

**SOME GROWTH ANALYSIS OF ITERATED ENTIRE
 FUNCTIONS ON THE BASIS OF THEIR MAXIMUM TERMS
 AND RELATIVE ORDERS**

TANMAY BISWAS

ABSTRACT. The main aim of this paper is to study some growth properties of iterated entire functions on the basis of their maximum terms and relative order.

1. Introduction and Definitions.

We denote \mathbb{C} by the set of all finite complex numbers. For any two entire functions f and g defined in \mathbb{C} , Lahiri and Banerjee [5] introduced the concept of the iteration of f with respect to g in the following manner:

$$\begin{aligned} f(z) &= f_1(z) \\ f(g(z)) &= f(g_1(z)) = f_2(z) \\ f(g(f(z))) &= f(g(f_1(z))) = f(g_2(z)) = f_3(z) \\ \dots & \dots \dots \dots \dots \dots \\ \dots & \dots \dots \dots \dots \dots \\ f(g(f \dots (f(z) \text{ or } g(z)) \dots)) &= f_n(z) \end{aligned}$$

according as n is odd or even, and so

$$\begin{aligned} g(z) &= g_1(z) \\ g(f(z)) &= g(f_1(z)) = g_2(z) \\ g(f(g(z))) &= g(f(g_1(z))) = g(f_2(z)) = g_3(z) \\ \dots & \dots \dots \dots \dots \dots \\ \dots & \dots \dots \dots \dots \dots \\ g(f(g_{n-2}(z))) &= g(f_{n-1}(z)) = g_n(z) . \end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

For any entire function f , the so called maximum modulus function denoted by $M_f(r)$ is defined for each non-negative real value of r as follows:

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

If an entire function f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1} : (|f(0)|, \infty) \rightarrow (0, \infty)$ exists and is such

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that $\lim_{s \rightarrow \infty} M_f^{-1}(s) = \infty$. The maximum term $\mu_f(r)$ of entire f can be defined as $\mu_f(r) = \max_{n \geq 0} (|a_n| r^n)$. Obviously $\mu_f(r)$ is also a real and increasing function of r . For another entire function g , the ratios $\frac{M_f(r)}{M_g(r)}$ when $r \rightarrow +\infty$ as well as $\frac{\mu_f(r)}{\mu_g(r)}$ when $r \rightarrow +\infty$ are called the comparative growth of f with respect to g in terms of their maximum moduli and the maximum term respectively. Actually the study of comparative growth properties of composite entire functions under some different directions is the prime concern of this paper. We use the standard notations and definitions of the theory of entire functions which are available in [4] and [9], and therefore we do not explain those in details. We begin by recalling the following definitions.

Definition 1. [1, 2] *The relative order and relative lower order of an entire function f with respect to another entire function g , denoted by $\rho_g(f)$ and $\lambda_g(f)$ respectively are defined as*

$$\rho_g(f) = \limsup_{r \rightarrow +\infty} \frac{\log M_g^{-1}(M_f(r))}{\log r} \text{ and } \lambda_g(f) = \liminf_{r \rightarrow +\infty} \frac{\log M_g^{-1}(M_f(r))}{\log r}.$$

Definition 1 coincides with the classical definitions of order and lower order of entire function [8] if $g(z) = \exp z$.

Datta and Maji [3] gave an alternative definition of relative order and relative lower order in terms of maximum term of an entire function with respect to another entire function in the following way:

Definition 2. [3] *The relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of an entire function f with respect to an entire function g are defined as follows:*

$$\rho_g(f) = \limsup_{r \rightarrow \infty} \frac{\log \mu_g^{-1}(\mu_f(r))}{\log r} \text{ and } \lambda_g(f) = \liminf_{r \rightarrow \infty} \frac{\log \mu_g^{-1}(\mu_f(r))}{\log r}.$$

In fact, Datta and Maji [3] also established the equivalence of Definition 1 and Definition 2.

Now for another two non-constant entire functions h and k , we may define the iteration of $\mu_h^{-1}(r)$ with respect to $\mu_k^{-1}(r)$ in the following manner:

$$\begin{aligned} \mu_h^{-1}(r) &= \mu_{h_1}^{-1}(r); \\ \mu_k^{-1}(\mu_h^{-1}(r)) &= \mu_{h_1}^{-1}(\mu_k^{-1}(r)) = \mu_{h_2}^{-1}(r); \\ \mu_h^{-1}(\mu_k^{-1}(\mu_h^{-1}(r))) &= \mu_{h_2}^{-1}(\mu_{h_2}^{-1}(r)) = \mu_{h_3}^{-1}(r); \\ &\dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \\ &\dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \\ \mu_h^{-1}(\dots\dots\dots(\mu_h^{-1}(\mu_k^{-1}(\mu_h^{-1}(r)))))) &= \mu_{h_n}^{-1}(r) \text{ when } n \text{ is odd and} \\ \mu_k^{-1}(\dots\dots\dots(\mu_h^{-1}(\mu_k^{-1}(\mu_h^{-1}(r)))))) &= \mu_{h_n}^{-1}(r) \text{ when } n \text{ is even.} \end{aligned}$$

Obviously $\mu_{h_n}^{-1}(r)$ is an increasing functions of r .

The main aim of this paper is to prove some results related to the growth rates of iterated entire functions on the basis of maximum term using the idea of the relative order of an entire function with respect to another entire function.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [6] *Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f \circ g}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f \left(\frac{\alpha R}{R - r} \mu_g(R) \right) .$$

Lemma 2. [7] *If f and g are any two entire functions. Then for all sufficiently large values of r ,*

$$\mu_{f \circ g}(r) \geq \frac{1}{2} \mu_f \left(\frac{1}{16} \mu_g \left(\frac{r}{4} \right) \right)$$

Lemma 3. [3] *If f be an entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r ,*

$$\mu_f(\alpha r) \geq \beta \mu_f(r) .$$

Lemma 4. *Let f be a non-constant entire function and a, b be real with $b > a > 1$. Then when r is large enough, one has*

$$[\mu_f(r)]^a < \mu_f(r^b) .$$

Proof. Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ be a non-constant entire function. Then for all $r > 0$ we have $\mu_f(r) = \max_{n \geq 0} (|a_n| r^n)$. So we obtain that

$$[\mu_f(r)]^a = \max_{n \geq 0} (|a_n|)^a r^{an} . \tag{1}$$

Further, we get that

$$\mu_f(r^b) = \max_{n \geq 0} |a_n| r^{bn} . \tag{2}$$

As we take the maximum value for large r , therefore $n \neq 0$. Since $b > a > 1$, so the lemma follows from (1) and (2). □

