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FIXED POINT THEOREMS SATISFY PROPERTY P IN G_b-METRIC SPACES

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Abstract. In this paper, we prove fixed point theorem for a contractive mapping which satisfy property P in G_b-metric space. Our results are supported by an example.

1. Introduction

The metric space is generalized by different authors by various ways. Czerwik [9] introduced b-metric space. Zead Mustafa and Brailey Sims [13] coined the concept of G-metric space. A. Aghajani, M. Abbas and J. R. Roshan [2] extended the G-metric space with b-metric space and develop the new structure of metric space, which we call G_b-metric space. They proved the fixed point theorems in G_b-metric spaces. A self map T of the space X with a nonempty fixed point set $F(T)$. Then we say that T has a property P if $F(T) = F(T^n)$ for each $n \in N$ [10]-[3]. We design the fixed point theorems for the self maps which satisfy property P for various contractions in G-metric spaces [10]- [8].

Definition 1 ([9]) Let X be a non empty set and the mapping $d : X \times X \to [0, \infty)$. The mapping d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. Then d is called a b-metric on X. The ordered pair (X, d) is called b-metric space with coefficient s.

G-metric space is defined as follows

Definition 2 ([13])Let X be a non empty set and the mapping $G : X \times X \times X \to [0, \infty)$. The mapping G satisfies

(i) $G(x, y, z) = 0$ if and only if $x = y = z$ for all $x, y, z \in X$;
(ii) $0 < G(x, x, y)$ for all $x, y \in X$;
(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

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(v) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality). Then G is called a G-metric on X and the ordered pair (X,G) is called G-metric space.

A. Aghajani, M. Abbas and J. R. Roshan [2] introduced G_b- metric space as follows

Definition 3 ([2]) Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G_b : X \times X \times X \to R^+$ satisfies:

(i) $G_b(x,y,z) = 0$ if $x = y = z$ for all $x,y,z \in X$;
(ii) $0 < G_b(x,x,y)$ for all $x,y,z \in X$ with $x \neq y$;
(iii) $G_b(x,x,y) \leq G_b(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$;
(iv) $G_b(x,y,z) = G_b(px,z,y)$, where p is a permutation of x,y,z (symmetry);
(v) $G_b(x,y,z) \leq s[G_b(x,a,a) + G_b(a,y,z)]$.

Then G_b is called a generalized b -metric or G_b metric on X. The ordered pair (X,G_b) is called generalized b metric or G_b-metric space.

Following example shows that a G_b-metric on X need not be a G-metric on X.

Example 1 ([2]) Let (X,G) be a G-metric space and $G_*(x,y,z) = G(x,y,z)^p$; where $p > 1$ is a real number. Note that G_* is a G_b-metric with $s = 2^{P-1}$. Obviously, G_* satisfies conditions (i) to (iv) of the G_b metric space. Now it suffices to show that condition (v) of G_b metric space to be hold. If $1 < p < \infty$, then the convexity of the function $f(x) = x^p (x > 0)$ implies that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. Thus for each $x,y,z,a \in X$ we obtain

$$\begin{aligned} G_*(x,y,z) = G(x,y,z)^p &\leq (G(x,a,a) + G(a,y,z))^p \\ &\leq 2^{p-1}(G(x,a,a)^p + G(a,y,z)^p) \\ &= 2^{p-1}(G_*(x,a,a) + G_*(a,y,z)). \end{aligned}$$

So G_* is a G_b-metric with $s = 2^{p-1}$.

Also in the above example, (X,G_*) is not necessarily a G-metric space.

Example 2 Let $X = R$ and let

$$G_b(x,y,z) = \max\{|x-y|^2, |y-z|^2, |x-z|^2\}.$$

Then (X,G_b) is a G_b-metric space with the coefficient $s = 2$.

Definition 4 ([2]) Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be :

(i) G_b-Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that, for all $m,n,l \geq n_0$, $G(x_n,x_m,x_l) < \epsilon$;
(ii) G_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that, for all $m,n \geq n_0$, $G(x_n,x_m,x) < \epsilon$.

Proposition 1 ([2]) Let (X,G_b) be a G_b-metric space. Then the following are equivalent:

(i) $\{x_n\}$ is G_b-converges to x.
(ii) $G_b(x_n,x_n,x) \to 0$, as $n \to \infty$.
(iii) $G_b(x_n,x,x) \to 0$, as $n \to \infty$.

Proposition 2 ([2]) Let (X,G_b) be a G_b-metric space. Then the following are equivalent:

(i) The sequence $\{x_n\}$ is G_b-Cauchy.
(ii) For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G_b(x_n,x_m,x_l) < \epsilon$, for all $n,m,l \geq n_0$.

Definition 5 ([2]) A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.

2. Main results

Our first main result is

Theorem 1 Let (X, G_b) be a complete G_b-metric space with $s \geq 1$ and let $T : X \to X$ be a mapping satisfying

$$\begin{aligned} G_b(Tx,Ty,Tz) \leq k \max \Big[& G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz), \\ & \frac{[G_b(x,Ty,Ty)+G_b(y,Tx,Tx)]}{2}, \frac{[G_b(y,Tz,Tz)+G_b(z,Ty,Ty)]}{2}, \\ & \frac{[G_b(x,Tz,Tz)+G_b(z,Tx,Tx)]}{2} \Big]; \end{aligned} \tag{1}$$

for all $x, y, z \in X$, where k is such that $sk \in [0, 1)$. Then T has a unique fixed point say p $(Tp = p)$ in X and T is G_b-continuous at p.

