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PRE-SCHWARZIAN NORM ESTIMATE FOR FUNCTIONS CONVEX IN ONE DIRECTION

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Abstract. For the normalized analytic function f in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, we consider the class $\mathcal{F}(\alpha)$ of functions f satisfying the analytic characterization $\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{\alpha}{2\alpha-3}$, where α is an arbitrary number and is not less than 3/2. For a locally univalent analytic function f defined on $\mathbb{D}$, we consider the pre-Schwarzian norm by $\|f\| = \sup_{|z|<1}\left(1-|z|^2\right)\left|\frac{f''(z)}{f'(z)}\right|$. In this paper, we find the sharp norm estimate for the functions f in the class $\mathcal{F}(\alpha)$.

1. Introduction

Let $\mathcal{H}$ denote the space of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the set of all functions $f \in \mathcal{H}$ satisfying the usual normalization $f(0) = f'(0) - 1 = 0$ with the Taylor's expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which are also univalent in $\mathbb{D}$.

A domain $D \subset \mathbb{C}$ is said to be convex if it is starlike with respect to each of its points, that is, if the line segment joining any two points of D lies entirely in D. A function $f \in \mathcal{S}$ is said to be convex function, if and only if $f(\mathbb{D})$ is a convex domain. It is well known that a function $f \in \mathcal{A}$ is called convex of order α $(0 \le \alpha < 1)$, if and only if $\Re\{1 + zf''(z)/f'(z)\} > \alpha$, $z \in \mathbb{D}$, and we denote this class of function by $\mathcal{K}(\alpha)$. In particular, it is well known that $\mathcal{K}(0) = \mathcal{K}$.

A domain $D \subset \mathbb{C}$ is called convex in the direction φ $(0 \le \varphi < \pi)$, if every line parallel to the line through 0 and $e^{i\varphi}$ has a connected or empty intersection with D. A function $f \in \mathcal{S}$ is said to be convex in the direction φ, if and only if $f(\mathbb{D})$ is convex in the direction φ.

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Umezawa [10, Theorem 1] studied that, if functions $f \in \mathcal{A}$ of the form (1) be meromorphic in $\mathbb{D}$ and satisfying the relation

$$\alpha > \Re\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{\alpha}{2\alpha-3}, \quad z \in \mathbb{D}, \tag{2}$$

where α is an arbitrary number not less than $3/2$, then $f(z)$ is analytic and univalent in $\mathbb{D}$. Moreover, $f(z)$ maps $|z|=r$ for every $r<1$ into a curve which is convex in one direction, and $|a_n| \le n$ for all n.

Several special cases of inequality (2) can be drawn by allowing different values of $\alpha \ge 3/2$ which have been studied for different contexts. Let the class of all functions $f \in \mathcal{A}$ satisfying the condition

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{\alpha}{2\alpha-3}, \quad z \in \mathbb{D}, \tag{3}$$

be denoted by $\mathcal{F}(\alpha)$. In particular, we denote the class $\mathcal{F} := \mathcal{F}(\alpha)|_{\alpha\to\infty}$, i.e., class $\mathcal{F}$ satisfies the analytic characterization

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}.$$

The class $\mathcal{F}$ plays an important role in the discussion on certain extremal problems for the classes of complex-valued and sense-preserving harmonic convex functions and some other related problems in determining univalence criteria for sense-preserving harmonic mappings. In view of Kaplan characterization [4, p.48, Theorem 2.18], functions in $\mathcal{F}$ are also close-to-convex (hence univalent) in $\mathbb{D}$. Recently, Ponnusamy *et al.* [9] have studied the radius of convexity of partial sums of functions $f \in \mathcal{F}$ and proved that every section $s_n(z) = z+\sum_{k=2}^{n} a_k z^k$ of function $f \in \mathcal{F}$ is convex in disk $|z| < 1/6$. Agrawal and Sahoo [1] proved that every section $s_{2n-1}(z) = z + \sum_{k=2}^{n} a_{2k-1} z^{2k-1}$ of odd univalent function $f \in \mathcal{F}$ is convex in the disk $|z| < \sqrt{2}/3$. Furthermore, authors have studied the bounds on third Hankel determinant for coefficients of Taylor's series expansion of functions in $\mathcal{F}$ [2].

