FIXED POINT THEOREMS SATISFY PROPERTY $P$ IN $G_b$-METRIC SPACES

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Abstract. In this paper, we prove fixed point theorem for a contractive mapping which satisfy property $P$ in $G_b$-metric space. Our results are supported by an example.

1. Introduction

The metric space is generalized by different authors by various ways. Czerwik [9] introduced $b$-metric space. Zead Mustafa and Brailey Sims [13] coined the concept of $G$-metric space. A. Aghajani, M. Abbas and J. R. Roshan [2] extended the $G$-metric space with $b$-metric space and develop the new structure of metric space, which we call $G_b$-metric space. They proved the fixed point theorems in $G_b$-metric spaces. A self map $T$ of the space $X$ with a nonempty fixed point set $F(T)$. Then we say that $T$ has a property $P$ if $F(T^n) = F(T)$ for each $n \in N$. We design the fixed point theorems for the self maps which satisfy property $P$ for various contractions in $G_b$-metric spaces [10]-[3].

Definition 1 ([9]) Let $X$ be a non empty set and the mapping $d : X \times X \to [0, \infty)$. The mapping $d$ satisfies

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$ ;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ ;
(iii) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. Then $d$ is called a $b$-metric on $X$. The ordered pair $(X, d)$ is called $b$-metric space with coefficient $s$.

$G$-metric space is defined as follows

Definition 2 ([13]) Let $X$ be a non empty set and the mapping $G : X \times X \times X \to [0, \infty)$. The mapping $G$ satisfies

(i) $G(x, y, z) = 0$ if and only if $x = y = z$ for all $x, y, z \in X$ ;
(ii) $0 < G(x, x, y)$ for all $x, y \in X$ ;
(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$ ;
(iv) $G(x, x, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

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(v) \( G(x, y, z) \leq G(x, a, a) + G(a, y, z) \) for all \( x, y, z, a \in X \) (rectangle inequality). Then \( G \) is called a \( G \)-metric on \( X \) and the ordered pair \((X, G)\) is called \( G \)-metric space.

A. Aghajani, M. Abbas and J. R. Roshan [2] introduced \( G_b \)-metric space as follows

**Definition 3** ([2]) Let \( X \) be a nonempty set and \( s \geq 1 \) be a given real number. Suppose that a mapping \( G_b : X \times X \times X \to R^+ \) satisfies:

(i) \( G_b(x, y, z) = 0 \) if \( x = y = z \) for all \( x, y, z \in X \);
(ii) \( 0 < G_b(x, x, y) \) for all \( x, y, z \in X \) with \( x \neq y \);
(iii) \( G_b(x, x, y) \leq G_b(x, y, z) \) for all \( x, y, z \in X \) with \( y \neq z \);
(iv) \( G_b(x, y, z) = G_b(px, z, y) \), where \( p \) is a permutation of \( x, y, z \) (symmetry);
(v) \( G_b(x, x, z) \leq s(G_b(x, a, a) + G_b(a, y, z)) \).

Then \( G_b \) is called a generalized \( b \)-metric or \( G_b \)-metric on \( X \). The ordered pair \((X, G_b)\) is called generalized \( b \) metric or \( G_b \)-metric space.

Following example shows that a \( G_b \)-metric on \( X \) need not be a \( G \)-metric on \( X \).

**Example 1** ([2]) Let \((X, G)\) be a \( G \)-metric space and \( G_*(x, y, z) = G(x, y, z)^p \); where \( p > 1 \) is a real number. Note that \( G_* \) is a \( G_b \)-metric with \( s = 2^{p-1} \). Obviously, \( G_* \) satisfies conditions (i) to (iv) of the \( G_b \) metric space. Now it suffices to show that condition (v) of \( G_b \) metric space to be hold. If \( 1 < p < \infty \), then the convexity of the function \( f(x) = x^p(x > 0) \) implies that \((a + b)^p \leq 2^{p-1}(a^p + b^p)\). Thus for each \( x, y, z, a \in X \) we obtain

\[
G_*(x, y, z) = G(x, y, z)^p \leq (G(x, a, a) + G(a, y, z))^p \\
\leq 2^{p-1}(G(x, a, a)^p + G(a, y, z)^p) \\
= 2^{p-1}(G_*(x, a, a) + G_*(a, y, z)).
\]

So \( G_* \) is a \( G_b \)-metric with \( s = 2^{p-1} \).

Also in the above example, \((X, G_*)\) is not necessarily a \( G \)-metric space.

**Example 2** Let \( X = R \) and let

\[
G_b(x, y, z) = \max\{|x - y|^2, |y - z|^2, |x - z|^2\}.
\]

Then \((X, G_b)\) is a \( G_b \)-metric space with the coefficient \( s = 2 \).

**Definition 4** ([2]) Let \( X \) be a \( G_b \)-metric space. A sequence \( \{x_n\} \) in \( X \) is said to be:

(i) \( G_b \)-Cauchy sequence if, for each \( \epsilon > 0 \), there exists a positive integer \( n_0 \in \mathbb{N} \) such that, for all \( m, n, l \geq n_0 \), \( G_b(x_n, x_m, x_l) < \epsilon \);
(ii) \( G_b \)-convergent to a point \( x \in X \) if, for each \( \epsilon > 0 \), there exists a positive integer \( n_0 \in \mathbb{N} \) such that, for all \( m \geq n_0 \), \( G_b(x_n, x_m, x) < \epsilon \).

**Proposition 1** ([2]) Let \((X, G_b)\) be a \( G_b \)-metric space. Then the following are equivalent:

(i) \( \{x_n\} \) is \( G_b \)-converges to \( x \).
(ii) \( G_b(x_n, x, x) \to 0 \), as \( n \to \infty \).
(iii) \( G_b(x_n, x, x) \to 0 \), as \( n \to \infty \).

**Proposition 2** ([2]) Let \((X, G_b)\) be a \( G_b \)-metric space. Then the following are equivalent:

(i) The sequence \( \{x_n\} \) is \( G_b \)-Cauchy.
(ii) For every \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( G_b(x_n, x_m, x_l) < \epsilon \), for all \( n, m, l \geq n_0 \).
Let $p$ say $(\text{Theorem 1} v)$ By property $(\text{Definition 5} EJMAA-2019/7(2) FIXED POINT THEOREMS 183)$

Case (i) Suppose

There are three cases:

for all $x, y, z$\n
sequence is $G_b$-convergent in $X$.