Proof. Let $x_0 \in X$. Since $T : X \to X$ be a self map, then we get a sequence (x_n) in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &= G_b(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq k \max \Big[G_b(x_{n-1}, x_n, x_n), G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \\ &\quad G_b(x_n, x_{n+1}, x_{n+1}), \frac{[G_b(x_{n-1}, x_{n+1}, x_{n+1}) + G_b(x_n, x_n, x_n)]}{2}, \\ &\quad \frac{[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_n, x_{n+1}, x_{n+1})]}{2}, \\ &\quad \frac{[G_b(x_{n-1}, x_{n+1}, x_{n+1}) + G_b(x_n, x_n, x_n)]}{2} \Big] \\ &\leq k \max \Big[G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \\ &\quad \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2} \Big]. \end{aligned}$$

There are three cases:

Case (i) Suppose

$$\max \Big[G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2} \Big] = \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2}.$$

Then

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq k \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2}.$$

By property (v) of G_b metric space, we have

$$G_b(x_{n-1}, x_{n+1}, x_{n+1}) \leq s\{G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})\}.$$

Then, we get

$$\begin{aligned}
G_b(x_n, x_{n+1}, x_{n+1}) &\leq sk\Big[\frac{G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})}{2}\Big] \\
&\leq \frac{sk}{(2-sk)} G_b(x_{n-1}, x_n, x_n) \\
&= \lambda G_b(x_{n-1}, x_n, x_n),
\end{aligned}$$

where $\lambda = \frac{sk}{(2-sk)}$. Therefore by continuing in this way, we get

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G_b(x_0, x_1, x_1).$$

As $n \to \infty, G_b(x_n, x_{n+1}, x_{n+1}) \to 0$, since $\lambda < 1$. Moreover for all $n, m \in \mathbb{N}, n < m$, and by (v) the property of G_b-metric space.

$$\begin{aligned}
G_b(x_n, x_m, x_m) &\leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, x_m, x_m)] \\
&\leq s[\lambda^n G_b(x_0, x_1, x_1) + G_b(x_{n+1}, x_m, x_m)] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2[G_b(x_{n+1}, x_{n+2}, x_{n+2}) + G_b(x_{n+2}, x_m, x_m)] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) + s^3[G_b(x_{n+2}, x_{n+3}, x_{n+3}) \\
&\quad + G_b(x_{n+3}, x_m, x_m)] \\
&= s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) + s^3\lambda^{n+2} G_b(x_0, x_1, x_1) + \\
&\quad s^3 G_b(x_{n+3}, x_m, x_m) \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) + s^3\lambda^{n+2} G_b(x_0, x_1, x_1) \\
&\quad + \cdots + s^{m-1}\lambda^{n+m-2} G_b(x_0, x_1, x_1) + s^{m-1}\lambda^{n+m-1} G_b(x_0, x_1, x_1) \\
&= s\lambda^n[(1 + s\lambda + (s\lambda)^2 + (s\lambda)^3 + ... + (s\lambda)^{m-2}) + (s\lambda)^{m-2}\lambda] \\
&\quad G_b(x_0, x_1, x_1) \\
&= s\lambda^n[\frac{1 - (s\lambda)^{n-(m-2)}}{(1 - s\lambda)} + (s\lambda)^{m-2}\lambda] G_b(x_0, x_1, x_1).
\end{aligned}$$

Letting $m, n \to \infty$, we get

$$\lim_{n,m \to \infty} G_b(x_n, x_m, x_m) = 0.$$

Case (ii) Suppose

$$\max\Big[G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2}\Big] = G_b(x_n, x_{n+1}, x_{n+1}).$$

Then

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq kG_b(x_n, x_{n+1}, x_{n+1}),$$

which is contradiction, since $k < 1$.
Case (iii) Suppose

$$\max\Big[G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2}\Big] = G_b(x_{n-1}, x_n, x_n).$$

Then

$$\begin{aligned}G_b(x_n, x_{n+1}, x_{n+1}) &\leq kG_b(x_{n-1}, x_n, x_n)\\ &\leq k^2 G_b(x_{n-2}, x_{n-1}, x_{n-1}) \leq \cdots \leq k^n G_b(x_0, x_1, x_1).\end{aligned}$$

As $n \to \infty, G_b(x_n, x_{n+1}, x_{n+1}) \to 0$, since $k < 1$. Also Since $sk < 1$ and by case(i) $\{x_n\}$ is a G_b -Cauchy sequence in X. Since X is G_b-complete, then there exists $p \in X$ such that $\{x_n\}$ is G_b-converges to $p \in X$. Now we claim that p is fixed point of T. Suppose that $Tp \neq p$.

$$\begin{aligned}G_b(x_n, Tp, Tp) &\leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tp, Tp)]\\ &\leq s\lambda^n G_b(x_0, x_1, x_1) + sk \max\Big[G_b(x_n, p, p), G_b(x_n, x_{n+1}, x_{n+1}),\\ &\quad G_b(p, Tp, Tp), G_b(p, Tp, Tp), \frac{[G_b(x_n, Tp, Tp) + G_b(Tp, x_{n+1}, x_{n+1})]}{2},\\ &\quad \frac{[G_b(p, Tp, Tp) + G_b(p, Tp, Tp)]}{2},\\ &\quad \frac{[G_b(x_n, Tp, Tp) + G_b(Tp, x_{n+1}, x_{n+1})]}{2}\Big].\end{aligned}$$

As $n \to \infty, \{x_n\} \to p$ and above inequality terns into

$$G_b(p, Tp, Tp) \leq skG_b(p, Tp, Tp).$$

It is contradiction, since $sk < 1$. Thus $Tp = p$. Therefore p is a fixed point of T. For uniqueness suppose $q \neq p$ and q is another fixed point of T, i.e. $Tq = q$. By (v) the property of G_b-metric space

$$G_b(x_n, Tq, Tq) \leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tq, Tq)].$$

Therefore

$$\begin{aligned}G_b(x_n, q, q) \leq s\lambda^n G_b(x_0, x_1, x_1) + sk \max\Big[&G_b(x_n, q, q), G_b(x_n, x_{n+1}, x_{n+1}),\\ &G_b(q, Tq, Tq), G_b(q, Tq, Tq), \frac{[G_b(x_n, Tq, Tq) + G_b(q, x_{n+1}, x_{n+1})]}{2},\\ &\frac{[G_b(q, Tq, Tq) + G_b(q, Tq, Tq)]}{2}, \frac{[G_b(x_n, Tq, Tq) + G_b(q, x_{n+1}, x_{n+1})]}{2}\Big].\end{aligned}$$

As $\lambda < 1$, we extend $n \to \infty$, so $\{x_n\} \to p$. Thus we get

$$G_b(p, q, q) \leq sk \max\Big[\frac{[G_b(p, q, q) + G_b(q, p, p)]}{2}, G_b(p, q, q)\Big].$$

There are two cases

Case (i) Suppose

$$\max\Big[\frac{[G_b(p, q, q) + G_b(q, p, p)]}{2}, G_b(p, q, q)\Big] = G_b(p, q, q).$$

Then

$$G_b(p, q, q) \leq skG_b(p, q, q),$$

which is contradiction, since $sk < 1$.