Lemma 5. *Let f, g, h, k be any four entire functions. Also let $\beta = \beta_1 \beta_2$ where $\beta_1 > 1$, $\beta_2 > \frac{\alpha \alpha_1 \beta_1}{\beta_1 - 1}$ for every $\alpha > 1$ and $\alpha_1 > \frac{\alpha}{\alpha - 1}$. Then for all sufficiently large values of r ,*

$$(I) \mu_{h_n}^{-1}(\mu_{f_n}(r)) < \mu_k^{-1}(\mu_g(Ar^\delta)) \text{ when } n \text{ is even}$$

and

$$(II) \mu_{h_n}^{-1}(\mu_{f_n}(r)) < \mu_h^{-1}(\mu_f(Ar^\delta)) \text{ when } n(n \neq 1) \text{ is odd}$$

where

$$\left\{ \begin{array}{l} (i) \delta = m^{n-1} \text{ and } A = \beta^{m+m^2+m^3+\dots+m^{n-1}} \text{ wherever } n \text{ is any integer with } n > 1 \text{ and } 1 \leq \min \{ \rho_h(f), \rho_k(g) \} \leq \max \{ \rho_h(f), \rho_k(g) \} < m; \\ (ii) \delta = m^{\frac{n}{2}-1} \text{ and } A = \beta^{1+2m+2m^2+\dots+2m^{\frac{n}{2}-1}} \text{ wherever } n \text{ is any even integer and } 0 < \rho_h(f) < 1 \leq \rho_k(g) < m; \\ (iii) \delta = m^{\frac{n-1}{2}} \text{ and } \beta^{2m+2m^2+\dots+2m^{\frac{n-1}{2}}} \text{ wherever } n (n \neq 1) \text{ is any odd integer and } 0 < \rho_h(f) < 1 \leq \rho_k(g) < m; \\ (iv) \delta = m^{\frac{n}{2}} \text{ and } A = \beta^{2m+2m^2+2m^3+\dots+2m^{\frac{n}{2}-1}+m^{\frac{n}{2}}} \text{ wherever } n \text{ is any even integer and } 0 < \rho_k(g) < 1 \leq \rho_h(f) < m; \\ (v) \delta = m^{\frac{n-1}{2}} \text{ and } A = \beta^{1+2m+2m^2+\dots+2m^{\frac{n-2}{2}}+m^{\frac{n-1}{2}}} \text{ wherever } n (n \neq 1) \text{ is any odd integer and } 0 < \rho_k(g) < 1 \leq \rho_h(f) < m; \\ (vi) \delta = 1 \text{ and } A = \beta^{n-1} \text{ wherever } n \text{ is any integer with } n > 1 \text{ and } 0 < \max \{ \rho_h(f), \rho_k(g) \} < 1. \end{array} \right.$$

Proof. Case I. Let $1 < \min \{ \rho_h(f), \rho_k(g) \} \leq \max \{ \rho_h(f), \rho_k(g) \} < \infty$. Now we consider m is such that $1 < \min \{ \rho_h(f), \rho_k(g) \} \leq \max \{ \rho_h(f), \rho_k(g) \} < m$. Also suppose that $\rho_h(f) + \varepsilon < m$ and $\rho_k(g) + \varepsilon < m$ respectively where $\varepsilon (> 0)$ is arbitrary. Now in view of Lemma 1, Lemma 3, Lemma 4 and for any even integer n , we get for all sufficiently large values of r that

$$\begin{aligned} \mu_{f_n}(r) &\leq \mu_f(\mu_{g_{n-1}}(\beta r)) \\ \text{i.e., } \mu_h^{-1}(\mu_{f_n}(r)) &\leq \mu_h^{-1}(\mu_f(\mu_{g_{n-1}}(\beta r))) \\ \text{i.e., } \mu_h^{-1}(\mu_{f_n}(r)) &\leq (\mu_{g_{n-1}}(\beta r))^{(\rho_h(f)+\varepsilon)} \tag{3} \\ \text{i.e., } \mu_h^{-1}(\mu_{f_n}(r)) &< \mu_{g_{n-1}}((\beta r)^m) \\ \text{i.e., } \mu_h^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m}}}{\beta}\right)\right) &< \mu_{g_{n-1}}(r) \leq \mu_g(\mu_{f_{n-2}}(\beta r)) \\ \text{i.e., } \mu_{h_2}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m}}}{\beta}\right)\right) &< \mu_k^{-1}(\mu_g(\mu_{f_{n-2}}(\beta r))) \\ \text{i.e., } \mu_{h_2}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m}}}{\beta}\right)\right) &< (\mu_{f_{n-2}}(\beta r))^{(\rho_k(g)+\varepsilon)} \\ \\ \text{i.e., } \mu_{h_2}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m}}}{\beta}\right)\right) &< \mu_{f_{n-2}}((\beta r)^m) \\ \text{i.e., } \mu_{h_2}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m^2}}}{(\beta)^{1+\frac{1}{m}}}\right)\right) &< \mu_{f_{n-2}}(r) \leq \mu_f(\mu_{g_{n-3}}(\beta r)), \end{aligned}$$

and so on. We thus have that, for even n

$$\begin{aligned} &\mu_{h_n}^{-1} \left(\mu_{f_n} \left(\frac{r^{\frac{1}{m^{n-1}}}}{(\beta)^{1+\frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}+\dots+\frac{1}{m^{n-2}}}} \right) \right) < \mu_k^{-1} (\mu_g(r)) \\ \text{i.e., } &\mu_{h_n}^{-1} (\mu_{f_n}(r)) < \mu_k^{-1} \left(\mu_g \left(\left((\beta)^{1+\frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}+\dots+\frac{1}{m^{n-2}} } r \right)^{m^{n-1}} \right) \right) \\ \text{i.e., } &\mu_{h_n}^{-1} (\mu_{f_n}(r)) < \mu_k^{-1} \left(\mu_g \left((\beta)^{m+m^2+m^3+\dots+m^{n-1}} \cdot r^{m^{n-1}} \right) \right). \end{aligned}$$

Similarly, we find that, for odd $n (n \neq 1)$,

$$\mu_{h_n}^{-1} (\mu_{f_n}(r)) < \mu_h^{-1} \left(\mu_f \left((\beta)^{m+m^2+m^3+\dots+m^{n-1}} \cdot r^{m^{n-1}} \right) \right).$$

Hence (i) of the lemma is established.

Case II. Let $0 < \min \{ \rho_h(f), \rho_k(g) \} < 1 \leq \max \{ \rho_h(f), \rho_k(g) \} < \infty$.