2. Main results

Our first main result is

**Theorem 1** Let $(X, G_b)$ be a complete $G_b$-metric space with $s \geq 1$ and let $T : X \to X$ be a mapping satisfying

$$G_b(Tx, Ty, Tz) \leq k \max \left\{G_b(x, y, z), \frac{G_b(x, Tx, Tx) + G_b(y, Ty, Ty) + G_b(z, Tz, Tz)}{2}, \frac{G_b(x, Tz, Tz) + G_b(z, Ty, Ty)}{2}, \frac{G_b(y, Ty, Ty)}{2} \right\}$$

(1)

for all $x, y, z \in X$, where $k$ is such that $sk \in [0, 1)$. Then $T$ has a unique fixed point say $p$ ($Tp = p$) in $X$ and $T$ is $G_b$-continuous at $p$.

**Proof.** Let $x_0 \in X$. Since $T : X \to X$ be a self map, then we get a sequence $(x_n)$ in $X$ such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$G_b(x_n, x_n+1, x_{n+1}) = G_b(Tx_{n-1}, Tx_n, Tx_n)$$

$$\leq k \max \left\{G_b(x_{n-1}, x_n, x_n), G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_n, x_n+1, x_{n+1}) \right\}$$

$$\leq k \max \left\{ \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1}) + G_b(x_n, x_n, x_n)}{2}, \frac{G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_n, x_n, x_n)}{2}, \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2} \right\}$$

There are three cases:

Case (i) Suppose

$$\max \left\{G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_n+1, x_{n+1}), \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2} \right\} = \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2}.$$ 

Then

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq k \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2}.$$ 

By property (v) of $G_b$ metric space, we have

$$G_b(x_{n-1}, x_{n+1}, x_{n+1}) \leq s \{G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})\}.$$
Then, we get
\[
G_b(x_n, x_{n+1}, x_{n+1}) \leq sk\left[\frac{G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})}{2}\right] \\
\leq \frac{sk}{(2-sk)}G_b(x_{n-1}, x_n) \\
= \lambda G_b(x_{n-1}, x, x_n),
\]
where \(\lambda = \frac{sk}{(2-sk)}\). Therefore by continuing in this way, we get
\[
G_b(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G_b(x_0, x_1).
\]
As \(n \to \infty\), \(G_b(x_n, x_{n+1}, x_{n+1}) \to 0\), since \(\lambda < 1\). Moreover for all \(n, m \in \mathbb{N}, n < m\), and by (v) the property of \(G_b\)-metric space.

\[
G_b(x_n, x_m, x_m) \leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, x_m, x_m)] \\
\leq s[\lambda^nG_b(x_0, x_1, x_1) + G_b(x_{n+1}, x_m, x_m)] \\
\leq s\lambda^nG_b(x_0, x_1, x_1) + s^2[G_b(x_{n+1}, x_{n+2}, x_{n+2}) + G_b(x_{n+2}, x_m, x_m)] \\
\leq s\lambda^nG_b(x_0, x_1, x_1) + s^2\lambda^{n+1}G_b(x_0, x_1, x_1) + s^3[G_b(x_{n+2}, x_{n+3}, x_{n+3}) \\
+ G_b(x_{n+3}, x_m, x_m)] \\
= s\lambda^nG_b(x_0, x_1, x_1) + s^2\lambda^{n+1}G_b(x_0, x_1, x_1) + s^3\lambda^{n+2}G_b(x_0, x_1, x_1) + \\
s^3G_b(x_{n+3}, x_m, x_m) \\
\leq s\lambda^nG_b(x_0, x_1, x_1) + s^2\lambda^{n+1}G_b(x_0, x_1, x_1) + s^3\lambda^{n+2}G_b(x_0, x_1, x_1) \\
+ \cdots + s^{m-1}\lambda^{n+m-2}G_b(x_0, x_1, x_1) + s^{m-1}\lambda^{n+m-1}G_b(x_0, x_1, x_1) \\
= s\lambda^n[(1 + s\lambda + (s\lambda)^2 + (s\lambda)^3 + \cdots + (s\lambda)^{m-2}) + (s\lambda)^{m-2}\lambda] \\
G_b(x_0, x_1, x_1) \\
= s\lambda^n\left[1 - \frac{(s\lambda)^{n-(m-2)}}{(1 - s\lambda)} + (s\lambda)^{m-2}\lambda\right]G_b(x_0, x_1, x_1).
\]

Letting \(m, n \to \infty\), we get
\[
\lim_{n,m \to \infty} G_b(x_n, x_m, x_m) = 0.
\]

Case (ii) Suppose
\[
\max \left[ G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2} \right] \\
= G_b(x_n, x_{n+1}, x_{n+1}).
\]

Then
\[
G_b(x_n, x_{n+1}, x_{n+1}) \leq kG_b(x_n, x_{n+1}, x_{n+1}),
\]
which is contradiction, since \(k < 1\).

Case (iii) Suppose
\[
\max \left[ G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2} \right] \\
= G_b(x_{n-1}, x_n, x_n).
\]
Then
\[ G_b(x_n, x_{n+1}, x_{n+1}) \leq kG_b(x_{n-1}, x_n, x_n) \]
\[ \leq k^2G_b(x_{n-2}, x_{n-1}, x_{n-1}) \leq \cdots \leq k^nG_b(x_0, x_1, x_1). \]

As \( n \to \infty, G_b(x_n, x_{n+1}, x_{n+1}) \to 0 \), since \( k < 1 \). Also Since \( sk < 1 \) and by case(i) \( \{x_n\} \) is a \( G_b \)-Cauchy sequence in \( X \). Since \( X \) is \( G_b \)-complete, then there exists \( p \in X \) such that \( \{x_n\} \) is \( G_b \)-converges to \( p \in X \). Now we claim that \( p \) is fixed point of \( T \). Suppose that \( Tp \neq p \).
\[
G_b(x_n, Tp, Tp) \leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tp, Tp)]
\[
\leq s\lambda^nG_b(x_0, x_1, x_1) + sk \max \left\{ G_b(x_n, p, p), G_b(x_n, x_{n+1}, x_{n+1}),
G_b(p, Tp, Tp), \frac{\left\{ G_b(x_n, Tp, Tp) + G_b(Tp, x_{n+1}, x_{n+1}) \right\}}{2},
\frac{\left\{ G_b(x_n, Tp, Tp) + G_b(Tp, x_{n+1}, x_{n+1}) \right\}}{2} \right\}.
\]

As \( n \to \infty, \{x_n\} \to p \) and above inequality turns into
\[
G_b(p, Tp, Tp) \leq skG_b(p, Tp, Tp).
\]
It is contradiction, since \( sk < 1 \). Thus \( Tp = p \). Therefore \( p \) is a fixed point of \( T \).