For two analytic functions f and g in $\mathbb{D}$, we say that the function f is subordinate to the function g, written as $f(z) \prec g(z)$, if there exists an analytic function w in $\mathbb{D}$ such that $|w(z)| < 1$, $z \in \mathbb{D}$, and $w(0) = 0$, with $f(z) = g(w(z))$ in $\mathbb{D}$. In particular, if $g \in \mathcal{S}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

For a locally univalent analytic function f, the pre-Schwarzian derivative of f is defined by

$$T_f = \frac{f''}{f'},$$

and its norm is defined by

$$\|T_f\| = \sup_{|z|<1} \left(1-|z|^2\right) |T_f(z)|.$$

The pre-Schwarzian derivative T_f and its norm $\|T_f\|$ have significant meanings in the theory of Teichmüller space [6, 15]. It is known that $\|T_f\| < \infty$ if and only if f is uniformly locally univalent, that is, there exists a constant $\rho = \rho(f) > 0$ such that f is univalent in each disk

$$\left\{ z \in \mathbb{C} : \left|\frac{z-a}{1-\bar{a}z}\right| < \rho, \;\; |a| < 1 \right\},$$

(see [11, 12]). It is well known that $\|T_f\| \leq 6$ for $f \in \mathcal{S}$, and $\|T_f\| \leq 4$ for $f \in \mathcal{K}$. Conversely, by Beckers theorem [3] it follows that if $f \in \mathcal{A}$ and $\|T_f\| \leq 1$, then $f \in \mathcal{S}$. For $f \in \mathcal{S}$ the Alexander transform $J[f](z) := \int_0^z (f(t)/t)\, dt$ is locally univalent and it has been obtain by Kim et al. [5] that $\|T_{J[f]}\| \leq 4$. Yamashita [14] proved that, if $f \in \mathcal{S}^*(\alpha)$ then $\|T_f\| \leq 6 - 4\alpha$ and $\|T_{J[f]}\| \leq 4(1-\alpha)$. Both the inequalities are sharp.

In this work we investigate the sharp norm estimates for functions in the class $\mathcal{F}(\alpha)$, and for the class $\mathcal{F}$ as its particular case. Consider the function

$$\Phi(z) = \frac{2\alpha - 3}{3 - 4\alpha}\left(1 - (1-z)^{\frac{3-4\alpha}{2\alpha-3}}\right), \tag{4}$$

for which

$$1 + \frac{z\Phi''(z)}{\Phi'(z)} = \frac{1 + \left(\frac{4\alpha-3}{2\alpha-3}\right) z}{1 - z}.$$

Then $\Phi(z) \in \mathcal{F}(\alpha)$. It is well known that the function $\Phi(z)$ is the extremal function for the following estimate of a_2. For all function $f \in \mathcal{F}(\alpha)$, we have $|a_2| \leq \frac{3\alpha-3}{2\alpha-3}$ and the equality holds if and only if

$$f(z) = \overline{\mu}\Phi(\mu z), \tag{5}$$

where μ is a unimodular constant, that is, μ is complex with $|\mu|^2 = \mu\overline{\mu} = 1$.

The class of Carathéodory functions $\mathcal{P}$, is the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{6}$$

having a positive real part in $\mathbb{D}$. Following are the well known results for the functions belonging to the class $\mathcal{P}$ and can be found in Duren [4] and Libera and Zlotkiewicz [7].

Lemma 1.1. *If $p \in \mathcal{P}$ be of the form* (6)*, then*

$$|c_n| \leq 2, \quad n \in \mathbb{N}, \tag{7}$$

and

$$2c_2 = c_1^2 + x(4 - c_1^2), \quad c_1 \in \mathbb{R}, \tag{8}$$

for some x with $|x| \leq 1$. The inequality (7) *is sharp and the equality holds for the function $\varphi(z) = \dfrac{1+z}{1-z}$, $z \in \mathbb{D}$.*

By using Lemma 1.1, we obtain the following result which is needful to obtain the Corollary 2.2.

