)-\sigma(\frac{\partial f(S^*,I^*)}{\partial I}I^*+f(S^*,I^*)) \\
&+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)((\mu+\gamma+\delta)+(\mu+\sigma))\Big) \\
&-(\mu+\sigma)(\mu+\gamma+\delta)(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)+\sigma(\mu+\delta)(\frac{\partial f(S^*,I^*)}{\partial I}I^*+f(S^*,I^*)) \\
&=(\mu+\delta)(\mu+\sigma)(\mu+\gamma+\delta)-\sigma(\mu+\delta)(\frac{\partial f(S^*,I^*)}{\partial I}I^*+f(S^*,I^*)) \\
&+(\gamma+(\mu+\sigma))\Big((\mu+\sigma)(\mu+\gamma+\delta)-\sigma(\frac{\partial f(S^*,I^*)}{\partial I}I^*+f(S^*,I^*))\Big) \\
&+(\mu+\sigma)(\mu+\gamma+\delta)(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)-\sigma(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)(\frac{\partial f(S^*,I^*)}{\partial I}I^*+f(S^*,I^*)) \\
&-(\mu+\sigma)(\mu+\gamma+\delta)(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)+\sigma(\mu+\delta)(\frac{\partial f(S^*,I^*)}{\partial I}I^*+f(S^*,I^*)) \\
&+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)(\mu+\sigma)\Big((\mu+\gamma+\delta)+(\mu+\sigma)+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\Big) \\
&+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)(\mu+\gamma+\delta)\Big((\mu+\gamma+\delta)+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\Big) \\
&+(\mu+\sigma)(\mu+\gamma+\delta)(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*) \\
&=(\mu+\delta)(\mu+\sigma)(\mu+\gamma+\delta)+\sigma(\gamma+(\mu+\sigma))\Big(\frac{(\mu+\sigma)(\mu+\gamma+\delta)}{\sigma}-f(S^*,I^*)\Big) \\
&-\sigma(\gamma+(\mu+\sigma))\frac{\partial f(S^*,I^*)}{\partial I}I^*-\sigma(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\frac{\partial f(S^*,I^*)}{\partial I}I^* \\
&+\sigma(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\Big(\frac{(\mu+\sigma)(\mu+\gamma+\delta)}{\sigma}-f(S^*,I^*)\Big) \\
&+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)(\mu+\sigma)\Big((\mu+\gamma+\delta)+(\mu+\sigma)+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\Big) \\
&+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)(\mu+\gamma+\delta)\Big((\mu+\gamma+\delta)+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\Big) \\
&=(\mu+\delta)(\mu+\sigma)(\mu+\gamma+\delta)+\sigma(\gamma+(\mu+\sigma))\frac{\delta}{\mu}f(S^*,I^*) \\
&-\sigma(\gamma+(\mu+\sigma))\frac{\partial f(S^*,I^*)}{\partial I}I^*-\sigma(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\frac{\partial f(S^*,I^*)}{\partial I}I^* \\
&+\sigma(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\frac{\delta}{\mu}f(S^*,I^*) \\
&+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)(\mu+\sigma)\Big((\mu+\gamma+\delta)+(\mu+\sigma)+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\Big) \\
&+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)(\mu+\gamma+\delta)\Big((\mu+\gamma+\delta)+(\mu+\frac{\partial f(S^*,I^*)}{\partial S}I^*)\Big)\geq 0.
\end{aligned}
$$

So, by the Routh-Hurwitz criterion, we obtain the local stability of P_1^*. This concludes the proof. □

By Theorem 3.1 and $N = S + E + I + R$ we have the following corollary.

Corollary 3.1. *Suppose the hypothesis* (H_1) *hold.*
If $R_0 > 1$*, then the endemic equilibrium* P^* *of system* (2) *is locally asymptotically stable.*

3.2. **Global stability of the disease-free equilibrium and the endemic equilibrium.** Now, we discuss the global stability of the disease-free equilibrium P_0 and the endemic equilibrium P^* of system (2).