For uniqueness suppose \( q \neq p \) and \( q \) is another fixed point of \( T \), i.e. \( Tq = q \). By (v) the property of \( G_b \)-metric space
\[
G_b(x_n, Tq, Tq) \leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tq, Tq)].
\]

Therefore
\[
G_b(x_n, q, q) \leq s\lambda^nG_b(x_0, x_1, x_1) + sk \max \left\{ G_b(x_n, q, q), G_b(x_n, x_{n+1}, x_{n+1}),
G_b(q, Tq, Tq), \frac{\left\{ G_b(x_n, Tq, Tq) + G_b(q, x_{n+1}, x_{n+1}) \right\}}{2},
\frac{\left\{ G_b(x_n, Tq, Tq) + G_b(q, x_{n+1}, x_{n+1}) \right\}}{2} \right\}.
\]

As \( \lambda < 1 \), we extend \( n \to \infty \), so \( \{x_n\} \to p \). Thus we get
\[
G_b(p, q, q) \leq sk \max \left[ \frac{[G_b(p, q, q) + G_b(q, p, p)]}{2}, G_b(p, q, q) \right].
\]

There are two cases

Case (i) Suppose
\[
\max \left[ \frac{[G_b(p, q, q) + G_b(q, p, p)]}{2}, G_b(p, q, q) \right] = G_b(p, q, q).
\]
Then
\[
G_b(p, q, q) \leq skG_b(p, q, q),
\]
which is contradiction, since \( sk < 1 \).
Case (ii) Suppose

\[
\max \left[ \frac{G_b(p, q, q) + G_b(q, p, p)}{2}, G_b(p, q, q) \right] = \frac{G_b(p, q, q) + G_b(q, p, p)}{2}. 
\]

Then

\[
G_b(p, q, q) \leq sk \left[ \frac{G_b(p, q, q) + G_b(q, p, p)}{2} \right].
\]

It implies that

\[
G_b(p, q, q) \leq \frac{sk}{2 - sk} G_b(q, p, p). \tag{2}
\]

Also consider,

\[
G_b(Tq, x_n, x_n) \leq k \max \left[ G_b(q, x_{n-1}, x_{n-1}), G_b(q, Tq, Tq), G_b(x_{n-1}, x_n, x_n), G_b(x_{n-1}, x_n, x_n), \frac{[G_b(q, x_n, x_n) + G_b(x_{n-1}, Tq, Tq)]}{2}, \frac{[G_b(x_{n-1}, x_n, x_n) + G_b(x_{n-1}, x_n, x_n)]}{2}, \frac{[G_b(q, x_n, x_n) + G_b(x_n, q, q)]}{2} \right].
\]

As \( n \to \infty \), we get

\[
G_b(q, p, p) \leq k \max \left[ G_b(q, p, p), \frac{G_b(q, p, p) + G_b(p, q, q)}{2} \right].
\]

There are two cases:

Case (i) Suppose

\[
\max \left[ G_b(q, p, p), \frac{G_b(q, p, p) + G_b(p, q, q)}{2} \right] = G_b(q, p, p).
\]

Then, we get

\[
G_b(q, p, p) \leq k G_b(q, p, p),
\]

which is contradiction, since \( k < 1 \).

Case (ii) Suppose

\[
\max \left[ G_b(q, p, p), \frac{G_b(q, p, p) + G_b(p, q, q)}{2} \right] = \frac{G_b(q, p, p) + G_b(p, q, q)}{2}.
\]

Then

\[
G_b(q, p, p) \leq k \left[ G_b(q, p, p) + \frac{G_b(p, q, q)}{2} \right].
\]

It implies that

\[
G_b(q, p, p) \leq \left( \frac{k}{2 - k} \right) G_b(p, q, q). \tag{3}
\]

Therefore from (2) and (3), we get

\[
G_b(p, q, q) \leq \left( \frac{sk}{2 - sk} \right) \left( \frac{k}{2 - k} \right) G_b(p, q, q). \tag{4}
\]
Then by property (2.4) is possible only when \( G_b(p, q, q) = 0 \). Thus \( p = q \). Now we claim that \( T \) is \( G_b \)-continuous at \( p \). Let \( (y_n) \) be a sequence in \( X \) such that

\[
\lim_{n \to \infty} y_n = p.
\]

Consider

\[
G_b(p, Ty_n, Ty_n) \leq k \max \left[ G_b(p, y_n, y_n), G_b(p, p, p), G_b(y_n, Ty_n, Ty_n), G_b(y_n, Ty_n, Ty_n), \frac{[G_b(p, Ty_n, Ty_n) + G_b(y_n, p, p)]}{2}, \frac{[G_b(y_n, Ty_n, Ty_n) + G_b(y_n, Ty_n, Ty_n)]}{2}, \frac{G_b(p, Ty_n, Ty_n) + G_b(y_n, p, p)}{2} \right].
\]

Letting \( n \to \infty \), we get

\[
G_b(p, Ty_n, Ty_n) \leq k \max \left[ G_b(p, Ty_n, Ty_n), \frac{G_b(p, Ty_n, Ty_n)}{2} \right] = kG_b(p, Ty_n, Ty_n).
\]

Since \( k < 1 \), it is possible only when \( G_b(p, Ty_n, Ty_n) = 0 \). Therefore \( T \) is \( G_b \)-continuous at \( p \).