Lemma 1.2. *If the function $f \in \mathcal{F}$ be of the form* (1)*, then*

$$|9a_3 - 8a_2^2| \leq 9/2. \tag{9}$$

Proof. If $f \in \mathcal{F}$ be of the form (1), then we may write

$$1 + \frac{zf''(z)}{f'(z)} = \frac{3}{2}p(z) - \frac{1}{2},$$

where $p \in \mathcal{P}$ be of the form (6). Substituting the series expansions of f and p and equating the coefficients, we get $a_2 = 3c_1/4$ and $a_3 = (3c_1^2 + 2c_2)/8$. Using these value of coefficients and Lemma 1.1 for some $c_1 = c \in [0,2]$ and x such that $|x| = \mu \leq 1$, we get

$$|9a_3 - 8a_2^2| \leq \frac{9}{8}\mu(4 - c^2) = H(c,\mu)$$

Let $\Omega = \{(c,\mu) : 0 \leq c \leq 2,\ 0 \leq \mu \leq 1\}$. To find the maximum value of H over the region Ω, note that H is increasing function of μ and decreasing function of c , hence the maximum value of $H(c,\mu)$ is attained at the point $(0,1)$ in Ω, that is $\max_{\Omega} H(c,\mu) = H(0,1) = 9/2$. This completes the proof. □

2. Main Result

Here we obtain the pre-Schwarzian norm estimation for function f in the class $\mathcal{F}(\alpha)$.

Theorem 2.1. *For $\alpha \geq \frac{3}{2}$, the following propositions holds good.*

- **(a)** *Suppose that $f \in \mathcal{F}(\alpha)$. Then $\|T_f\| = 2\zeta$, where $\zeta = \dfrac{6\alpha - 6}{2\alpha - 3}$, if and only if f is of the form (5).*
- **(b)** *Suppose that $f \in \mathcal{F}(\alpha)$ is not of the form (5). Then*

$$\|T_f\| \leq 2\zeta \left(\frac{1 + A + B}{3 - A + B}\right), \quad for \quad \zeta = \frac{6\alpha - 6}{2\alpha - 3}, \tag{10}$$

where

$$0 \leq A = \frac{2}{\zeta}\,|a_2| < 1, \tag{11}$$

and

$$0 \leq B = \frac{2}{\zeta}\,\frac{\left|3\zeta a_3 - 2(\zeta + 1)a_2^2\right|}{\zeta - 2|a_2|} \leq 1 + A < 2, \tag{12}$$

so that

$$\frac{1}{3} \leq \frac{1 + A + B}{3 - A + B} \leq \frac{1 + A}{2} < 1.$$

Proof. Consider the function

$$F(z) \equiv F_\alpha(z) = \frac{1 + \eta z}{1 - z}, \quad \text{where} \quad \eta = \frac{4\alpha - 3}{2\alpha - 3}. \tag{13}$$

The function $F(z)$ is univalent in $\mathbb{D}$, and satisfying the conditions

$$F'(0) = \zeta, \quad F''(0) = 2\zeta,$$

and

$$F(\mathbb{D}) = \left\{z \in \mathbb{C} : \Re(z) > -\frac{\alpha}{2\alpha - 3}\right\},$$

where $\zeta := 1 + \eta = \dfrac{6\alpha - 6}{2\alpha - 3}$. For the function $f \in \mathcal{F}(\alpha)$, set

$$g(z) = 1 + \frac{zf''(z)}{f'(z)}, \quad z \in \mathbb{D}.$$

Note here that $F^{-1}(w) = \dfrac{w - 1}{w + \eta}$, then the composed function

$$\phi \equiv F^{-1} \circ g : \mathbb{D} \to \mathbb{D},$$

is analytic in $\mathbb{D}$ with $\phi(0) = 0$ and $g = F \circ \phi$. In particular, for all $f \in \mathcal{F}(\alpha)$, g is subordinate to F. It is clear that