Sub case (A). Let $0 < \rho_h(f) < 1 \leq \rho_k(g) < \infty$. Now we consider m is such that $0 < \rho_h(f) < 1 \leq \rho_k(g) < m$. Also suppose that $\rho_h(f) + \varepsilon < 1$ and $\rho_k(g) + \varepsilon < m$ respectively where $\varepsilon (> 0)$ is arbitrary. Now in view of Lemma 1, Lemma 3, Lemma 4 and for any even integer n , we obtain from (3) for all sufficiently large values of r that

$$\begin{aligned} &\mu_{h_n}^{-1} (\mu_{f_n}(r)) \leq \left(\mu_{g_{n-1}}(\beta r) \right)^{(\rho_h(f)+\varepsilon)} < \mu_{g_{n-1}}(\beta r) \\ \text{i.e., } &\mu_{h_n}^{-1} \left(\mu_{f_n} \left(\frac{r}{\beta} \right) \right) < \mu_{g_{n-1}}(r) \leq \mu_g(\mu_{f_{n-2}}(\beta r)) \\ \text{i.e., } &\mu_{h_2}^{-1} \left(\mu_{f_n} \left(\frac{r}{\beta} \right) \right) < \mu_k^{-1} \left(\mu_g(\mu_{f_{n-2}}(\beta r)) \right) \\ \text{i.e., } &\mu_{h_2}^{-1} \left(\mu_{f_n} \left(\frac{r}{\beta} \right) \right) < \left(\mu_{f_{n-2}}(2\beta r) \right)^{(\rho_k(g)+\varepsilon)} < \mu_{f_{n-2}}((\beta r)^m) \\ \text{i.e., } &\mu_{h_2}^{-1} \left(\mu_{f_n} \left(\frac{r^{\frac{1}{m}}}{(\beta)^2} \right) \right) < \mu_{f_{n-2}}(r) \leq \mu_f(\mu_{g_{n-3}}(\beta r)) \\ \text{i.e., } &\mu_{h_3}^{-1} \left(\mu_{f_n} \left(\frac{r^{\frac{1}{m}}}{(\beta)^2} \right) \right) < \left(\mu_{g_{n-3}}(\beta r) \right)^{(\rho_h(f)+\varepsilon)} < \mu_{g_{n-3}}(\beta r) \\ \text{i.e., } &\mu_{h_3}^{-1} \left(\mu_{f_n} \left(\frac{r^{\frac{1}{m}}}{(\beta)^{2+\frac{1}{m}}} \right) \right) < \mu_{g_{n-3}}(r) \leq \mu_g(\mu_{f_{n-4}}(\beta r)), \end{aligned}$$

and so on. We finally arrive at the following inequality when n is even

$$\begin{aligned} &\mu_{h_n}^{-1} \left(\mu_{f_n} \left(\frac{r^{\frac{1}{m^{\frac{n}{2}-1}}}}{(\beta)^{2+\frac{2}{m}+\frac{2}{m^2}+\frac{2}{m^3}+\dots+\frac{2}{m^{\frac{n}{2}-2}+\frac{1}{m^{\frac{n}{2}-1}}}} \right) \right) < \mu_k^{-1} (\mu_g(r)) \\ \text{i.e., } &\mu_{h_n}^{-1} (\mu_{f_n}(r)) < \mu_k^{-1} \left(\mu_g \left(\left((\beta)^{2+\frac{2}{m}+\frac{2}{m^2}+\frac{2}{m^3}+\dots+\frac{2}{m^{\frac{n}{2}-2}+\frac{1}{m^{\frac{n}{2}-1}} } r \right)^{m^{\frac{n}{2}-1}} \right) \right) \\ \text{i.e., } &\mu_{h_n}^{-1} (\mu_{f_n}(r)) < \mu_k^{-1} \left(\mu_g \left((\beta)^{1+2m+2m^2+\dots+2m^{\frac{n}{2}-1}} \cdot r^{m^{\frac{n}{2}-1}} \right) \right). \end{aligned}$$

Similarly, when n is odd and $n \neq 1$ and $0 < \rho_h(f) < 1 \leq \rho_k(g) < m$, we find

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) < \mu_h^{-1}\left(\mu_f\left((\beta)^{2m+2m^2+\dots+2m\frac{n-1}{2}} \cdot r^{m\frac{n-1}{2}}\right)\right).$$

Hence (ii) and (iii) of the lemma are proved.

Sub case (B). Let $0 < \rho_k(g) < 1 \leq \rho_h(f) < \infty$. Now we consider m is such that $0 < \rho_k(g) < 1 \leq \rho_h(f) < m$. Also suppose that $\rho_k(g) + \varepsilon < 1$ and $\rho_h(f) + \varepsilon < m$ respectively where $\varepsilon (> 0)$ is arbitrary. Now in view of Lemma 1, Lemma 3, Lemma 4 and for any even integer n , we get from (3) for all sufficiently large values of r that

$$\begin{aligned} \mu_h^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m}}}{\beta}\right)\right) &< \mu_{g_{n-1}}(r) \leq \mu_g(\mu_{f_{n-2}}(\beta r)) \\ \text{i.e., } \mu_{h_2}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m}}}{\beta}\right)\right) &< \mu_k^{-1}\left(\mu_g(\mu_{f_{n-2}}(\beta r))\right) \\ \text{i.e., } \mu_{h_2}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m}}}{\beta}\right)\right) &< (\mu_{f_{n-2}}(\beta r))^{\rho_k(g)+\varepsilon} < \mu_{f_{n-2}}(\beta r) \\ \text{i.e., } \mu_{h_2}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m}}}{(\beta)^{1+\frac{1}{m}}}\right)\right) &< \mu_{f_{n-2}}(r) \leq \mu_f(\mu_{g_{n-3}}(\beta r)) \\ \text{i.e., } \mu_{h_3}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m}}}{(\beta)^{1+\frac{1}{m}}}\right)\right) &< (\mu_{g_{n-3}}(\beta r))^{\rho_h(f)+\varepsilon} < \mu_{g_{n-3}}((\beta r)^m) \\ \text{i.e., } \mu_{h_3}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{m^2}}}{(\beta)^{1+\frac{2}{m}}}\right)\right) &< \mu_{g_{n-3}}(r) \leq \mu_g(\mu_{f_{n-4}}(\beta r)), \end{aligned}$$

and so on.

We finally have the following inequality when n is even

$$\begin{aligned} \mu_{h_n}^{-1}\left(\mu_{f_n}\left(\frac{r^{\frac{1}{\frac{n}{2}}}}{(\beta)^{1+\frac{2}{m}+\frac{2}{m^2}+\frac{2}{m^3}+\dots+\frac{2}{m^{\frac{n}{2}-1}}}}\right)\right) &< \mu_k^{-1}(\mu_g(r)) \\ \text{i.e., } \mu_{h_n}^{-1}(\mu_{f_n}(r)) &< \mu_k^{-1}\left(\mu_g\left(\left((\beta)^{1+\frac{2}{m}+\frac{2}{m^2}+\frac{2}{m^3}+\dots+\frac{2}{m^{\frac{n}{2}-1}}}\right) r\right)^{m^{\frac{n}{2}}}\right) \\ \text{i.e., } \mu_{h_n}^{-1}(\mu_{f_n}(r)) &< \mu_k^{-1}\left(\mu_g\left((\beta)^{2m+2m^2+2m^3+\dots+2m^{\frac{n}{2}-1}+m^{\frac{n}{2}}}\right) \cdot r^{m^{\frac{n}{2}}}\right). \end{aligned}$$

Likewise, when n is odd ($n \neq 1$) and $0 < \rho_k(g) < 1 \leq \rho_h(f) < m$, we get

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) < \mu_h^{-1}\left(\mu_f\left((\beta)^{1+2m+2m^2+\dots+2m\frac{n-2}{2}+m\frac{n-1}{2}} \cdot r^{m\frac{n-1}{2}}\right)\right).$$

Hence (iv) and (v) of the lemma are established.