**Theorem 2** Let \( (X, G_b) \) be a complete \( G_b \)-metric space with \( s \geq 1 \) and let \( T : X \to X \) be a mapping satisfying

\[
G_b(Tx, Ty, Tz) \leq k \max \left[ G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Ty), G_b(x, Ty, Ty), G_b(y, Tx, Tx), G_b(z, Tz, Tz) \right];
\]

(5)

for all \( x, y, z \in X \) and where \( k \) is such that \( sk \in [0, \frac{1}{s}] \). Then \( T \) has a unique fixed point say \( p \) in \( X \) (i.e. \( Tp = p \)) and \( T \) is \( G_b \)-continuous at \( p \).

**Proof.** Let \( x_0 \in X \) and \( T : X \to X \) be a self map. Then, we get a sequence \( \{x_n\} \) in \( X \) such that \( x_n = Tx_{n-1} = T^n x_0 \). Consider

\[
G_b(x_n, x_{n+1}, x_{n+1}) = G_b(Tx_{n-1}, Tx_n, Tx_n)
\]

\[
\leq k \max \left[ G_b(x_{n-1}, x_n, x_n), G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_{n-1}, x_{n+1}, x_{n+1}), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_{n-1}, x_{n+1}, x_{n+1}) \right]
\]

\[
= k \max \left[ G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}) \right].
\]

There are three cases:

Case (i) Suppose

\[
\max \left[ G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_{n-1}, x_{n+1}, x_{n+1}) \right]
\]

\[
= G_b(x_{n-1}, x_{n+1}, x_{n+1}).
\]

Then, we get

\[
G_b(x_n, x_{n+1}, x_{n+1}) \leq kG_b(x_{n-1}, x_{n+1}, x_{n+1}).
\]

Then by property \( (v) \) of \( G_b \) metric space, we have

\[
G_b(x_{n-1}, x_{n+1}, x_{n+1}) \leq s \{ G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1}) \}.
\]
Thus
\[ G_b(x_n, x_{n+1}, x_{n+1}) \leq sk \left[ G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1}) \right]. \]

It gives that
\[ G_b(x_n, x_{n+1}, x_{n+1}) \leq \frac{sk}{1-sk} G_b(x_{n-1}, x_n, x_n) = \lambda G_b(x_{n-1}, x_n, x_n). \]

where \( \lambda = \frac{sk}{1-sk} \). Therefore by continuing in this way, we get
\[ G_b(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G_b(x_0, x_1, x_1). \]

Since \( k < 1 \), letting \( n \to \infty \), we have \( n \to \infty, G_b(x_n, x_{n+1}, x_{n+1}) \to 0 \). Moreover for all \( n, m \in \mathbb{N}, n < m \), since \( k < \lambda < 1 \) and by (v) the property of \( G_b \) metric space, we have
\[ G_b(x_n, x_m, x_m) \leq s \left[ G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, x_m, x_m) \right] \]
\[ \leq s \left[ \lambda^n G_b(x_0, x_1, x_1) + G_b(x_{n+1}, x_m, x_m) \right] \]
\[ \leq s\lambda^n G_b(x_0, x_1, x_1) + s^2 \left[ G_b(x_{n+1}, x_{n+2}, x_{n+2}) + G_b(x_{n+2}, x_m, x_m) \right] \]
\[ \leq s\lambda^n G_b(x_0, x_1, x_1) + s^2 \lambda^{n+1} G_b(x_0, x_1, x_1) \]
\[ + s^3 \left[ G_b(x_{n+2}, x_{n+3}, x_{n+3}) + G_b(x_{n+3}, x_m, x_m) \right] \]
\[ \leq s\lambda^n G_b(x_0, x_1, x_1) + s^2 \lambda^{n+1} G_b(x_0, x_1, x_1) + s^3 \lambda^{n+2} G_b(x_0, x_1, x_1) \]
\[ + \cdots + s^{m-1} \lambda^{n+m-2} G_b(x_0, x_1, x_1) + s^m \lambda^{n+m-1} G_b(x_0, x_1, x_1) \]
\[ = s\lambda^n \left[ 1 + s\lambda + (s\lambda)^2 + (s\lambda)^3 + \cdots + (s\lambda)^{m-2} \right] + (s\lambda)^{m-2} \lambda \]
\[ G_b(x_0, x_1, x_1) \]
\[ = s\lambda^n \left[ \frac{1 - (s\lambda)^{n-(m-2)}}{1 - s\lambda} \right] + (s\lambda)^{m-2} \lambda \]
\[ G_b(x_0, x_1, x_1). \]

Letting \( m, n \to \infty \), we get \( \lim_{m,n \to \infty} G_b(x_n, x_m, x_m) = 0 \), since \( sk < 1 \). This shows that \( \{x_n\} \) is a \( G_b \)-Cauchy sequence in \( X \).

Case (ii) Suppose
\[ \max \left[ G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_{n-1}, x_{n+1}, x_{n+1}) \right] \]
\[ = G_b(x_n, x_{n+1}, x_{n+1}). \]

Then
\[ G_b(x_n, x_{n+1}, x_{n+1}) \leq kG_b(x_{n+1}, x_{n+1}), \]

which is a contradiction, since \( k < 1 \).

Case (iii) Suppose
\[ \max \left[ G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_{n-1}, x_{n+1}, x_{n+1}) \right] \]
\[ = G_b(x_{n-1}, x_n, x_n). \]
Then
\[ G_b(x_n, x_{n+1}, x_{n+1}) \leq k G_b(x_{n-1}, x_n, x_n) \]
\[ \leq k^2 G_b(x_{n-2}, x_{n-1}, x_{n-1}) \leq \cdots \leq k^n G_b(x_0, x_1, x_1). \]

Since \( k < 1 \), as \( n \to \infty \), we have \( G_b(x_n, x_{n+1}, x_{n+1}) \to 0 \). Thus in this case also \( \{x_n\} \) is a \( G_b \)-Cauchy sequence in \( X \). Since \( X \) is \( G_b \)-complete, then there exists \( p \in X \) such that \( \{x_n\} \to p \). Now, we claim that \( p \) is fixed point of \( T \). Suppose that \( Tp \neq p \), by (v) the property of \( G_b \)-metric space and by (2.5), we get
\[ G_b(x_n, Tp, Tp) \leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_n, Tp, Tp)] \]
\[ \leq s \lambda^n G_b(x_0, x_1, x_1) + sk \max\left[G_b(x_n, p, p), G_b(x_n, x_{n+1}, x_{n+1})\right] + G_b(p, Tp, Tp), G_b(x_n, Tp, Tp), G_b(p, x_{n+1}, x_{n+1}), G_b(p, Tp, Tp) \].