Proposition 3.2. *Suppose the hypothesis* (H_1) *hold.*
If $R_0 \leq 1$*, then the disease-free equilibrium* P_0 *is globally asymptotically stable.*

Proof. Define a Lyapunov functional

$$W_0(t) = V_0(t) + U_0(t),$$

where

$$V_0(t) = \int_{\frac{A}{\mu}}^{S} (1 - \frac{f(\frac{A}{\mu}, 0)}{f(u, 0)}) du,$$

and

$$U_0(t) = E + \frac{\sigma + \mu}{\sigma} I + \frac{\sigma + \mu}{\sigma} \frac{\delta}{\mu + \delta} R$$

We will show that $\frac{dW_0(t)}{dt} \leq 0$ for all $t \geq 0$. We have :

$$\begin{aligned}
\frac{dV_0(t)}{dt} &= (1 - \frac{f(\frac{A}{\mu}, 0)}{f(S, 0)}) \dot{S} \\
&= (1 - \frac{f(\frac{A}{\mu}, 0)}{f(S, 0)})(A - \mu S - f(S, I)I) \\
&= \mu(1 - \frac{f(\frac{A}{\mu}, 0)}{f(S, 0)})(\frac{A}{\mu} - S) - f(S, I)I + \frac{f(\frac{A}{\mu}, 0)}{f(S, 0)} f(S, I)I
\end{aligned}$$

and

$$\begin{aligned}
\frac{dU_0(t)}{dt} &= f(S, I)I - (\mu + \sigma)E + (\mu + \sigma)E - \frac{(\mu + \gamma)(\mu + \sigma)}{\sigma} I \\
&\quad + \frac{(\sigma + \mu)\delta}{\sigma} R + \frac{(\sigma + \mu)}{\sigma} \frac{\delta\gamma}{(\mu + \delta)} I - \frac{(\sigma + \mu)\delta}{\sigma} R \\
&= f(S, I)I - \frac{(\sigma + \mu)}{\sigma(\mu + \delta)} \mu(\mu + \delta + \gamma) I
\end{aligned}$$

Then:

$$\begin{aligned}\frac{dW_0(t)}{dt} &= \mu(1-\frac{f(\frac{A}{\mu},0)}{f(S,0)})(\frac{A}{\mu}-S)\\ &+\frac{f(\frac{A}{\mu},0)}{f(S,0)}f(S,I)I-\frac{(\sigma+\mu)}{\sigma(\mu+\delta)}\mu(\mu+\delta+\gamma)I\\ &= \mu(1-\frac{f(\frac{A}{\mu},0)}{f(S,0)})(\frac{A}{\mu}-S)\\ &+\frac{(\sigma+\mu)}{\sigma(\mu+\delta)}\mu(\mu+\delta+\gamma)I(\frac{f(\frac{A}{\mu},0)}{\frac{(\sigma+\mu)}{\sigma(\mu+\delta)}\mu(\mu+\delta+\gamma)}\frac{f(S,I)}{f(S,0)}-1)\\ &= \mu(1-\frac{f(\frac{A}{\mu},0)}{f(S,0)})(\frac{A}{\mu}-S)\\ &+\frac{(\sigma+\mu)}{\sigma(\mu+\delta)}\mu(\mu+\delta+\gamma)I(R_0\frac{f(S,I)}{f(S,0)}-1)\end{aligned}$$

By the hypothesis (H_1), we obtain that

$$(1-\frac{f(\frac{A}{\mu},0)}{f(S,0)})(\frac{A}{\mu}-S)\leq 0$$

Where equality holds if and only if $S=\frac{A}{\mu}$.
Furthermore, it follows from the hypothesis (H_1) that

$$\begin{aligned}R_0\frac{f(S,I)}{f(S,0)} &\leq R_0\frac{f(S,0)}{f(S,0)}\\ &\leq R_0\end{aligned}$$

Therefore, $R_0\leq 1$ ensures that $\frac{dW_0(t)}{dt}\leq 0$ for all $t\geq 0$, where $\frac{dW_0(t)}{dt}=0$ holds if $(S,E,I,R)=(\frac{A}{\mu},0,0,0)$. Hence, it follows from system (2) that $\{P_0\}$ is the largest invariant set in $\left\{(S,E,I,R)|\frac{dW_0(t)}{dt}=0\right\}$. From the Lyapunov-LaSalle asymptotic stability, we obtain that P_0 is globally asymptotically stable. This completes the proof. □

Theorem 3.2. *Suppose the hypotheses* (H_1) *and* (H_2) *hold.*
If $R_0>1$, *then the endemic equilibrium* P^* *is globally asymptotically stable.*

Proof. To prove global stability of the endemic equilibrium, we define a Lyapunov functional
$W(t)=V_1(t)+V_2(t)+V_3(t)$, with

$$V_1(t)=\int_{S^*}^{S}(1-\frac{f(S^*,I^*)}{f(u,I^*)})du,$$

$$V_2(t)=(E-E^*-E^*\ln\frac{E}{E^*})+\frac{\sigma+\mu}{\sigma}(I-I^*-I^*\ln\frac{I}{I^*}),$$

and

$$V_3(t)=\frac{(\sigma+\mu)\delta}{\sigma}[\frac{R^*}{\gamma I^*}(R-R^*-R^*\ln\frac{R}{R^*})].$$