Case III. Let $0 < \max\{\rho_h(f), \rho_k(g)\} < 1$.

In this case we can choose an arbitrary $\varepsilon (> 0)$ in such a manner so that $\rho_h(f) + \varepsilon < 1$ and $\rho_k(g) + \varepsilon < 1$ hold. Now reasoning similarly as in the proof stated above one can easily deduce the conclusion of (vi) of lemma, so its proof is omitted.

This completes the proof of the lemma. \square

Lemma 6. *Let f, g, h, k be any four entire functions. Also let $\beta = 4\beta_1$ where $\beta_1 > 16\beta_2$ for every $\beta_2 > 2$. Then for all sufficiently large values of r ,*

$$(I) \mu_{h_n}^{-1}(\mu_{f_n}(r)) > \mu_k^{-1}\left(\mu_g\left(Ar^{\frac{1}{\delta}}\right)\right) \text{ when } n \text{ is even}$$

and

$$(II) \mu_{h_n}^{-1}(\mu_{f_n}(r)) > \mu_h^{-1}\left(\mu_f\left(Ar^{\frac{1}{\delta}}\right)\right) \text{ when } n (n \neq 1) \text{ is odd}$$

where

$$\left\{ \begin{array}{l} (i) \delta = m^{n-1} \text{ and } A = \beta^{\frac{m^{n-1}}{1+m+m^2+\dots+m^{n-2}}} \text{ whenever } n \text{ is any integer with } n > 1 \text{ and } \\ \frac{1}{m} < \min\{\lambda_h(f), \lambda_k(g)\} \leq \max\{\lambda_h(f), \lambda_k(g)\} < 1; \\ (ii) \delta = m^{\frac{n}{2}-1} \text{ and } A = \beta^{\frac{m^{\frac{n}{2}-1}}{2+2m+2m^2+\dots+2m^{\frac{n}{2}-2}+m^{\frac{n}{2}-1}}} \text{ whenever } n \text{ is any even integer} \\ \text{and } 0 < \lambda_k(g) \leq 1 < \lambda_h(f) < \infty; \\ (iii) \delta = m^{\frac{n-1}{2}} \text{ and } A = \beta^{\frac{m^{\frac{n-1}{2}}}{2+2m+2m^2+\dots+2m^{\frac{n-1}{2}-1}}} \text{ whenever } n (n \neq 1) \text{ is any odd integer} \\ \text{and } 0 < \lambda_k(g) \leq 1 < \lambda_h(f) < \infty;; \\ (iv) \delta = m^{\frac{n}{2}} \text{ and } A = \beta^{\frac{m^{\frac{n}{2}}}{1+2m+2m^2+\dots+2m^{\frac{n}{2}-1}}} \text{ whenever } n \text{ is any even integer and} \\ 0 < \lambda_h(f) \leq 1 < \lambda_k(g) < \infty; \\ (v) \delta = m^{\frac{n-1}{2}} \text{ and } A = \beta^{\frac{m^{\frac{n-1}{2}}}{1+2m+2m^2+\dots+m^{\frac{n-1}{2}}}} \text{ whenever } n (n \neq 1) \text{ is any odd integer} \\ 0 < \lambda_h(f) \leq 1 < \lambda_k(g) < \infty; \\ (vi) \delta = 1 \text{ and } A = \beta^{\frac{1}{n-1}} \text{ whenever } n \text{ is any integer with } n > 1 \text{ and} \\ 1 < \min\{\lambda_h(f), \lambda_k(g)\} < \infty . \end{array} \right.$$

Proof. Case I. Let $0 < \min\{\lambda_h(f), \lambda_k(g)\} \leq \max\{\lambda_h(f), \lambda_k(g)\} < 1$. Now we consider that m is such that $\frac{1}{m} < \min\{\lambda_h(f), \lambda_k(g)\} \leq \max\{\lambda_h(f), \lambda_k(g)\} < 1$. Also suppose that $\lambda_h(f) - \varepsilon > \frac{1}{m}$ and $\lambda_k(g) - \varepsilon > \frac{1}{m}$ respectively where $\varepsilon (> 0)$ is arbitrary. Now in view of Lemma 2, Lemma 3, Lemma 4 and for any even integer n , we get for all sufficiently large values of r that

$$\begin{aligned} \mu_{f_n}(r) &\geq \mu_f\left(\mu_{g_{n-1}}\left(\frac{r}{\beta}\right)\right) \\ \text{i.e., } \mu_h^{-1}(\mu_{f_n}(r)) &\geq \mu_h^{-1}\left(\mu_f\left(\mu_{g_{n-1}}\left(\frac{r}{\beta}\right)\right)\right) \\ \text{i.e., } \mu_h^{-1}(\mu_{f_n}(r)) &\geq \left(\mu_{g_{n-1}}\left(\frac{r}{\beta}\right)\right)^{(\lambda_h(f)-\varepsilon)} > \mu_{g_{n-1}}\left(\left(\frac{r}{\beta}\right)^{\frac{1}{m}}\right) \quad (4) \\ \text{i.e., } \mu_h^{-1}(\mu_{f_n}(\beta r^m)) &> \mu_{g_{n-1}}(r) \geq \mu_g\left(\mu_{f_{n-2}}\left(\frac{r}{\beta}\right)\right) \end{aligned}$$

$$\begin{aligned}
 i.e., \mu_{h_2}^{-1}(\mu_{f_n}(\beta r^m)) &> \mu_k^{-1}\left(\mu_g\left(\mu_{f_{n-2}}\left(\frac{r}{\beta}\right)\right)\right) \\
 i.e., \mu_{h_2}^{-1}(\mu_{f_n}(\beta r^m)) &> \left(\mu_{f_{n-2}}\left(\frac{r}{\beta}\right)\right)^{(\lambda_k(g)-\varepsilon)} > \mu_{f_{n-2}}\left(\left(\frac{r}{\beta}\right)^{\frac{1}{m}}\right) \\
 i.e., \mu_{h_2}^{-1}(\mu_{f_n}(\beta^{1+m} \cdot r^{m^2})) &> \mu_{f_{n-2}}(r) \geq \mu_f\left(\mu_{g_{n-3}}\left(\frac{r}{\beta}\right)\right),
 \end{aligned}$$

and so on. We thus have that, for even n

$$\begin{aligned}
 \mu_{h_n}^{-1}(\mu_{f_n}(\beta^{1+m+m^2+\dots+m^{n-2}} \cdot r^{m^{n-1}})) &> \mu_k^{-1}(\mu_g(r)) \\
 i.e., \mu_{h_n}^{-1}(\mu_{f_n}(r)) &> \mu_k^{-1}\left(\mu_g\left(\left(\frac{r}{\beta^{1+m+m^2+\dots+m^{n-2}}}\right)^{\frac{1}{m^{n-1}}}\right)\right) \\
 i.e., \mu_{h_n}^{-1}(\mu_{f_n}(r)) &> \mu_k^{-1}\left(\mu_g\left(\left(\frac{r^{\frac{1}{m^{n-1}}}}{\beta^{\frac{1+m+m^2+\dots+m^{n-2}}{m^{n-1}}}}\right)\right)\right).
 \end{aligned}$$

Similarly, we find that, for odd $n (n \neq 1)$,

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) > \mu_h^{-1}\left(\mu_f\left(\left(\frac{r^{\frac{1}{m^{n-1}}}}{\beta^{\frac{1+m+m^2+\dots+m^{n-2}}{m^{n-1}}}}\right)\right)\right).$$

Hence (i) of the lemma is established.