Letting \( n \to \infty \), we have \( \{x_n\} \to p \). Then
\[ G_b(p, Tp, Tp) \leq sk G_b(p, Tp, Tp). \]

Since \( sk < 1 \), the above inequality is true only if \( G_b(p, Tp, Tp) = 0 \). Thus \( p = Tp \). Therefore \( p \) is a fixed point of \( T \). For uniqueness, suppose \( q \neq p \) and \( q \) is another fixed point of \( T \) i.e. \( Tq = q \). Consider
\[ G_b(x_n, Tq, Tq) \leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_n, Tq, Tq)]. \]

It gives that
\[ G_b(x_n, q, q) \leq s \lambda^n G_b(x_0, x_1, x_1) + sk \max\left[G_b(x_n, q, q), G_b(x_n, x_{n+1}, x_{n+1})\right] + G_b(q, Tq, Tq), G_b(x_n, Tq, Tq), G_b(q, x_{n+1}, x_{n+1}), G_b(q, Tq, Tq) \].

Letting \( n \to \infty \), we have \( \{x_n\} \to p \) with \( Tq = q \). Then, we get
\[ G_b(p, q, q) \leq sk \max\left[G_b(p, q, q), G_b(q, p, p)\right]. \]

There are two cases:
Case (a) Suppose \( \max\left[G_b(p, q, q), G_b(q, p, p)\right] = G_b(p, q, q) \). Then
\[ G_b(p, q, q) \leq sk G_b(p, q, q), \]
which is contradiction, since \( sk < 1 \).

Case (b) Suppose \( \max\left[G_b(p, q, q), G_b(q, p, p)\right] = G_b(q, p, p) \). Then
\[ G_b(p, q, q) \leq sk G_b(q, p, p). \]

Now, consider
\[ G_b(Tq, x_n, x_n) \leq k \max\left[G_b(q, x_{n-1}, x_{n-1}), G_b(q, Tq, Tq), G_b(x_{n-1}, x_n), G_b(q, x_n, x_n), G_b(x_{n-1}, Tq, Tq), G_b(x_{n-1}, x_n, x_n)\right]. \]

Letting \( n \to \infty \), it implies that
\[ G_b(q, p, p) \leq k \max\left[G_b(q, p, p), G_b(p, q, q)\right]. \]

There are two cases:
Case (c) Suppose \( \max \left[ G_b(q, p, p), G_b(p, q, q) \right] = G_b(q, p, p) \). Then
\[
G_b(q, p, p) \leq kG_b(q, p, p),
\]
which is contradiction, since \( k < 1 \).

Case (d) Suppose \( \max \left[ G_b(q, p, p), G_b(p, q, q) \right] = G_b(p, q, q) \). Then
\[
G_b(q, p, p) \leq kG_b(p, q, q).
\]
Using inequality (7) in (6), we have
\[
G_b(p, q, q) \leq sk^2G_b(p, q, q).
\]
Since \( sk < 1 \). Thus (8) is true only if \( G_b(p, q, q) = 0 \). Thus \( p = q \). Therefore \( p \) is a fixed point of \( T \) in \( X \).

To show that \( T \) is \( G_b \)-continuous at \( p \), let \( \{y_n\} \) be a sequence in \( X \) such that \( \lim_{n \to \infty} y_n = p \). Consider
\[
G_b(p, Ty_n, Ty_n) \leq k \max \left[ G_b(p, y_n, y_n), G_b(p, p, p), G_b(y_n, Ty_n, Ty_n), G_b(p, Ty_n, Ty_n), G_b(y_n, Ty_n, Ty_n), G_b(p, Ty_n, Ty_n) \right].
\]
Letting \( n \to \infty \), we get
\[
G_b(p, Ty_n, Ty_n) \leq \max \left[ G_b(p, p, p), G_b(p, p, p), G_b(p, Ty_n, Ty_n), G_b(p, Ty_n, Ty_n), G_b(p, Ty_n, Ty_n) \right].
\]
Thus
\[
G_b(p, Ty_n, Ty_n) \leq k \max G_b(p, Ty_n, Ty_n).
\]
It is possible only if \( G_b(p, Ty_n, Ty_n) = 0 \). Thus \( Ty_n = p = Tp \). It is proved that \( T \) is \( G_b \)-continuous at \( p \).

**Theorem 3** Let \( (X, G_b) \) be a complete \( G_b \)-metric space with \( s \geq 1 \) and let \( T : X \to X \) be a mapping satisfying
\[
G_b(Tx, Ty, Tz) \leq \alpha G_b(x, y, z) + \beta G_b(x, Tx, Tx) + \gamma G_b(y, Ty, Ty) + \delta G_b(z, Tz, Tz)
\]
for all \( x, y, z \in X \) and where \( \alpha + \beta + \gamma + \delta < 1 \). Then \( T \) has a unique fixed point say \( p \) (i.e. \( Tp = p \)) and \( T \) is \( G_b \)-continuous at \( p \).

**Proof.** Let \( x_0 \in X \) and the mapping \( T : X \to X \) then we get a sequence \( \{x_n\} \) in \( X \) such that \( x_n = Tx_{n-1} = T^n x_0 \). Consider
\[
G_b(x_n, x_{n+1}, x_{n+1}) = G_b(Tx_{n-1}, Tx_n, Tx_n)
\]
\[
\leq \alpha G_b(x_{n-1}, x_n, x_n) + \beta G_b(x_{n-1}, x_n, x_{n+1}) + \gamma G_b(x_n, x_{n+1}, x_{n+1}) + \delta G_b(x_n, x_{n+1}, x_{n+1})
\]
\[
\leq (\alpha + \beta) G_b(x_{n-1}, x_n, x_n) + (\gamma + \delta) G_b(x_n, x_{n+1}, x_{n+1})
\]
\[
\leq \frac{\alpha + \beta}{1 - (\gamma + \delta)} G_b(x_{n-1}, x_n, x_n)
\]
\[
\leq \lambda G_b(x_{n-1}, x_n, x_n),
\]
where \( \lambda = \frac{\alpha + \beta}{1 - (\gamma + \delta)} \). Therefore, continuing in this way, we get
\[
G_b(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G_b(x_0, x_1, x_1).
\]
Moreover for all \( n, m \in \mathbb{N}, n < m \) and by (v) th property of \( G_b \) metric space, we have