Case (ii) Suppose

$$\max\Big[\frac{[G_b(p,q,q)+G_b(q,p,p)]}{2}, G_b(p,q,q)\Big] = \frac{[G_b(p,q,q)+G_b(q,p,p)]}{2}.$$

Then

$$G_b(p,q,q) \leq sk\Big[\frac{G_b(p,q,q)+G_b(q,p,p)}{2}\Big].$$

It implies that

$$G_b(p,q,q) \leq \frac{sk}{2-sk} G_b(q,p,p). \tag{2}$$

Also consider,

$$\begin{aligned} G_b(Tq,x_n,x_n) \leq k\max\Big[&G_b(q,x_{n-1},x_{n-1}), G_b(q,Tq,Tq), G_b(x_{n-1},x_n,x_n),\\ &G_b(x_{n-1},x_n,x_n), \frac{[G_b(q,x_n,x_n)+G_b(x_{n-1},Tq,Tq)]}{2},\\ &\frac{[G_b(x_{n-1},x_n,x_n)+G_b(x_{n-1},x_n,x_n)]}{2},\\ &\frac{[G_b(q,x_n,x_n)+G_b(x_n,q,q)]}{2}\Big]. \end{aligned}$$

As $n\to\infty$, we get

$$G_b(q,p,p) \leq k\max\Big[G_b(q,p,p), \frac{[G_b(q,p,p)+G_b(p,q,q)]}{2}\Big].$$

There are two cases:
Case (i)Suppose

$$\max\Big[G_b(q,p,p), \frac{[G_b(q,p,p)+G_b(p,q,q)]}{2}\Big] = G_b(q,p,p).$$

Then, we get

$$G_b(q,p,p) \leq kG_b(q,p,p),$$

which is contradiction, since $k<1$.

Case (ii) Suppose

$$\max\Big[G_b(q,p,p), \frac{[G_b(q,p,p)+G_b(p,q,q)]}{2}\Big] = \frac{[G_b(q,p,p)+G_b(p,q,q)]}{2}.$$

$$G_b(q,p,p) \leq k\Big[G_b(q,p,p)+\frac{G_b(p,q,q)}{2}\Big].$$

It implies that

$$G_b(q,p,p) \leq \Big(\frac{k}{2-k}\Big)G_b(p,q,q). \tag{3}$$

Therefore from (2) and (3), we get

$$G_b(p,q,q) \leq \Big(\frac{sk}{2-sk}\Big)\Big(\frac{k}{2-k}\Big)G_b(p,q,q). \tag{4}$$

Since $(\frac{sk}{2-sk})(\frac{k}{2-k}) < 1$. So (2.4) is possible only when $G_b(p,q,q) = 0$. Thus $p = q$. Now we claim that T is G_b-continuous at p. Let (y_n) be a sequence in X such that $\lim_{n\to\infty} y_n = p$. Consider

$$\begin{aligned} G_b(p,Ty_n,Ty_n) \leq k \max \Big[& G_b(p,y_n,y_n), G_b(p,p,p), G_b(y_n,Ty_n,Ty_n), \\ & G_b(y_n,Ty_n,Ty_n), \frac{[G_b(p,Ty_n,Ty_n) + G_b(y_n,p,p)]}{2}, \\ & \frac{[G_b(y_n,Ty_n,Ty_n) + G_b(y_n,Ty_n,Ty_n)]}{2}, \\ & \frac{[G_b(p,Ty_n,Ty_n) + G_b(y_n,p,p)]}{2} \Big]. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} G_b(p,Ty_n,Ty_n) &\leq k \max \Big[G_b(p,Ty_n,Ty_n), \frac{G_b(p,Ty_n,Ty_n)}{2} \Big] \\ &= kG_b(p,Ty_n,Ty_n). \end{aligned}$$

Since $k < 1$, It is possible only when $G_b(p,Ty_n,Ty_n) = 0$ i.e. $Ty_n = p = Tp$. Therefore T is G_b-continuous at p.

Theorem 2 Let (X, G_b) be a complete G_b-metric space with $s \geq 1$ and let $T : X \to X$ be a mapping satisfying

$$\begin{aligned} G_b(Tx,Ty,Tz) \leq k \max \Big[& G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(x,Ty,Ty), \\ & G_b(y,Tx,Tx), G_b(z,Tz,Tz), \Big]; \end{aligned} \tag{5}$$

for all $x,y,z \in X$ and where k is such that $sk \in [0, \frac{1}{2})$. Then T has a unique fixed point say p in X(i.e. $Tp = p$)and T is G_b-continuous at p.