Case II. Let $0 < \min\{\lambda_h(f), \lambda_k(g)\} \leq 1 < \max\{\lambda_h(f), \lambda_k(g)\} < \infty$.

Sub case (A). Let $0 < \lambda_k(g) \leq 1 < \lambda_h(f) < \infty$. Now we consider m is such that $\frac{1}{m} < \lambda_k(g) \leq 1 < \lambda_h(f) < \infty$. Also suppose that $\lambda_k(g) - \varepsilon > \frac{1}{m}$ and $\lambda_h(f) - \varepsilon > 1$ respectively where $\varepsilon (> 0)$ is arbitrary. Now in view of Lemma 2, Lemma 3, Lemma 4 and for any even integer n , we obtain from (4) for all sufficiently large values of r that

$$\begin{aligned}
 \mu_h^{-1}(\mu_{f_n}(r)) &\geq \left(\mu_{g_{n-1}}\left(\frac{r}{\beta}\right)\right)^{(\lambda_h(f)-\varepsilon)} > \mu_{g_{n-1}}\left(\frac{r}{\beta}\right) \\
 i.e., \mu_h^{-1}(\mu_{f_n}(\beta r)) &> \mu_{g_{n-1}}(r) \geq \mu_g\left(\mu_{f_{n-2}}\left(\frac{r}{\beta}\right)\right) \\
 i.e., \mu_{h_2}^{-1}(\mu_{f_n}(\beta r)) &> \mu_k^{-1}\left(\mu_g\left(\mu_{f_{n-2}}\left(\frac{r}{\beta}\right)\right)\right) \\
 i.e., \mu_{h_2}^{-1}(\mu_{f_n}(\beta r)) &> \left(\mu_{f_{n-2}}\left(\frac{r}{\beta}\right)\right)^{(\lambda_k(g)-\varepsilon)} > \mu_{f_{n-2}}\left(\left(\frac{r}{\beta}\right)^{\frac{1}{m}}\right) \\
 i.e., \mu_{h_2}^{-1}(\mu_{f_n}(\beta^2 \cdot r^m)) &> \mu_{f_{n-2}}(r) \geq \mu_f\left(\mu_{g_{n-3}}\left(\frac{r}{\beta}\right)\right) \\
 i.e., \mu_{h_3}^{-1}(\mu_{f_n}(\beta^2 \cdot r^m)) &> \left(\mu_{g_{n-3}}\left(\frac{r}{\beta}\right)\right)^{(\lambda_h(f)-\varepsilon)} > \mu_{g_{n-3}}\left(\frac{r}{\beta}\right) \\
 i.e., \mu_{h_3}^{-1}(\mu_{f_n}(\beta^{2+m} \cdot r^m)) &> \mu_{g_{n-3}}\left(\frac{r}{\beta}\right) > \mu_g\left(\mu_{f_{n-4}}\left(\frac{r}{\beta}\right)\right),
 \end{aligned}$$

and so on. We finally arrive at the following inequality when n is even

$$\mu_{h_n}^{-1}(\mu_{f_n}(\beta^{2+2m+2m^2+\dots+2m^{\frac{n}{2}-2}+m^{\frac{n}{2}-1} \cdot r^{m^{\frac{n}{2}-1}})) > \mu_k^{-1}(\mu_g(r))$$

$$\begin{aligned}
 \text{i.e., } \mu_{h_n}^{-1}(\mu_{f_n}(r)) &> \mu_k^{-1} \left(\mu_g \left(\left(\frac{r}{\beta^{2+2m+2m^2+\dots+2m^{\frac{n}{2}-2}+m^{\frac{n}{2}-1}} \right)^{\frac{1}{m^{\frac{n}{2}-1}}} \right) \right) \\
 \text{i.e., } \mu_{h_n}^{-1}(\mu_{f_n}(r)) &> \mu_k^{-1} \left(\mu_g \left(\left(\frac{r^{\frac{1}{m^{\frac{n}{2}-1}}}}{\beta^{\frac{2+2m+2m^2+\dots+2m^{\frac{n}{2}-2}+m^{\frac{n}{2}-1}}{m^{\frac{n}{2}-1}}}} \right) \right) \right).
 \end{aligned}$$

Similarly, when n is odd and $n \neq 1$ and $\frac{1}{m} < \lambda_k(g) \leq 1 < \lambda_h(f) < \infty$, we find

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) > \mu_h^{-1} \left(\mu_f \left(\frac{r^{\frac{1}{m^{\frac{n-1}{2}}}}}{\beta^{\frac{2+2m+2m^2+\dots+2m^{\frac{n-1}{2}}}{m^{\frac{n-1}{2}}}}} \right) \right).$$

Hence (ii) and (iii) of the lemma are proved.

Subcase (B). Let $0 < \lambda_h(f) \leq 1 < \lambda_k(g) < \infty$. Now we consider m is such that $\frac{1}{m} < \lambda_h(f) \leq 1 < \lambda_k(g) < \infty$. Also suppose that $\lambda_k(g) - \varepsilon > 1$ and $\lambda_h(f) - \varepsilon > \frac{1}{m}$ respectively where $\varepsilon (> 0)$ is arbitrary. Now in view of Lemma 2, Lemma 3, Lemma 4 and for any even integer n , we obtain from (4) for all sufficiently large values of r that

$$\begin{aligned}
 \text{i.e., } \mu_h^{-1}(\mu_{f_n}(\beta r^m)) &> \mu_{g_{n-1}}(r) \geq \mu_g \left(\mu_{f_{n-2}} \left(\frac{r}{\beta} \right) \right) \\
 \text{i.e., } \mu_{h_2}^{-1}(\mu_{f_n}(\beta r^m)) &> \mu_k^{-1} \left(\mu_g \left(\mu_{f_{n-2}} \left(\frac{r}{\beta} \right) \right) \right) \\
 \text{i.e., } \mu_{h_2}^{-1}(\mu_{f_n}(\beta r^m)) &> \left(\mu_{f_{n-2}} \left(\frac{r}{\beta} \right) \right)^{(\lambda_k(g)-\varepsilon)} > \mu_{f_{n-2}} \left(\frac{r}{\beta} \right) \\
 \text{i.e., } \mu_{h_2}^{-1}(\mu_{f_n}(\beta^{1+m} \cdot r^m)) &> \mu_{f_{n-2}}(r) \geq \mu_f \left(\mu_{g_{n-3}} \left(\frac{r}{\beta} \right) \right) \\
 \text{i.e., } \mu_{h_3}^{-1}(\mu_{f_n}(\beta^{1+m} \cdot r^m)) &> \left(\mu_{g_{n-3}} \left(\frac{r}{\beta} \right) \right)^{(\lambda_h(f)-\varepsilon)} > \mu_{g_{n-3}} \left(\left(\frac{r}{\beta} \right)^{\frac{1}{m}} \right) \\
 \text{i.e., } \mu_{h_3}^{-1}(\mu_{f_n}(\beta^{1+2m} \cdot r^{m^2})) &> \mu_{g_{n-3}}(r) > \mu_g \left(\mu_{f_{n-4}} \left(\frac{r}{\beta} \right) \right),
 \end{aligned}$$