Using the relations

$$A=\mu S^*+f(S^*,I^*)I^*\ \textit{and}\ (\mu+\sigma)E^*=f(S^*,I^*)I^*,$$

a simple calculation gives:

$$\begin{aligned}\frac{dV_1(t)}{dt} &= \Big(1-\frac{f(S^*,I^*)}{f(S,I^*)}\Big)\Big(A-\mu S-f(S,I)I\Big)\\ &= \mu\Big(1-\frac{f(S^*,I^*)}{f(S,I^*)}\Big)(S^*-S)+\Big(1-\frac{f(S^*,I^*)}{f(S,I^*)}\Big)(f(S^*,I^*)I^*-f(S,I)I)\\ &= \mu\Big(1-\frac{f(S^*,I^*)}{f(S,I^*)}\Big)(S^*-S)+(\sigma+\mu)E^*\Big(1-\frac{f(S^*,I^*)}{f(S,I^*)}\Big)\Big(1-\frac{f(S,I)I}{f(S^*,I^*)I^*}\Big),\end{aligned}$$

$$\begin{aligned}\frac{dV_2(t)}{dt} &= (1-\frac{E^*}{E})(f(S,I)I-(\sigma+\mu)E)+\frac{(\sigma+\mu)}{\sigma}(1-\frac{I^*}{I})(\sigma E-(\mu+\gamma)I+\delta R)\\ &= (\sigma+\mu)E^*(1-\frac{E^*}{E})(\frac{f(S,I)I}{f(S^*,I^*)I^*}-\frac{E}{E^*})+(\sigma+\mu)E^*(1-\frac{I^*}{I})(\frac{E}{E^*}-\frac{(\mu+\gamma)}{\sigma E^*}I+\frac{\delta R}{\sigma E^*})\end{aligned}$$

Moreover, the two relationships:

$$\frac{(\sigma E^*+\delta R^*)I}{I^*}=(\mu+\gamma)I \ \textit{and}\ R^*=\frac{\gamma I^*}{\mu+\delta}$$

are used to find:

$$\begin{aligned}\frac{dV_2(t)}{dt} &= (\sigma+\mu)E^*(1-\frac{E^*}{E})(\frac{f(S,I)I}{f(S^*,I^*)I^*}-\frac{E}{E^*})+(\sigma+\mu)E^*(1-\frac{I^*}{I})(\frac{E}{E^*}-\frac{I}{I^*})\\ &\quad+(\sigma+\mu)E^*(1-\frac{I^*}{I})(\frac{\delta R}{\sigma E^*}-\frac{\delta R^*}{\sigma E^*}\frac{I}{I^*})\\ &= (\sigma+\mu)E^*[(1-\frac{E^*}{E})(\frac{f(S,I)I}{f(S^*,I^*)I^*}-\frac{E}{E^*})+(1-\frac{I^*}{I})(\frac{E}{E^*}-\frac{I}{I^*})]\\ &\quad+\frac{(\sigma+\mu)\delta}{\sigma}[R-\frac{R^*I}{I^*}-\frac{RI^*}{I}+R^*]\\ &= (\sigma+\mu)E^*[(1-\frac{E^*}{E})(\frac{f(S,I)I}{f(S^*,I^*)I^*}-\frac{E}{E^*})+(1-\frac{I^*}{I})(\frac{E}{E^*}-\frac{I}{I^*})]\\ &\quad-\frac{(\sigma+\mu)\delta}{\sigma}R^*(\sqrt{\frac{RI^*}{IR^*}}-\sqrt{\frac{IR^*}{RI^*}})^2\\ &\quad+\frac{(\sigma+\mu)\delta}{\sigma}[R-\frac{R^*I}{I^*}+\frac{I(R^*)^2}{RI^*}-R^*],\end{aligned}$$

and

$$\begin{aligned}\frac{dV_3(t)}{dt} &= \frac{(\sigma+\mu)\delta}{\sigma}[\frac{R^*}{\gamma I^*}(1-\frac{R^*}{R})(\gamma I-(\mu+\delta)R)]\\ &= \frac{(\sigma+\mu)\delta}{\sigma}[\frac{IR^*}{I^*}-\frac{(\mu+\delta)}{\gamma}\frac{RR^*}{I^*}-\frac{(R^*)^2I}{I^*R}+\frac{(\mu+\delta)}{\gamma}\frac{(R^*)^2}{I^*}]\\ &= \frac{(\sigma+\mu)\delta}{\sigma}[-R+\frac{R^*I}{I^*}-\frac{I(R^*)^2}{RI^*}+R^*]\end{aligned}$$