Proof. Let $x_0 \in X$ and $T : X \to X$ be a self map. Then, we get a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$\begin{aligned} G_b(x_n,x_{n+1},x_{n+1}) &= G_b(Tx_{n-1},Tx_n,Tx_n) \\ &\leq k \max \Big[G_b(x_{n-1},x_n,x_n), G_b(x_{n-1},x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}), \\ &\quad G_b(x_{n-1},x_{n+1},x_{n+1}), G_b(x_n,x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}) \Big] \\ &= k \max \Big[G_b(x_{n-1},x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}), \\ &\quad G_b(x_{n-1},x_{n+1},x_{n+1}) \Big]. \end{aligned}$$

There are three cases:

Case (i) Suppose

$$\begin{aligned} \max \Big[G_b(x_{n-1},x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}), G_b(x_{n-1},x_{n+1},x_{n+1}) \Big] \\ = G_b(x_{n-1},x_{n+1},x_{n+1}). \end{aligned}$$

Then, we get

$$G_b(x_n,x_{n+1},x_{n+1}) \leq kG_b(x_{n-1},x_{n+1},x_{n+1}).$$

Then by property (v) of G_b metric space, we have

$$G_b(x_{n-1},x_{n+1},x_{n+1}) \leq s\{G_b(x_{n-1},x_n,x_n) + G_b(x_n,x_{n+1},x_{n+1})\}.$$

Thus

$$G_b(x_n,x_{n+1},x_{n+1}) \leq sk\Big[G_b(x_{n-1},x_n,x_n)+G_b(x_n,x_{n+1},x_{n+1})\Big].$$

It gives that

$$\begin{aligned}G_b(x_n,x_{n+1},x_{n+1}) &\leq \frac{sk}{(1-sk)}G_b(x_{n-1},x_n,x_n)\\ &= \lambda G_b(x_{n-1},x_n,x_n).\end{aligned}$$

where $\lambda = \dfrac{sk}{1-sk}$. Therefore by continuing in this way, we get

$$G_b(x_n,x_{n+1},x_{n+1}) \leq \lambda^n G_b(x_0,x_1,x_1).$$

Since $k<1$, letting $n\to\infty$, we have $n\to\infty, G_b(x_n,x_{n+1},x_{n+1})\to 0$. Moreover for all $n,m\in\mathbb{N}, n<m$, since $k<\lambda<1$ and by (v) the property of G_b metric space, we have

$$\begin{aligned}G_b(x_n,x_m,x_m) &\leq s\Big[G_b(x_n,x_{n+1},x_{n+1})+G_b(x_{n+1},x_m,x_m)\Big]\\
&\leq s\Big[\lambda^n G_b(x_0,x_1,x_1)+G_b(x_{n+1},x_m,x_m)\Big]\\
&\leq s\lambda^n G_b(x_0,x_1,x_1)+s^2\Big[G_b(x_{n+1},x_{n+2},x_{n+2})+G_b(x_{n+2},x_m,x_m)\Big]\\
&\leq s\lambda^n G_b(x_0,x_1,x_1)+s^2\lambda^{n+1}G_b(x_0,x_1,x_1)\\
&\quad + s^3\Big[G_b(x_{n+2},x_{n+3},x_{n+3})+G_b(x_{n+3},x_m,x_m)\Big]\\
&\leq s\lambda^n G_b(x_0,x_1,x_1)+s^2\lambda^{n+1}G_b(x_0,x_1,x_1)+s^3\lambda^{n+2}G_b(x_0,x_1,x_1)\\
&\quad +\cdots+s^{m-1}\lambda^{n+m-2}G_b(x_0,x_1,x_1)+s^{m-1}\lambda^{n+m-1}G_b(x_0,x_1,x_1)\\
&= s\lambda^n\Big[(1+s\lambda+(s\lambda)^2+(s\lambda)^3+\cdots+(s\lambda)^{m-2})+(s\lambda)^{m-2}\lambda\Big]\\
&\quad G_b(x_0,x_1,x_1)\\
&= s\lambda^n\Big[\frac{1-(s\lambda)^{n-(m-2)}}{(1-s\lambda)}+(s\lambda)^{m-2}\lambda\Big]G_b(x_0,x_1,x_1).\end{aligned}$$

Letting $m,n\to\infty$, we get, $\lim_{n,m\to\infty} G_b(x_n,x_m,x_m)=0$, since $sk<1$. This show that $\{x_n\}$ is a G_b-Cauchy sequence in X.
Case (ii) Suppose

$$\begin{aligned}\max\Big[G_b(x_{n-1},x_n,x_n),G_b(x_n,x_{n+1},x_{n+1}),G_b(x_{n-1},x_{n+1},x_{n+1})\Big]&\\ = G_b(x_n,x_{n+1},x_{n+1}).&\end{aligned}$$

Then

$$G_b(x_n,x_{n+1},x_{n+1}) \leq kG_b(x_n,x_{n+1},x_{n+1}),$$

which is a contradiction, since $k<1$.
Case (iii) Suppose

$$\begin{aligned}\max\Big[G_b(x_{n-1},x_n,x_n),G_b(x_n,x_{n+1},x_{n+1}),G_b(x_{n-1},x_{n+1},x_{n+1})\Big]&\\ = G_b(x_{n-1},x_n,x_n).&\end{aligned}$$

Then

$$\begin{aligned}G_b(x_n, x_{n+1}, x_{n+1}) &\leq kG_b(x_{n-1}, x_n, x_n)\\ &\leq k^2 G_b(x_{n-2}, x_{n-1}, x_{n-1}) \leq \cdots \leq k^n G_b(x_0, x_1, x_1).\end{aligned}$$

Since $k < 1$, as $n \to \infty$, we have $G_b(x_n, x_{n+1}, x_{n+1}) \to 0$. Thus in this case also $\{x_n\}$ is a G_b-Cauchy sequence in X. Since X is G_b-complete, then there exists $p \in X$ such that $\{x_n\} \to p$. Now, we claim that p is fixed point of T. Suppose that $Tp \neq p$, by (v) the property of G_b-metric space and by (2.5), we get

$$\begin{aligned}G_b(x_n, Tp, Tp) &\leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tp, Tp)]\\ &\leq s\lambda^n G_b(x_0, x_1, x_1) + sk \max \Big[G_b(x_n, p, p), G_b(x_n, x_{n+1}, x_{n+1}),\\ &\qquad G_b(p, Tp, Tp), G_b(x_n, Tp, Tp), G_b(p, x_{n+1}, x_{n+1}), G_b(p, Tp, Tp)\Big].\end{aligned}$$