$$\phi'(z) = (F^{-1})'(g(z)) \cdot g'(z) = \frac{\zeta\, g'(z)}{(g(z)+\eta)^2},$$

and

$$\phi''(z) = \zeta\, \frac{g''(z)\,(g(z)+\eta) - 2(g'(z))^2}{(g(z)+\eta)^3}.$$

Since

$$g'(0) = 2a_2 \quad \text{and} \quad g''(0) = 12a_3 - 8a_2^2,$$

it follows that

$$\phi'(0) = \frac{2}{\zeta}\, a_2 \quad \text{and} \quad \phi''(0) = \frac{1}{\zeta^2}\left(12\zeta a_3 - 8(\zeta+1)a_2^2\right). \tag{14}$$

Clearly, the function $\phi(z)$ satisfy the conditions of Schwarz lemma. Hence the Schwarz lemma for ϕ shows that

$$|\phi'(0)| := A = \frac{2}{\zeta}\, |a_2| \leq 1,$$

and further $A = 1$ if and only if

$$\phi(z) = \mu z, \tag{15}$$

for a unimodular constant μ, or f is of the form (5). On the other hand, it follows from the fact $g = F \circ \phi$ that

$$\frac{f''(z)}{f'(z)} = \zeta\, \frac{\phi(z)}{z(1-\phi(z))} \tag{16}$$

is in the unit disk $\mathbb{D}$.

For the proof of (b), remark that ϕ is not of the form (15). Thus form [13, p.313, (6.8**(a)] it follows that

$$|\phi(z)| \leq |z| Q(|z|), \quad z \in \mathbb{D}, \tag{17}$$

where

$$Q(x) = \frac{x^2 + Bx + A}{Ax^2 + Bx + 1}, \quad 0 \leq x \leq 1.$$

Here

$$B = \frac{|\phi''(0)|}{2(1 - |\phi'(0)|)},$$

which together with (14), provides the value of B in terms of a_2 and a_3. By the help of Schwarz-Pick inequality at 0 applied to $\chi(z) = \phi(z)/z$, where $|\chi| < 1$, we observe that

$$\frac{B}{1 + |\phi'(0)|} = \frac{|\chi'(0)|}{1 - |\chi(0)|^2} \leq 1.$$

Hence

$$B \leq 1 + |\phi'(0)| = 1 + A < 2,$$

by $|\phi'(0)| = A < 1$. By using (16) and (17), one can compute

$$\left(1 - |z|^2\right) \left|\frac{f''(z)}{f'(z)}\right| \leq \zeta\, \frac{\left(1 - |z|^2\right) Q(|z|)}{1 - |z| Q(|z|)} = \zeta\, G(|z|), \tag{18}$$

where

$$\begin{aligned} G(x) &= \frac{(1-x^2)Q(x)}{1-xQ(x)} \\ &= \frac{(1+x)(x^2+Bx+A)}{x^2+(1-A+B)x+1}, \quad 0 \le x \le 1. \end{aligned}$$

To prove that

$$G(x) \le G(1) = \frac{2(1+A+B)}{3-A+B}, \quad 0 \le x \le 1, \tag{19}$$

let $H(x)$ be the numerator of the derivative $G'(x)$. Then,

$$\begin{aligned} &H(0) = (1-A)B + A^2 \ge 0, \\ &H'(0) = 2(1-A+B) > 0, \\ &H''(0) = 2(B^2 + (1-A)B + 2(2-A)) > 0, \end{aligned}$$

and,

$$H'''(x) = 12(2x + B - A + 1) > 0 \quad \text{for} \quad 0 \le x \le 1.$$

Hence, $H(x) \ge 0$ or $G(x)$ is nondecreasing in $0 \le x \le 1$, which prove the condition (19). Combining (18) and (19), we finally get the result (10), that is

$$\|T_f\| \le 2\zeta \left(\frac{1+A+B}{3-A+B} \right).$$

Now it remains to prove that

$$\|T_f\| = 2\zeta,$$

for f be of the form (5). If f is of the form (5), then

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{\mu z \, \Phi''(\mu z)}{\Phi'(\mu z)} = F(\mu z).$$