and so on. We thus have that, for even n

$$\begin{aligned}
 \mu_{h_n}^{-1} \left(\mu_{f_n} \left(\beta^{1+2m+2m^2+\dots+2m^{\frac{n}{2}-1} \cdot r^{m^{\frac{n}{2}}} \right) \right) &> \mu_k^{-1}(\mu_g(r)) \\
 \text{i.e., } \mu_{h_n}^{-1}(\mu_{f_n}(r)) &> \mu_k^{-1} \left(\mu_g \left(\left(\frac{r}{\beta^{1+2m+2m^2+\dots+2m^{\frac{n}{2}-1}}} \right)^{\frac{1}{m^{\frac{n}{2}}}} \right) \right) \\
 \text{i.e., } \mu_{h_n}^{-1}(\mu_{f_n}(r)) &> \mu_k^{-1} \left(\mu_g \left(\left(\frac{r^{\frac{1}{m^{\frac{n}{2}}}}}{\beta^{\frac{1+2m+2m^2+\dots+2m^{\frac{n}{2}-1}}{m^{\frac{n}{2}}}}} \right) \right) \right).
 \end{aligned}$$

Likewise, when n is odd ($n \neq 1$) and $\frac{1}{m} < \lambda_h(f) \leq 1 < \lambda_k(g) < \infty$, we get

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) > \mu_h^{-1} \left(\mu_f \left(\left(\frac{r m^{\frac{n-1}{2}}}{\beta \frac{1+2m+2m^2+\dots+m^{\frac{n-1}{2}}}{m^{\frac{n-1}{2}}}} \right) \right) \right).$$

Hence (iv) and (v) of the lemma are established.

Case III. Let $1 < \min\{\rho_h(f), \rho_k(g)\} < \infty$.

In this case we can choose an arbitrary $\varepsilon (> 0)$ in such a manner so that $\lambda_k(g) - \varepsilon > 1$ and $\lambda_h(f) - \varepsilon > 1$ hold. Now reasoning similarly as in the proof stated above one can easily deduce the conclusion of (vi) of lemma, so its proof is omitted.

This completes the proof of the lemma. \square

3. Main Results

In this section we present the main results of the paper.

Theorem 1. Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Also let γ be a positive continuous function defined on $[0, +\infty)$ increasing to $+\infty$ as $r \rightarrow +\infty$. Then for every real number α ,

$$\lim_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\{\log \mu_h^{-1}(\mu_f(\exp \gamma(r)))\}^{1+\alpha}} = \infty,$$

when $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$ and n is any integer such that $n > 1$.

Proof. First let us consider n to be an even integer. If α be such that $1 + \alpha \leq 0$ then the theorem is trivial. So we suppose that $1 + \alpha > 0$. Now it follows from the first part of Lemma 6, for all sufficiently large values of r that

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) > \left(Ar^{\frac{1}{\delta}} \right)^{\lambda_k(g) - \varepsilon}, \quad (5)$$

where A and δ satisfy the conditions of Lemma 6.

Again from the definition of $\rho_h(f)$, it follows for all sufficiently large values of r that

$$\{\log \mu_h^{-1}(\mu_f(\exp \gamma(r)))\}^{1+\alpha} \leq (\rho_h(f) + \varepsilon)^{1+\alpha} (\gamma(r))^{1+\alpha}. \quad (6)$$

Now from (5) and (6), it follows for all sufficiently large values of r that

$$\frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\{\log \mu_h^{-1}(\mu_f(\exp \gamma(r)))\}^{1+\alpha}} > \frac{\left(Ar^{\frac{1}{\delta}} \right)^{\lambda_k(g) - \varepsilon}}{(\rho_h(f) + \varepsilon)^{1+\alpha} (\gamma(r))^{1+\alpha}}.$$

Since $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$, therefore $\frac{r^{\frac{\lambda_k(g) - \varepsilon}{\delta}}}{(\gamma(r))^{1+\alpha}} \rightarrow +\infty$ as $r \rightarrow +\infty$, then from above it follows that

$$\liminf_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\{\log \mu_h^{-1}(\mu_f(\exp \gamma(r)))\}^{1+\alpha}} = \infty \text{ for any even number } n.$$

Similarly, with the help of the second part of Lemma 6 one can easily derive the same conclusion for any odd integer $n (\neq 1)$.

Thus the theorem follows from above. \square

Remark 1. Theorem 1 is still valid with “limit superior” instead of “limit” if we replace the condition “ $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ ” by “ $0 < \lambda_h(f) < \infty$ ”.

In the line of Theorem 1, one may state the following theorem without its proof:

Theorem 2. *Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Also let γ be a positive continuous function defined on $[0, +\infty)$ increasing to $+\infty$ as $r \rightarrow +\infty$. Then for every real number α ,*

$$\lim_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\{\log \mu_k^{-1}(\mu_g(\exp \gamma(r)))\}^{1+\alpha}} = \infty,$$

when $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = 0$ and n is any integer such that $n > 1$.

Remark 2. *In Theorem 2 if we take the condition $0 < \lambda_k(g) < \infty$ instead of $0 < \lambda_k(g) \leq \rho_k(g) < \infty$, then also Theorem 2 remains true with “limit superior” in place of “limit”.*

Theorem 3. *Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Also let γ be a positive continuous function defined on $[0, +\infty)$ increasing to $+\infty$ as $r \rightarrow +\infty$. Then for each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r \rightarrow +\infty} \frac{(\mu_{h_n}^{-1}(\mu_{f_n}(r)))^{1+\alpha}}{\log \mu_h^{-1}(\mu_f(\exp \gamma(r)))} = 0,$$

when $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$ and n is any integer such that $n > 1$.

Proof. If $1 + \alpha \leq 0$, then the theorem is obvious. We consider that $1 + \alpha > 0$. Now for any even integer n , it follows from the first part of Lemma 5 for all sufficiently large values of r that

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) < (Ar^\delta)^{(\rho_k(g)+\varepsilon)}, \tag{7}$$

where A and δ satisfy the conditions of Lemma 5.