Letting $n \to \infty$, we have $\{x_n\} \to p$. Then

$$G_b(p, Tp, Tp) \leq skG_b(p, Tp, Tp).$$

Since $sk < 1$, the above inequality is true only if $G_b(p, Tp, Tp) = 0$. Thus $p = Tp$. Therefore p is a fixed point of T. For uniqueness, suppose $q \neq p$ and q is another fixed point of T i.e. $Tq = q$. Consider

$$G_b(x_n, Tq, Tq) \leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tq, Tq)].$$

It gives that

$$\begin{aligned}G_b(x_n, q, q) &\leq s\lambda^n G_b(x_0, x_1, x_1) + sk \max \Big[G_b(x_n, q, q), G_b(x_n, x_{n+1}, x_{n+1}),\\ &\qquad G_b(q, Tq, Tq), G_b(x_n, Tq, Tq), G_b(q, x_{n+1}, x_{n+1}), G_b(q, Tq, Tq)\Big].\end{aligned}$$

Letting $n \to \infty$, we have $\{x_n\} \to p$ with $Tq = q$. Then, we get

$$G_b(p, q, q) \leq sk \max \Big[G_b(p, q, q), G_b(q, p, p)\Big].$$

There are two cases:

Case (a) Suppose $\max \Big[G_b(p, q, q), G_b(q, p, p)\Big] = G_b(p, q, q)$. Then

$$G_b(p, q, q) \leq skG_b(p, q, q),$$

which is contradiction, since $sk < 1$.

Case (b) Suppose $\max \Big[G_b(p, q, q), G_b(q, p, p)\Big] = G_b(q, p, p)$. Then

$$G_b(p, q, q) \leq skG_b(q, p, p). \tag{6}$$

Now, consider

$$\begin{aligned}G_b(Tq, x_n, x_n) &\leq k \max \Big[G_b(q, x_{n-1}, x_{n-1}), G_b(q, Tq, Tq), G_b(x_{n-1}, x_n, x_n),\\ &\qquad G_b(q, x_n, x_n), G_b(x_{n-1}, Tq, Tq), G_b(x_{n-1}, x_n, x_n)\Big].\end{aligned}$$

Letting $n \to \infty$, it implies that

$$G_b(q, p, p) \leq k \max \Big[G_b(q, p, p), G_b(p, q, q)\Big].$$

There are two cases:

Case (c) Suppose $\max\Big[G_b(q,p,p), G_b(p,q,q)\Big] = G_b(q,p,p)$. Then

$$G_b(q,p,p) \leq kG_b(q,p,p),$$

which is contradiction, since $k < 1$.

Case (d) Suppose $\max\Big[G_b(q,p,p), G_b(p,q,q)\Big] = G_b(p,q,q)$. Then

$$G_b(q,p,p) \leq kG_b(p,q,q). \tag{7}$$

Using inequality (7) in (6), we have

$$G_b(p,q,q) \leq sk^2 G_b(p,q,q). \tag{8}$$

Since $sk < 1$. Thus (8) is true only if $G_b(p,q,q) = 0$. Thus $p = q$. Therefore p is a fixed point of T in X.

To show that T is G_b-continuous at p, let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} y_n = p$. Consider

$$\begin{aligned} G_b(p,Ty_n,Ty_n) \leq k\max\Big[&G_b(p,y_n,y_n), G_b(p,p,p), G_b(y_n,Ty_n,Ty_n),\\ &G_b(p,Ty_n,Ty_n), G_b(y_n,p,p), G_b(y_n,Ty_n,Ty_n)\Big]. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} G_b(p,Ty_n,Ty_n) \leq \max\Big[&G_b(p,p,p), G_b(p,p,p), G_b(p,Ty_n,Ty_n), G_b(p,Ty_n,Ty_n),\\ &G_b(p,p,p), G_b(p,Ty_n,Ty_n)\Big]. \end{aligned}$$

Thus

$$G_b(p,Ty_n,Ty_n) \leq k\max G_b(p,Ty_n,Ty_n).$$

It is possible only if $G_b(p,Ty_n,Ty_n) = 0$. Thus $Ty_n = p = Tp$. It is proved that T is G_b-continuous at p.

Theorem 3 Let (X, G_b) be a complete G_b-metric space with $s \geq 1$ and let $T : X \to X$ be a mapping satisfying

$$G_b(Tx,Ty,Tz) \leq \alpha G_b(x,y,z) + \beta G_b(x,Tx,Tx) + \gamma G_b(y,Ty,Ty) + \delta G_b(z,Tz,Tz) \tag{9}$$

for all $x, y, z \in X$ and where $\alpha + \beta + \gamma + \delta < 1$. Then T has a unique fixed point say p (i.e. $Tp = p$) and T is G_b-continuous at p.

Proof. Let $x_0 \in X$ and the mapping $T : X \to X$ then we get a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$\begin{aligned} G_b(x_n,x_{n+1},x_{n+1}) &= G_b(Tx_{n-1},Tx_n,Tx_n)\\ &\leq \alpha G_b(x_{n-1},x_n,x_n) + \beta G_b(x_{n-1},x_n,x_n) + \gamma G_b(x_n,x_{n+1},x_{n+1})\\ &\quad + \delta G_b(x_n,x_{n+1},x_{n+1})\\ &\leq (\alpha+\beta) G_b(x_{n-1},x_n,x_n) + (\gamma+\delta) G_b(x_n,x_{n+1},x_{n+1})\\ &\leq \frac{\alpha+\beta}{1-(\gamma+\delta)} G_b(x_{n-1},x_n,x_n)\\ &\leq \lambda G_b(x_{n-1},x_n,x_n), \end{aligned}$$

where $\lambda = \frac{\alpha+\beta}{1-(\gamma+\delta)}$. Therefore, continuing in this way, we get $G_b(x_n,x_{n+1},x_{n+1}) \leq \lambda^n G_b(x_0,x_1,x_1)$.