Thus it follows that

$$\left(1-|z|^2\right) \left| \frac{f''(z)}{f'(z)} \right| = \zeta \, \frac{\left(1-|z|^2\right)|\phi(z)|}{|z|\left(|1-\phi(z)|\right)} = \zeta \, \frac{1-|z|^2}{|1-\mu z|} \le 2\zeta.$$

Since $\left(1-|z|^2\right) \left| \frac{f''(z)}{f'(z)} \right| = \zeta(1+x)$ for $z = \overline{\mu} x,\ 0 < x < 1$, tends to 2ζ as $x \to 1^-$, we finally get the result that $\|T_f\| = 2\zeta$. This is what we wanted to proof. □

In particular case, when α approaches ∞ in Theorem 2.1, together with Lemma 1.1 and Lemma 1.2, we get the following results:

Corollary 2.2. *The following propositions holds good.*

(a) *Suppose that $f \in \mathcal{F}$. Then $\|f\| = 6$ if and only if f is of the form*

$$f(z) = \overline{\mu}\Phi(\mu z)$$

where $\Phi(z) = \dfrac{z - z^2/2}{(1-z)^2}$.

(b) *Suppose that* $f \in \mathcal{F}$ *is not of the form*

$$f(z) = \overline{\mu}\Phi(\mu z)$$

where $\Phi(z) = \dfrac{z - z^2/2}{(1-z)^2}$. *Then*

$$\|f\| \leq 6\left(\frac{1+A+B}{3-A+B}\right), \tag{20}$$

where $A \in [0,1)$ *and* $B \in \left(0, \frac{1}{1-\alpha}\right)$.

References

[1] S. Agrawal and S. K. Sahoo, Radius of convexity of partial sums of odd functions in the close-to-convex family, Filomat, 31(11), 3519–3529, 2017.

[2] D. Bansal, S. Maharana, and J. K. Prajapat, Third order Hankel determinant for certain univalent functions, J. Korean Math. Soc., 52(6), 1139–1148, 2015.

[3] J. Becker, Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen (German), J. Reine Angew. Math., 255, 23–43, 1972.

[4] P. L. Duren, Univalent Functions, vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, New York, 1983.

[5] Y. C. Kim, S. Ponnusamy, and T. Sugawa, Mapping properties of nonlinear integral opertaors and pre-Schwarzian derivatives, J. Math. Anal. Appl., 299, 433–447, 2004.

[6] O. Lehto, Univalent functions and Teichmüller spaces, Springer-Verlag, 1987.

[7] R. J. Libera and E. J. Zlotkiewicz, Early coefficients of inverse of a regular convex function, Proc. Amer. Math. Soc., 85(2), 225–230, 1982.

[8] S. Ozaki, On the theory of multivalent functions II, Science Reports of the Tokyo Bunrika Daigaku, 4, 45–87, 1941.

[9] S. Ponnusamy, S. K. Sahoo, and H. Yanagihara, Radius of convexity of partial sums of functions in the close-to-convex family, Nonlinear Anal., 95, 219–228, 2014.

[10] T. Umezawa, Analytic functions convex in one direction, J. Math. Soc. Japan, 4, 194–202, 1952.

[11] S. Yamashita, Banach spaces of locally schlicht functions with the Hornich operations, Manuscripta Math., 16, 261–275, 1975.

[12] S. Yamashita, Almost locally univalent functions, Monatsh. Math., 81, 235–240, 1976.

[13] S. Yamashita, The Pick version of Schwarz lemma and comparision of the Poincaré densities, Ann. Acad. Sci. Fenn. Ser. A. I. Math., 19, 291–322, 1994.

[14] S. Yamashita, Norm estimates for function starlike or convex of order alpha, Hokkadio Math. J., 28, 217–230, 1999.

[15] I. V. Zhuravlev, Model of the universal Teichmüller space, Siberian Math. J., 27, 691–697, 1986.

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