Then, the time derivative of the function $W(t)$ along the positive solution of system (2) is :

$$\begin{aligned}
\frac{d(W(t))}{dt} &= \mu(1-\frac{f(S^*,I^*)}{f(S,I^*)})(S^*-S)\\
&+(\sigma+\mu)E^*(4-\frac{f(S,I^*)}{f(S,I)}-\frac{f(S^*,I^*)}{f(S,I^*)}-\frac{E^*}{E}\frac{f(S,I)I}{f(S^*,I^*)I^*}-\frac{I^*}{I}\frac{E}{E^*})\\
&+(\sigma+\mu)E^*(-1+\frac{f(S,I^*)}{f(S,I)}+\frac{f(S,I)I}{f(S,I^*)I^*}-\frac{I}{I^*})\\
&-\frac{(\sigma+\mu)\delta}{\sigma}R^*(\sqrt{\frac{RI^*}{IR^*}}-\sqrt{\frac{IR^*}{RI^*}})^2\\
&= \mu(1-\frac{f(S^*,I^*)}{f(S,I^*)})(S^*-S)\\
&+(\sigma+\mu)E^*(4-\frac{f(S,I^*)}{f(S,I)}-\frac{f(S^*,I^*)}{f(S,I^*)}-\frac{E^*}{E}\frac{f(S,I)I}{f(S^*,I^*)I^*}-\frac{I^*}{I}\frac{E}{E^*})\\
&+(\sigma+\mu)E^*\frac{I}{I^*}(1-\frac{f(S,I)}{f(S,I^*)})(\frac{\phi(S,I^*)}{\phi(S,I)}-1)\\
&-\frac{(\sigma+\mu)\delta}{\sigma}R^*(\sqrt{\frac{RI^*}{IR^*}}-\sqrt{\frac{IR^*}{RI^*}})^2
\end{aligned}$$

Now, according to the assumptions (H_1) and (H_2), we have

$$(1-\frac{f(S^*,I^*)}{f(S,I^*)})\mu(S^*-S)\leq 0$$

and

$$(1-\frac{f(S,I)}{f(S,I^*)})(\frac{\phi(S,I^*)}{\phi(S,I)}-1)\leq 0$$

Moreover, since the arithmetic mean is greater than or equal to the geometric mean, we obtain that

$$(4-\frac{f(S,I^*)}{f(S,I)}-\frac{f(S^*,I^*)}{f(S,I^*)}-\frac{E^*}{E}\frac{f(S,I)I}{f(S^*,I^*)I^*}-\frac{I^*}{I}\frac{E}{E^*})\leq 0$$

Therefore, $\frac{dW(t)}{dt}\leq 0$ for all $t\geq 0$, where the equality holds only at the equilibrium point (S^*,E^*,I^*,R^*). It follows from system (2) that $\{P^*\}$ is the largest invariant set in $\left\{(S,E,I,R)|\frac{dW(t)}{dt}=0\right\}$.
Consequently, we obtain, by the Lyapunov-LaSalle asymptotic stability theorem, that P^* is globally asymptotically stable. This completes the proof. □

4. Concluding Remarks

In this paper we propose an SEIRI epidemic model with a general incidence function, latent period and a relapse rate, (see system (2)).

We found the local stability of the endemic equilibrium and the global stability for the disease-free and endemic equilibrium, by Routh-Hurwitz criterion and by constructing Lyapunov functionals. When $R_0\leq 1$, the disease-free steady state is globally asymptotically stable, and no other equilibria exist. When $R_0>1$, a unique endemic equilibrium P^* appears. Using Routh-Hurwitz criterion and Lyapunov functional technique, the endemic equilibrium is locally and globally asymptotically stable.

Note that the basic reproduction number written as $R_0 = \frac{\sigma}{(\mu+\sigma)} \frac{(\mu+\delta)f(\frac{A}{\mu},0)}{\mu(\mu+\delta+\gamma)}$, where $\frac{\sigma}{(\mu+\sigma),}$ is the probability of survival of the exposed class, $\frac{A}{\mu}$ represents the number of susceptible individuals at the beginning of the infectious process and $f(\frac{A}{\mu}, 0)$ represents the value of the function f when all individuals are susceptible, μ is the natural death rate, and the constant rate δ at which an individual in the recovered class reverts to the infective class, the recovery rate γ of infectious individuals and the latent period.

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