Moreover for all $n, m \in \mathbb{N}, n < m$ and by (v) th property of G_b metric space, we have

$$\begin{aligned}
G_b(x_n, x_m, x_m) &\leq s\Big[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, x_m, x_m)\Big] \\
&\leq s\Big[\lambda^n G_b(x_0, x_1, x_1) + G_b(x_{n+1}, x_m, x_m)\Big] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\Big[G_b(x_{n+1}, x_{n+2}, x_{n+2}) + G_b(x_{n+2}, x_m, x_m)\Big] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) \\
&\quad + s^3\Big[G_b(x_{n+2}, x_{n+3}, x_{n+3}) + G_b(x_{n+3}, x_m, x_m)\Big] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) + s^3\lambda^{n+2} G_b(x_0, x_1, x_1) + \\
&\quad \cdots + s^{m-1}\lambda^{n+m-2} G_b(x_0, x_1, x_1) + s^{m-1}\lambda^{n+m-1} G_b(x_0, x_1, x_1) \\
&= s\lambda^n\Big[(1 + s\lambda + (s\lambda)^2 + (s\lambda)^3 + \cdots + (s\lambda)^{m-2}) + (s\lambda)^{m-2}\lambda\Big] \\
&\quad G_b(x_0, x_1, x_1) \\
&= s\lambda^n\Big[\frac{1 - (s\lambda)^{n-(m-2)}}{(1 - s\lambda)} + (s\lambda)^{m-2}\lambda\Big] G_b(x_0, x_1, x_1).
\end{aligned}$$

Letting $m, n \to \infty$, we have $\lim_{n,m\to\infty} G_b(x_n, x_m, x_m) = 0$. Hence $\{x_n\}$ is a G_b -Cauchy sequence in X. Since X is a G_b-complete, therefore there exists $p \in X$ such that $\{x_n\}$ is G_b -converges to p. Now we will show here p is fixed point of T. Suppose that $Tp \neq p$.

$$\begin{aligned}
G_b(x_n, Tp, Tp) &\leq s\Big[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tp, Tp)\Big] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s\Big[\alpha G_b(x_n, p, p) + \beta G_b(x_n, x_{n+1}, x_{n+1}) \\
&\quad + \gamma G_b(p, Tp, Tp) + \delta G_b(p, Tp, Tp)\Big].
\end{aligned}$$

Letting $n \to \infty$, since $\lambda < 1$, so $\lambda^n \to 0$ and $x_n \to p$. It gives that

$$G_b(p, Tp, Tp) \leq s(\gamma + \delta) G_b(p, Tp, Tp).$$

Since $s(\gamma + \delta) < 1$. The above inequality is true only if $G_b(p, Tp, Tp) = 0$ i.e. $p = Tp$. Thus p is a fixed point of T.

Suppose $q \neq p$ and q is another fixed point of T, i.e. $Tq = q$. Then consider

$$\begin{aligned}
G_b(x_n, Tq, Tq) &\leq s\Big[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tq, Tq)\Big] \\
G_b(x_n, q, q) &\leq s\lambda^n G_b(x_0, x_1, x_1) + s\Big[\alpha G_b(x_n, q, q) + \beta G_b(x_n, x_{n+1}, x_{n+1}) + \\
&\quad \gamma G_b(q, Tq, Tq) + \delta G_b(q, Tq, Tq)\Big].
\end{aligned}$$

As $n \to \infty, x_n \to p$, $\lambda^n \to 0$ as $\lambda < 1$ and $Tq = q$. We get

$$G_b(p, q, q) \leq s\alpha G_b(p, q, q), \tag{10}$$

since $s\alpha < 1$. The inequality (10) is true only when $G_b(p, q, q) = 0$. i.e.$p = q$. Thus p is a unique fixed point of T.

To show that T is G_b-continuous at p, let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} y_n = p$. Consider

$$\begin{aligned} G_b(p, Ty_n, Ty_n) &\leq \alpha G_b(p, y_n, y_n) + \beta G_b(p, p, p) + \gamma G_b(y_n, Ty_n, Ty_n) \\ &\quad + \delta G_b(y_n, Ty_n, Ty_n) \\ &= \alpha G_b(p, y_n, y_n) + (\gamma + \delta) G_b(y_n, Ty_n, Ty_n) \\ &\leq \alpha G_b(p, y_n, y_n) + s(\gamma + \delta)\{G_b(y_n, p, p) + G_b(p, Ty_n, Ty_n)\}. \end{aligned}$$

It is implies that

$$G_b(p, Ty_n, Ty_n) \leq \Big[\frac{\alpha}{1 - s(\gamma+\delta)} G_b(p, y_n, y_n) + \frac{s(\gamma+\delta)}{1 - s(\gamma+\delta)} G_b(p, Ty_n, Ty_n)\Big].$$

As $n \to \infty, \{y_n\} \to p$, we get

$$G_b(p, Ty_n, Ty_n) = \frac{s(\gamma+\delta)}{1 - s(\gamma+\delta)} G_b(p, Ty_n, Ty_n). \tag{11}$$

Since $\frac{s(\gamma+\delta)}{1-s(\gamma+\delta)} < 1$. The inequality (11) is true only if $G_b(p, Ty_n, Ty_n) = 0$. i.e. $Ty_n = p = Tp$, as $n \to \infty$. It shows that T is G_b-continuous at p.