\[
G_b(x_n, x_m, x_m) \leq s \left[ G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, x_m, x_m) \right]
\leq s \left[ \lambda^n G_b(x_0, x_1, x_1) + G_b(x_{n+1}, x_m, x_m) \right]
\leq s \lambda^n G_b(x_0, x_1, x_1) + s^2 \left[ G_b(x_{n+1}, x_{n+2}, x_{n+2}) + G_b(x_{n+2}, x_m, x_m) \right]
\leq s \lambda^n G_b(x_0, x_1, x_1) + s^2 \lambda^{n+1} G_b(x_0, x_1, x_1)
\quad + s^3 \left[ G_b(x_{n+2}, x_{n+3}, x_{n+3}) + G_b(x_{n+3}, x_m, x_m) \right]
\leq s \lambda^n G_b(x_0, x_1, x_1) + s^2 \lambda^{n+1} G_b(x_0, x_1, x_1) + s^3 \lambda^{n+2} G_b(x_0, x_1, x_1)
\quad + \ldots + s^{m-1} \lambda^{n+m-2} G_b(x_0, x_1, x_1) + s^{m-1} \lambda^{n+m-1} G_b(x_0, x_1, x_1)
= s \lambda^n \left[ 1 + \lambda + (s \lambda)^2 + (s \lambda)^3 + \ldots + (s \lambda)^{m-2} \right] + (s \lambda)^{m-2} \lambda
\]

\[
G_b(x_0, x_1, x_1) = s \lambda^n \left[ \frac{1 - (s \lambda)^{n-(m-2)}}{1 - s \lambda} \right] + (s \lambda)^{m-2} \lambda \]

Letting \( m, n \to \infty \), we have \( \lim_{m, n \to \infty} G_b(x_n, x_m, x_m) = 0 \). Hence \( \{x_n\} \) is a \( G_b \)-Cauchy sequence in \( X \). Since \( X \) is a \( G_b \)-complete, therefore there exists \( p \in X \) such that \( \{x_n\} \) is \( G_b \)-converges to \( p \). Now we will show here \( p \) is fixed point of \( T \).

Suppose that \( Tp \neq p \).

\[
G_b(x_n, Tp, Tp) \leq s \left[ G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tp, Tp) \right]
\leq s \lambda^n G_b(x_0, x_1, x_1) + s \left[ \alpha G_b(x_n, p, p) + \beta G_b(x_n, x_{n+1}, x_{n+1}) \right]
\quad + \gamma G_b(p, Tp, Tp) + \delta G_b(p, Tp, Tp).
\]

Letting \( n \to \infty \), since \( \lambda < 1 \), so \( \lambda^n \to 0 \) and \( x_n \to p \). It gives that

\[
G_b(p, Tp, Tp) \leq s(\gamma + \delta) G_b(p, Tp, Tp).
\]

Since \( s(\gamma + \delta) < 1 \). The above inequality is true only if \( G_b(p, Tp, Tp) = 0 \) i.e. \( p = Tp \). Thus \( p \) is a fixed point of \( T \).

Suppose \( q \neq p \) and \( q \) is another fixed point of \( T \), i.e. \( Tq = q \). Then consider

\[
G_b(x_n, Tq, Tq) \leq s \left[ G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tq, Tq) \right]
\]

\[
G_b(x_n, q, q) \leq s \lambda^n G_b(x_0, x_1, x_1) + s \left[ \alpha G_b(x_n, q, q) + \beta G_b(x_n, x_{n+1}, x_{n+1}) + \gamma G_b(q, Tq, Tq) + \delta G_b(q, Tq, Tq) \right].
\]

As \( n \to \infty, x_n \to p, \lambda^n \to 0 \) as \( \lambda < 1 \) and \( Tq = q \). We get

\[
G_b(p, q, q) \leq s \alpha G_b(p, q, q),
\]

since \( s \alpha < 1 \). The inequality (10) is true only when \( G_b(p, q, q) = 0 \). i.e.\( p = q \). Thus \( p \) is a unique fixed point of \( T \).
To show that $T$ is $G_b$-continuous at $p$, let $\{y_n\}$ be a sequence in $X$ such that $\lim_{n \to \infty} y_n = p$. Consider
\[
G_b(p, Ty_n, Ty_n) \leq \alpha G_b(p, y_n, y_n) + \beta G_b(p, p, p) + \gamma G_b(y_n, Ty_n, Ty_n) + \delta G_b(y_n, Ty_n, Ty_n).
\]
As $\lim_{n \to \infty} y_n = p$, we get
\[
G_b(p, Ty_n, Ty_n) = \frac{s(\gamma + \delta)}{1 - s(\gamma + \delta)} G_b(p, Ty_n, Ty_n).
\]
It is implies that
\[
G_b(p, Ty_n, Ty_n) \leq \left[ \frac{\alpha}{1 - s(\gamma + \delta)} G_b(p, y_n, y_n) + \frac{s(\gamma + \delta)}{1 - s(\gamma + \delta)} G_b(p, Ty_n, Ty_n) \right].
\]
Since $\frac{s(\gamma + \delta)}{1 - s(\gamma + \delta)} < 1$. The inequality (11) is true only if $G_b(p, Ty_n, Ty_n) = 0$, i.e.
$Ty_n = p = Tp$, as $n \to \infty$. It shows that $T$ is $G_b$-continuous at $p$.

3. Property P

Let $T$ be a self map of a complete $G_b$ metric space with non-empty fixed point set $F(T)$. Then $T$ is said to satisfy property $P$ if $F(T) = F(T^n)$, for each $n \in \mathbb{N}$.

Theorem 4 Under the contraction of theorem 1, $T$ has property $P$.