Again for all sufficiently large values of r we get that

$$\log \mu_h^{-1}(\mu_f(\exp \gamma(r))) \geq (\lambda_h(f) - \varepsilon) \gamma(r). \tag{8}$$

Hence for all sufficiently large values of r , we obtain from (7) and (8) that

$$\frac{(\mu_{h_n}^{-1}(\mu_{f_n}(r)))^{1+\alpha}}{\log \mu_h^{-1}(\mu_f(\exp \gamma(r)))} < \frac{(Ar^\delta)^{(\rho_k(g)+\varepsilon)}}{(\lambda_h(f) - \varepsilon) \gamma(r)}. \tag{9}$$

Since $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$, therefore $\frac{r^{\delta(\rho_k(g)+\varepsilon)(1+\alpha)}}{\gamma(r)} \rightarrow +\infty$ as $r \rightarrow +\infty$. So from (9) we obtain that

$$\lim_{r \rightarrow +\infty} \frac{(\mu_{h_n}^{-1}(\mu_{f_n}(r)))^{1+\alpha}}{\log \mu_h^{-1}(\mu_f(\exp \gamma(r)))} = 0 \text{ for any even number } n.$$

Similarly, with the help of the second part of Lemma 5 one can easily derive the same conclusion for any odd integer $n (\neq 1)$.

This proves the theorem. □

Remark 3. *In Theorem 3 if we take the condition $0 < \rho_h(f) < \infty$ instead of $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, the theorem remains true with “limit inferior” in place of “limit”.*

In view of Theorem 3, the following theorem can be carried out :

Theorem 4. Let f, g, k and h be any four entire functions such that $0 < \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Also let γ be a positive continuous function defined on $[0, +\infty)$ increasing to $+\infty$ as $r \rightarrow +\infty$. Then for each $\alpha \in (-\infty, \infty)$,

$$\lim_{r \rightarrow +\infty} \frac{(\mu_{h_n}^{-1}(\mu_{f_n}(r)))^{1+\alpha}}{\log \mu_k^{-1}(\mu_g(\exp \gamma(r)))} = 0,$$

when $\lim_{r \rightarrow +\infty} \frac{\log \gamma(r)}{\log r} = +\infty$ and n is any integer such that $n > 1$.

The proof is omitted.

Remark 4. In Theorem 4 if we take the condition $0 < \rho_k(g) < \infty$ instead of $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ then the theorem remains true with “limit inferior” in place of “limit”.

Theorem 5. Let f, g, k and h be any four entire functions such that $\lambda_k(g) < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Then for any even number n ,

$$\liminf_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\mu_h^{-1}(\mu_f(r^\delta))} = 0,$$

where δ satisfies the conditions of Lemma 5.

Proof. From the first part of Lemma 5, we obtain for a sequence of values of r tending to infinity that

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) < (Ar^\delta)^{(\lambda_k(g)+\varepsilon)}. \quad (10)$$

Again from the definition of relative order, we obtain for all sufficiently large values of r that

$$\mu_h^{-1}(\mu_f(r^\delta)) \geq r^{\delta(\lambda_h(f)-\varepsilon)}. \quad (11)$$

Now in view of (10) and (11), we get for a sequence of values of r tending to infinity that

$$\frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\mu_h^{-1}(\mu_f(r^\delta))} < \frac{(Ar^\delta)^{(\lambda_k(g)+\varepsilon)}}{r^{\delta(\lambda_h(f)-\varepsilon)}}. \quad (12)$$

Now as $\lambda_k(g) < \lambda_h(f)$, we can choose $\varepsilon (> 0)$ in such a way that $\lambda_k(g) + \varepsilon < \lambda_h(f) - \varepsilon$ and the theorem follows from (12). \square

Remark 5. If we take $0 < \rho_k(g) < \lambda_h(f) \leq \rho_h(f) < \infty$ instead of “ $\lambda_k(g) < \lambda_h(f) \leq \rho_h(f) < \infty$ and $\rho_k(g) < \infty$ ” and the other conditions remain the same, the conclusion of Theorem 5 remains valid with “limit inferior” replaced by “limit”.

Theorem 6. Let f, g, k and h be any four entire functions such that $\lambda_h(f) < \lambda_k(g) \leq \rho_k(g) < \infty$ and $0 < \rho_h(f) < \infty$. Then for any odd number $n (\neq 1)$,

$$\liminf_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\mu_k^{-1}(\mu_g(r^\delta))} = 0,$$

where δ satisfies the conditions of Lemma 5.

The proof of Theorem 6 is omitted as it can be carried out in the line of Theorem 5 and with the help of the second part of Lemma 5.

Remark 6. If we consider $0 < \rho_h(f) < \lambda_k(g) \leq \rho_k(g) < \infty$ instead of " $\lambda_h(f) < \lambda_k(g) \leq \rho_k(g) < \infty$ and $\rho_h(f) < \infty$ " and the other conditions remain the same, the conclusion of Theorem 5 remains valid with "limit inferior" replaced by "limit".

Theorem 7. Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Then

$$(i) \limsup_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(r^\delta))} \leq \frac{\rho_k(g)}{\lambda_h(f)} \text{ when } n \text{ is even,}$$

and

$$(ii) \limsup_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(r^\delta))} \leq \frac{\rho_h(f)}{\lambda_h(f)} \text{ when } n (\neq 1) \text{ is any odd integer}$$

where δ satisfies the conditions of Lemma 5.

Proof. From the first part of Lemma 5, it follows for all sufficiently large values of r that $\mu_{h_n}^{-1}(\mu_{f_n}(r)) < \mu_k^{-1}(\mu_g(Ar^\delta))$

$$\frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(r^\delta))} < \frac{\log \mu_k^{-1}(\mu_g(Ar^\delta))}{\log \mu_h^{-1}(\mu_f(r^\delta))}$$

$$i.e., \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(r^\delta))} < \frac{\log \mu_k^{-1}(\mu_g(Ar^\delta))}{\log(Ar^\delta)} \cdot \frac{\log(Ar^\delta)}{\log \mu_h^{-1}(\mu_f(r^\delta))}$$

$$i.e., \limsup_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(r^\delta))} \leq \limsup_{r \rightarrow +\infty} \frac{\log \mu_k^{-1}(\mu_g(Ar^\delta))}{\log(Ar^\delta)} \cdot \limsup_{r \rightarrow +\infty} \frac{\log(Ar^\delta)}{\log \mu_h^{-1}(\mu_f(r^\delta))}$$

$$i.e., \limsup_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(r^\delta))} \leq \rho_k(g) \cdot \frac{1}{\lambda_h(f)} = \frac{\rho_k(g)}{\lambda_h(f)}.$$

Thus the first part of theorem follows from above.

Likewise, with the help of the second part of Lemma 5 one can easily derive conclusion of the second part of theorem.

This proves the theorem. □

Theorem 8. Let f, g, k and h be any four entire functions such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty$ and $0 < \rho_h(f) < \infty$. Then

$$(i) \limsup_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_k^{-1}(\mu_g(r^\delta))} \leq \frac{\rho_k(g)}{\lambda_k(g)} \text{ when } n \text{ is even,}$$

and

$$(ii) \limsup_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_k^{-1}(\mu_g(r^\delta))} \leq \frac{\rho_h(f)}{\lambda_k(g)} \text{ when } n (\neq 1) \text{ is any odd integer}$$

where δ satisfies the conditions of Lemma 5.

The proof of Theorem 8 is omitted as it can be carried out in the line of Theorem 7.

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 7 and Theorem 8 respectively and with the help of Lemma 5.