3. Property P

Let T be a self map of a complete G_b metric space with non-empty fixed point set $F(T)$. Then T is said to satisfy property P if $F(T) = F(T^n)$, for each $n \in \mathbb{N}$.
Theorem 4 Under the contraction of theorem 1, T has property P.
Proof. By Theorem 1, T has a fixed point. Therefore $F(T^n) \neq \emptyset$, each $n \in \mathbb{N}$. Fix $n > 1$ and assume that $p \in F(T^n)$. To show that $p \in F(T)$. Suppose that $p \neq Tp$. Then we have

$$\begin{aligned} G_b(p, Tp, Tp) &= G_b(T^n p, T^{n+1} p, T^{n+1} p) \\ &\leq k \max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^{n-1}p, T^n p, T^n p), \\ &\quad G_b(T^n p, T^{n+1} p, T^{n+1} p), G_b(T^n p, T^{n+1} p, T^{n+1} p), \\ &\quad \frac{[G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p) + G_b(T^n p, T^n p, T^n p)]}{2}, \\ &\quad \frac{[G_b(T^n p, T^{n+1}p, T^{n+1}p) + G_b(T^n p, T^{n+1} p, T^{n+1} p)]}{2}, \\ &\quad \frac{[G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p) + G_b(T^n p, T^n p, T^n p)]}{2}\Big] \\ &= k \max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), \\ &\quad \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}\Big]. \end{aligned}$$

Here, we have three cases:
Case (i) Suppose

$$\begin{aligned} \max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}\Big] \\ = G_b(T^{n-1}p, T^n p, T^n p). \end{aligned}$$

Then, we get

$$G_b(T^np, T^{n+1}p, T^{n+1}p) \leq kG_b(T^{n-1}p, T^np, T^np) \leq \cdots \leq k^nG_b(p, Tp, Tp).$$

Case (ii) Suppose

$$\max\Big[G_b(T^{n-1}p, T^np, T^np), G_b(T^np, T^{n+1}p, T^{n+1}p), \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}\Big] = G_b(T^np, T^{n+1}p, T^{n+1}p).$$

Then, we get

$$G_b(T^np, T^{n+1}p, T^{n+1}p) \leq kG_b(T^np, T^{n+1}p, T^{n+1}p),$$

which is contradiction, since $k < 1$.
Case (iii) Suppose

$$\max\Big[G_b(T^{n-1}p, T^np, T^np), G_b(T^np, T^{n+1}p, T^{n+1}p), \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}\Big] = \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}.$$

Then, we get

$$G_b(T^np, T^{n+1}p, T^{n+1}p) \leq k\frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}. \tag{12}$$

By property (v) of G_b-metric space, we have

$$G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p) \leq s\Big[G_b(T^{n-1}p, T^np, T^np) + G_b(T^np, T^{n+1}p, T^{n+1}p)\Big]. \tag{13}$$

Using inequality (13) in (12), we get

$$G_b(T^np, T^{n+1}p, T^{n+1}p) \leq sk\Big[\frac{[G_b(T^{n-1}p, T^np, T^np) + G_b(T^np, T^{n+1}p, T^{n+1}p)]}{2}\Big].$$

It gives that

$$\begin{aligned} G_b(T^np, T^{n+1}p, T^{n+1}p) &\leq \frac{sk}{2-sk}G_b(T^{n-1}p, T^np, T^np) \\ &= \lambda G_b(T^{n-1}p, T^np, T^np) \leq \cdots \leq \lambda^n G_b(p, Tp, Tp), \end{aligned}$$

where $\lambda = \frac{sk}{2-sk}$. Since $k, \lambda < 1$, so as $n \to \infty$, we get $G_b(p, Tp, Tp) = 0$ and hence in all cases $p = Tp$ i.e. $p \in F(T)$. Hence T has property P.
Theorem 5 Under the contraction of theorem 2, T has property P.
Proof. By theorem 2, T has a fixed point. Therefore $F(T^n) \neq \emptyset$, each $n \in \mathbb{N}$. Fix

$n > 1$ and assume that $p \in F(T^n)$. To show that $p \in F(T)$. Suppose that $P \neq Tp$.

$$\begin{aligned}
G_b(p, Tp, Tp &= G_b(T^n p, T^{n+1}p, T^{n+1}p) \\
&\leq k \max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^{n-1}p, T^n p, T^n p), \\
&\quad G_b(T^n p, T^{n+1}p, T^{n+1}p), G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p), G_b(T^n p, T^n p, T^n p), \\
&\quad G_b(T^n p, T^{n+1}p, T^{n+1}p)\Big] \\
&\leq k \max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p), \\
&\quad G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)\Big].
\end{aligned}$$

Here we have three cases,

Case (i)Suppose

$$\max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p), G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)\Big] = G_b(T^{n-1}p, T^n p, T^n p).$$

Then we get,

$$G_b(T^n p, T^{n+1}p, T^{n+1}p) \leq k G_b(T^{n-1}p, T^n p, T^n p) \leq \cdots \leq k^n G_b(p, Tp, Tp).$$

Case (ii)Suppose

$$\max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p), G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)\Big] = G_b(T^n p, T^{n+1}p, T^{n+1}p).$$

Then we get,

$$G_b(T^n p, T^{n+1}p, T^{n+1}p) \leq k G_b(T^n p, T^{n+1}p, T^{n+1}p),$$

which is contradiction, since $k < 1$.

Case (iii)Suppose

$$\max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p), G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)\Big] = G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p).$$

Then we get,

$$G_b(T^n p, T^{n+1}p, T^{n+1}p) \leq k G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p). \tag{14}$$

By property (v) of G_b-metric space, we have

$$G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p) \leq s\Big[G_b(T^{n-1}p, T^n p, T^n p) + G_b(T^n p, T^{n+1}p, T^{n+1}p)\Big]. \tag{15}$$

Using inequality (15) in (14), we get

$$G_b(T^n p, T^{n+1}p, T^{n+1}p) \leq sk\Big[G_b(T^{n-1}p, T^n p, T^n p) + G_b(T^n p, T^{n+1}p, T^{n+1}p)\Big].$$

It gives that

$$\begin{aligned}
G_b(T^n p, T^{n+1}p, T^{n+1}p) &\leq \frac{sk}{1 - sk} G_b(T^{n-1}p, T^n p, T^n p) \\
&= \lambda G_b(T^{n-1}p, T^n p, T^n p) \leq \cdots \leq \lambda^n G_b(p, Tp, Tp),
\end{aligned}$$

where $\lambda = \frac{sk}{1-sk}$. Since $k, \lambda < 1$, so as $n \to \infty$, we get $G_b(p, Tp, Tp) = 0$ and hence in all cases $p = Tp$ i.e. $p \in F(T)$. Hence T has property P.