Proof. By Theorem 1, $T$ has a fixed point. Therefore $F(T^n) \neq \emptyset$, each $n \in \mathbb{N}$. Fix $n > 1$ and assume that $p \in F(T^n)$. To show that $p \in F(T)$. Suppose that $p \neq Tp$. Then we have
\[
G_b(p, Tp, Tp) = G_b(T^n p, T^{n+1} p, T^{n+1} p) \leq k \max \left[ G_b(T^{n-1} p, T^n p, T^n p), G_b(T^{n-1} p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), \right.
\]
\[
\left. G_b(T^n p, T^{n+1} p, T^{n+1} p), \frac{1}{2} \left[ G_b(T^{n-1} p, T^{n+1} p, T^{n+1} p) + G_b(T^n p, T^n p, T^n p) \right], \frac{1}{2} \left[ G_b(T^{n-1} p, T^{n+1} p, T^{n+1} p) + G_b(T^n p, T^n p, T^n p) \right], \right]
\]
\[
= k \max \left[ G_b(T^{n-1} p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), \frac{1}{2} G_b(T^{n-1} p, T^{n+1} p, T^{n+1} p) \right].
\]

Here, we have three cases:

Case (i) Suppose
\[
\max \left[ G_b(T^{n-1} p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), \frac{1}{2} G_b(T^{n-1} p, T^{n+1} p, T^{n+1} p) \right] = G_b(T^{n-1} p, T^n p, T^n p).
\]
Then, we get

\[ G_b(T^n p, T^{n+1} p, T^{n+1} p) \leq k G_b(T^{n-1} p, T^n p, T^n p) \leq \cdots \leq k^n G_b(p, T p, T p). \]

Case (ii) Suppose

\[
\begin{align*}
&\max \left[ G_b(T^{n-1} p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), \frac{G_b(T^{n-1} p, T^{n+1} p, T^{n+1} p)}{2} \right] \\
&= G_b(T^n p, T^{n+1} p, T^{n+1} p).
\end{align*}
\]

Then, we get

\[ G_b(T^n p, T^{n+1} p, T^{n+1} p) \leq k G_b(T^n p, T^{n+1} p, T^{n+1} p), \]

which is contradiction, since \( k < 1. \)

Case (iii) Suppose

\[
\begin{align*}
&\max \left[ G_b(T^{n-1} p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), \frac{G_b(T^{n-1} p, T^{n+1} p, T^{n+1} p)}{2} \right] \\
&= \frac{G_b(T^{n-1} p, T^{n+1} p, T^{n+1} p)}{2}.
\end{align*}
\]

Then, we get

\[ G_b(T^n p, T^{n+1} p, T^{n+1} p) \leq \frac{k G_b(T^n p, T^{n+1} p, T^{n+1} p)}{2}. \quad (12) \]

By property (v) of \( G_b \)-metric space, we have

\[ G_b(T^{n-1} p, T^{n+1} p, T^{n+1} p) \leq s \left[ G_b(T^{n-1} p, T^n p, T^n p) + G_b(T^n p, T^{n+1} p, T^{n+1} p) \right]. \quad (13) \]

Using inequality (13) in (12), we get

\[ G_b(T^n p, T^{n+1} p, T^{n+1} p) \leq sk \left[ \frac{G_b(T^{n-1} p, T^n p, T^n p) + G_b(T^n p, T^{n+1} p, T^{n+1} p)}{2} \right]. \]

It gives that

\[ G_b(T^n p, T^{n+1} p, T^{n+1} p) \leq \frac{sk}{2} G_b(T^{n-1} p, T^n p, T^n p) \]

\[ = \lambda G_b(T^{n-1} p, T^n p, T^n p) \leq \cdots \leq \lambda^n G_b(p, T p, T p), \]

where \( \lambda = \frac{sk}{2}. \) Since \( k, \lambda < 1, \) so as \( n \to \infty, \) we get \( G_b(p, T p, T p) = 0 \) and hence in all cases \( p = T p \) i.e. \( p \in F(T). \) Hence \( T \) has property \( P. \)

**Theorem 5** Under the contraction of theorem 2, \( T \) has property \( P. \)

**Proof.** By theorem 2, \( T \) has a fixed point. Therefore \( F(T^n) \neq \emptyset, \) each \( n \in \mathbb{N}. \) Fix
\(n > 1\) and assume that \(p \in F(T^n)\). To show that \(p \in F(T)\). Suppose that \(P \neq Tp\).

\[ G_b(p, Tp, Tp) = G_b(T^n p, T^{n+1} p, T^{n+1} p) \]
\[ \leq k \max \left[ G_b(T^{n-1} p, T^n p, T^n p) \right. \]
\[ G_b(T^n p, T^{n+1} p, T^{n+1} p), G_b(T^{n-1} p, T^{n+1} p, T^n p), G_b(T^n p, T^n p, T^n p), \]
\[ G_b(T^n p, T^{n+1} p, T^{n+1} p) \]
\[ \left. \leq k \max \left[ G_b(T^{n-1} p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^n p) \right. \]
\[ G_b(T^{n-1} p, T^{n+1} p, T^n p) \right]. \]

Here we have three cases,

Case (i) Suppose

\[ \max \left[ G_b(T^{n-1} p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), G_b(T^{n-1} p, T^{n+1} p, T^n p) \right. \]
\[ \leq G_b(T^{n-1} p, T^n p, T^n p). \]

Then we get,

\[ G_b(T^n p, T^{n+1} p, T^n p) \leq k G_b(T^{n-1} p, T^n p, T^n p) \leq \cdots \leq k^n G_b(p, Tp, Tp). \]

Case (ii) Suppose

\[ \max \left[ G_b(T^{n-1} p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), G_b(T^{n-1} p, T^{n+1} p, T^n p) \right. \]
\[ = G_b(T^n p, T^{n+1} p, T^{n+1} p). \]

Then we get,

\[ G_b(T^n p, T^{n+1} p, T^n p) \leq k G_b(T^n p, T^{n+1} p, T^{n+1} p), \]

which is contradiction, since \(k < 1\).

Case (iii) Suppose

\[ \max \left[ G_b(T^{n-1} p, T^n p, T^n p), G_b(T^n p, T^{n+1} p, T^{n+1} p), G_b(T^{n-1} p, T^{n+1} p, T^n p) \right. \]
\[ = G_b(T^{n-1} p, T^{n+1} p, T^n p). \]

Then we get,

\[ G_b(T^n p, T^{n+1} p, T^{n+1} p) \leq k G_b(T^{n-1} p, T^{n+1} p, T^n p). \]

By property (v) of \(G_b\)-metric space, we have

\[ G_b(T^{n-1} p, T^{n+1} p, T^n p) \leq s \left[ G_b(T^{n-1} p, T^n p, T^n p) + G_b(T^n p, T^{n+1} p, T^{n+1} p) \right]. \]

Using inequality (15) in (14), we get

\[ G_b(T^n p, T^{n+1} p, T^n p) \leq sk \left[ G_b(T^{n-1} p, T^n p, T^n p) + G_b(T^n p, T^{n+1} p, T^{n+1} p) \right]. \]

It gives that

\[ G_b(T^n p, T^{n+1} p, T^n p) \leq \frac{sk}{1 - sk} G_b(T^{n-1} p, T^n p, T^n p) \]
\[ = \lambda G_b(T^{n-1} p, T^n p, T^n p) \leq \cdots \leq \lambda^n G_b(p, Tp, Tp), \]
Proof. By theorem 3, \( \lambda < 1 \), so as \( n \to \infty \), we get \( G_b(p, Tp, Tp) = 0 \) and hence in all cases \( p = Tp \) i.e. \( p \in F(T) \). Hence \( T \) has property \( P \).

**Theorem 6** Under the contraction of theorem 3, \( T \) has property \( P \).

**Proof.** By theorem 3, \( T \) has a fixed point. Therefore \( F(T^n) \neq \emptyset \), each \( n \in \mathbb{N} \). Fix \( n > 1 \) and assume that \( p \in F(T^n) \). To show that \( p \in F(T) \). Suppose that \( P \neq Tp \).

\[
G_b(p, Tp, Tp) = G_b(T^n p, T^{n+1} p, T^{n+1} p)
\]
\[
\leq \left[ \alpha G_b(T^{n-1} p, T^n p, T^n p) + \beta G_b(T^{n-1} p, T^n p, T^n p) \right.
\]
\[
+ \gamma G_b(T^n p, T^{n+1} p, T^{n+1} p) + \delta G_b(T^n p, T^{n+1} p, T^{n+1} p) \left. \right]
\]
\[
\leq (\alpha + \beta) G_b(T^{n-1} p, T^n p, T^n p) + (\gamma + \delta) G_b(T^n p, T^{n+1} p, T^{n+1} p).
\]

It gives that
\[
G_b(p, Tp, Tp) \leq \frac{\alpha + \beta}{1 - (\gamma + \delta)} G_b(T^{n-1} p, T^n p, T^n p)
\]
\[
= \lambda G_b(T^{n-1} p, T^n p, T^n p) \leq \cdots \leq \lambda^n G_b(p, Tp, Tp),
\]

where \( \lambda = \frac{\alpha + \beta}{1 - (\gamma + \delta)} \). Since \( \lambda < 1 \), so as \( n \to \infty \), we get \( G_b(p, Tp, Tp) = 0 \) and hence \( p = Tp \) i.e. \( p \in F(T) \). Hence \( T \) has property \( P \).

**Example 3** Let us define \( G_b(x, y, z) = |x - y| + |y - z| + |x - z| \) and let \( x \in X \). Then \( (X, G_b) \) be a complete \( G_b \)-metric space. Let \( T(x) = \frac{x}{3} \). Without loss of generality, we assume \( x > y > z \). Then

(i)

\[
G_b(T(x), T(y), T(z)) = \left| \frac{x}{3} - \frac{y}{3} \right| + \left| \frac{y}{3} - \frac{z}{3} \right| + \left| \frac{x}{3} - \frac{z}{3} \right|
\]
\[
= \frac{1}{3} \left[ |x - y| + |y - z| + |x - z| \right]
\]
\[
\leq k \max \left[ G_b(x, y, z), G_b(x, T(x), T(x)), G_b(y, T(y), T(y)), G_b(z, T(z), T(z)), \right.
\]
\[
\frac{2}{2} \left[ G_b(x, T(y), T(y)) + G_b(z, T(x), T(x)) \right], \frac{2}{2} \left[ G_b(x, T(y), T(y)) + G_b(y, T(z), T(z)) \right],
\]
\[
\frac{2}{2} \left[ G_b(y, T(z), T(z)) + G_b(z, T(x), T(x)) \right].
\]

(ii)

\[
G_b(T(x), T(y), T(z)) = \left| \frac{x}{3} - \frac{y}{3} \right| + \left| \frac{y}{3} - \frac{z}{3} \right| + \left| \frac{x}{3} - \frac{z}{3} \right|
\]
\[
= \frac{1}{3} \left[ |x - y| + |y - z| + |x - z| \right]
\]
\[
\leq k \max \left[ G_b(x, y, z), G_b(x, T(x), T(x)), G_b(y, T(y), T(y)), G_b(x, T(y), T(y)), \right.
\]
\[
G_b(y, T(x), T(x)), G_b(z, T(z), T(z)) \right].
\]
(iii)

\[ G_b(T(x), T(y), T(z)) = \frac{1}{3} \left| x - y \right| + \left| y - z \right| + \left| z - x \right| \]

\[ \leq \frac{1}{9} \left[ \left| x - y \right| + \left| y - z \right| + \left| z - x \right| \right] + \frac{2}{9} \left[ \left| x - y \right| + \left| y - z \right| + \left| z - x \right| \right] + \frac{1}{3} \left[ \frac{2}{3} \left| x - y \right| + \frac{2}{3} \left| y - z \right| + \frac{1}{3} \left| z - x \right| \right] \]

\[ \leq \frac{1}{9} \left[ \left| x - y \right| + \left| y - z \right| + \left| z - x \right| \right] + \frac{1}{3} \left[ \frac{2}{3} \left| x - y \right| + \frac{2}{3} \left| y - z \right| + \frac{1}{3} \left| z - x \right| \right] \]

\[ \leq \frac{1}{9} \left[ \left| x - y \right| + \left| y - z \right| + \left| z - x \right| \right] + \frac{1}{6} \left[ \left| x - y \right| + \left| y - z \right| + \left| z - x \right| \right] \]

\[ \leq \alpha G_b(x, y, z) + \beta G_b(x, T(x), T(x)) + \gamma G_b(y, T(y), T(y)) + \delta G_b(z, T(z), T(z)) \]

where \( \alpha = \frac{1}{3}, \beta = \frac{1}{6}, \gamma = \frac{1}{6}, \delta = \frac{1}{6} \) and \( \alpha + \beta + \gamma + \delta = \frac{17}{18} < 1 \).

**References**


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