Theorem 6 Under the contraction of theorem 3, T has property P.

Proof. By theorem 3, T has a fixed point. Therefore $F(T^n) \neq \emptyset$, each $n \in \mathbb{N}$. Fix $n > 1$ and assume that $p \in F(T^n)$. To show that $p \in F(T)$. Suppose that $P \neq Tp$.

$$\begin{aligned}
G_b(p, Tp, Tp) &= G_b(T^n p, T^{n+1} p, T^{n+1} p) \\
&\leq \Big[\alpha G_b(T^{n-1} p, T^n p, T^n p) + \beta G_b(T^{n-1} p, T^n p, T^n p) \\
&\quad + \gamma G_b(T^n p, T^{n+1} p, T^{n+1} p) + \delta G_b(T^n p, T^{n+1} p, T^{n+1} p)\Big] \\
&\leq (\alpha + \beta) G_b(T^{n-1} p, T^n p, T^n p) + (\gamma + \delta) G_b(T^n p, T^{n+1} p, T^{n+1} p).
\end{aligned}$$

It gives that

$$\begin{aligned}
G_b(p, Tp, Tp) &\leq \frac{(\alpha + \beta)}{1 - (\gamma + \delta)} G_b(T^{n-1} p, T^n p, T^n p) \\
&= \lambda G_b(T^{n-1} p, T^n p, T^n p) \leq \cdots \leq \lambda^n G_b(p, Tp, Tp),
\end{aligned}$$

where $\lambda = \frac{(\alpha+\beta)}{1-(\gamma+\delta)}$. Since $\lambda < 1$, so as $n \to \infty$, we get $G_b(p, Tp, Tp) = 0$ and hence $p = Tp$ i.e. $p \in F(T)$. Hence T has property P.

Example 3 Let us define $G_b(x, y, z) = |x-y| + |y-z| + |x-z|$ and let $x \in X$. Then (X, G_b) be a complete G_b-metric space. Let $T(x) = \frac{x}{3}$. Without loss of generality, we assume $x > y > z$. Then

(i)

$$\begin{aligned}
&G_b(T(x), T(y), T(z)) = |\frac{x}{3} - \frac{y}{3}| + |\frac{y}{3} - \frac{z}{3}| + |\frac{x}{3} - \frac{z}{3}| \\
&= \frac{1}{3}[|x-y| + |y-z| + |x-z|] \\
&\leq k \max \Big[G_b(x, y, z), G_b(x, T(x), T(x)), G_b(y, T(y), T(y)), G_b(z, T(z), T(z)), \\
&\quad \frac{[G_b(x, T(y), T(y)) + G_b(z, T(x), T(x))]}{2}, \frac{[G_b(x, T(y), T(y)) + G_b(y, T(x), T(x))]}{2}, \\
&\quad \frac{[G_b(y, T(z), T(z)) + G_b(z, T(y), T(y))]}{2}, \frac{[G_b(x, T(z), T(z)) + G_b(z, T(x), T(x))]}{2}\Big].
\end{aligned}$$

(ii)

$$\begin{aligned}
&G_b(T(x), T(y), T(z)) = |\frac{x}{3} - \frac{y}{3}| + |\frac{y}{3} - \frac{z}{3}| + |\frac{x}{3} - \frac{z}{3}| \\
&= \frac{1}{3}[|x-y| + |y-z| + |x-z|] \\
&\leq k \max \Big[G_b(x, y, z), G_b(x, T(x), T(x)), G_b(y, T(y), T(y)), G_b(x, T(y), T(y)), \\
&\quad G_b(y, T(x), T(x)), G_b(z, T(z), T(z)), \Big].
\end{aligned}$$

(iii)

$$
\begin{aligned}
&G_b(T(x),T(y),T(z)) = |\frac{x}{3}-\frac{y}{3}| + |\frac{y}{3}-\frac{z}{3}| + |\frac{x}{3}-\frac{z}{3}| \\
&= \frac{1}{3}[|x-y| + |y-z| + |x-z|] \\
&= \frac{1}{9}[|x-y| + |y-z| + |x-z|] + \frac{2}{9}[|x-y| + |y-z| + |x-z|] \\
&\leq \frac{1}{9}\Big[|x-y| + |y-z| + |x-z|\Big] + \frac{2}{9}\Big[|x| + |y| + 3|z|\Big] \\
&\leq \frac{1}{9}\Big[|x-y| + |y-z| + |x-z|\Big] + \frac{1}{3}\Big[|\frac{2x}{3}| + |\frac{2y}{3}|] + \frac{1}{3}|6z|\Big] \\
&\leq \frac{1}{9}\Big[|x-y| + |y-z| + |x-z|\Big] + \frac{1}{3}\Big[|x-\frac{x}{3}| + |y-\frac{y}{3}|\Big] + \frac{1}{3}\Big[9|z-\frac{z}{3}\Big] \\
&\leq \frac{1}{9}\Big[|x-y| + |y-z| + |x-z|\Big] + \frac{1}{6}\Big[2|x-\frac{x}{3}|\Big] + \frac{1}{6}\Big[2|y-\frac{y}{3}|\Big] + \frac{1}{2}\Big[2|z-\frac{z}{3}\Big] \\
&\leq \Big[\alpha G_b(x,y,z) + \beta G_b(x,T(x),T(x)) + \gamma G_b(y,T(y),T(y)) + \delta G_b(z,T(z),T(z))\Big],
\end{aligned}
$$

where $\alpha = \frac{1}{9}, \beta = \frac{1}{6}, \gamma = \frac{1}{6}, \delta = \frac{1}{2}$ and $\alpha + \beta + \gamma + \delta = \frac{17}{18} < 1$.

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