Theorem 9. Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then

$$(i) \liminf_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(r^\delta))} \leq \frac{\lambda_k(g)}{\lambda_h(f)} \text{ when } n \text{ is even,}$$

and

$$(ii) \liminf_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(r^\delta))} \leq 1 \text{ when } n (\neq 1) \text{ is any odd integer}$$

where δ satisfies the conditions of Lemma 5.

Theorem 10. Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then

$$(i) \liminf_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_k^{-1}(\mu_g(r^\delta))} \leq 1 \text{ when } n \text{ is even,}$$

and

$$(ii) \liminf_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_k^{-1}(\mu_g(r^\delta))} \leq \frac{\lambda_h(f)}{\lambda_k(g)} \text{ when } n (\neq 1) \text{ is any odd integer}$$

where δ satisfies the conditions of Lemma 5.

Theorem 11. Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for any even number n ,

$$(i) \liminf_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(r))} \geq \frac{\lambda_k(g)}{\rho_h(f)} \text{ when } 0 < \rho_h(f) < \infty$$

and

$$(ii) \liminf_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_k^{-1}(\mu_g(r))} \geq \frac{\lambda_k(g)}{\rho_k(g)} \text{ when } 0 < \rho_k(g) < \infty,$$

where δ satisfies the conditions of Lemma 6.

Proof. From the first part of Lemma 6, we obtain for all sufficiently large values of r that

$$\log \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta)) > (\lambda_k(g) - \varepsilon) \log r + (\lambda_k(g) - \varepsilon) \log A. \quad (13)$$

Also from the definition of $\rho_h(f)$, we obtain for all sufficiently large values of r that

$$\log \mu_h^{-1}(\mu_f(r)) \leq (\rho_h(f) + \varepsilon) \log r. \quad (14)$$

Analogously, from the definition of $\rho_k(g)$, it follows for all sufficiently large values of r that

$$\log \mu_k^{-1}(\mu_g(r)) \leq (\rho_k(g) + \varepsilon) \log r. \quad (15)$$

Now from (13) and (14), it follows for all sufficiently large values of r that

$$\frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(r))} > \frac{(\lambda_k(g) - \varepsilon) \log r + (\lambda_k(g) - \varepsilon) \log A}{(\rho_h(f) + \varepsilon) \log r}$$

$$\text{i.e., } \liminf_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(r))} \geq \frac{\lambda_k(g)}{\rho_h(f)}. \quad (16)$$

Thus the first part of theorem follows from (16).

Likewise, the conclusion of the second part of theorem can easily be derived from (13) and (15).

Hence the theorem follows. □

Theorem 12. *Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for any odd number $n (\neq 1)$,*

$$(i) \liminf_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(r))} \geq \frac{\lambda_h(f)}{\rho_h(f)} \text{ when } 0 < \rho_h(f) < \infty$$

and

$$(ii) \liminf_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_k^{-1}(\mu_g(r))} \geq \frac{\lambda_h(f)}{\rho_k(g)} \text{ when } 0 < \rho_k(g) < \infty,$$

where δ satisfies the conditions of Lemma 6.

The proofs of Theorem 12 is omitted as it can be carried out in the line of Theorem 11 and with the help of the second part of Lemma 6.

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 11 and Theorem 12 respectively and with the help of Lemma 6.

Theorem 13. *Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$. Then for any even number n ,*

$$(i) \limsup_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(r))} \geq \frac{\rho_k(g)}{\rho_h(f)} \text{ when } 0 < \rho_h(f) < \infty$$

and

$$(ii) \limsup_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_k^{-1}(\mu_g(r))} \geq 1,$$

where δ satisfies the conditions of Lemma 6.

Theorem 14. *Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) < \infty$. Then for any odd number $n (\neq 1)$,*

$$(i) \limsup_{r \rightarrow +\infty} \frac{\log \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_k^{-1}(\mu_g(r))} \geq \frac{\rho_h(f)}{\rho_k(g)} \text{ when } 0 < \rho_k(g) < \infty$$

and

$$(ii) \limsup_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(r))} \geq 1 \text{ when } \rho_h(f) < \infty,$$

where δ satisfies the conditions of Lemma 6.

Theorem 15. *Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $0 < \gamma < \rho_k(g) < \infty$. Then for any even number n ,*

$$\limsup_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(\exp r^\gamma))} = \infty,$$

where δ satisfies the conditions of Lemma 6.

Proof. From the first part of Lemma 6, we get for a sequence of values of r tending to infinity that

$$\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta)) > (Ar)^{(\rho_k(g)-\varepsilon)}. \quad (17)$$

Again from the definition of $\rho_h(f)$, we obtain for all sufficiently large values of r that

$$\log \mu_h^{-1}(\mu_f(\exp r^\gamma)) \leq (\rho_h(f) + \varepsilon) r^\gamma. \quad (18)$$

Now from (17) and (18), it follows for a sequence of values of r tending to infinity that

$$\frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(\exp r^\gamma))} > \frac{(Ar)^{(\rho_k(g)-\varepsilon)}}{(\rho_h(f) + \varepsilon) r^\gamma}. \quad (19)$$

As $\gamma < \rho_k(g)$, we can choose $\varepsilon (> 0)$ in such a way that

$$\gamma < \rho_k(g) - \varepsilon. \quad (20)$$

Thus from (19) and (20) we get that

$$\limsup_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(\exp r^\gamma))} = \infty. \quad (21)$$

Hence the theorem follows from (21). \square

Theorem 16. Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$, $0 < \lambda_k(g) < \infty$ and $0 < \gamma < \rho_k(g) < \infty$. Then for any even number n ,

$$\limsup_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_k^{-1}(\mu_g(\exp r^\gamma))} = \infty,$$

where δ satisfies the conditions of Lemma 6.

Proof. Let $0 < \gamma < \gamma_0 < \rho_k(g)$. Then from (21), we obtain for a sequence of values of r tending to infinity and $A > 1$ that

$$\begin{aligned} \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta)) &> A \log \mu_h^{-1}(\mu_f(\exp r^{\gamma_0})) \\ \text{i.e., } \mu_{h_n}^{-1}(\mu_{f_n}(r^\delta)) &> A(\lambda_h(f) - \varepsilon) r^{\gamma_0}. \end{aligned} \quad (22)$$

Again from the definition of $\rho_k(g)$, we obtain for all sufficiently large values of r that

$$\log \mu_k^{-1}(\mu_g(\exp r^\gamma)) \leq (\rho_k(g) + \varepsilon) r^\gamma. \quad (23)$$

So combining (22) and (23), we obtain for a sequence of values of r tending to infinity that

$$\frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_k^{-1}(\mu_g(\exp r^\gamma))} > \frac{A(\lambda_h(f) - \varepsilon) r^{\gamma_0}}{(\rho_k(g) + \varepsilon) r^\gamma}. \quad (24)$$

Since $\gamma_0 > \gamma$, from (24) it follows that

$$\limsup_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_k^{-1}(\mu_g(\exp r^\gamma))} = \infty.$$

Thus the theorem follows. \square

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 15 and Theorem 16 respectively and with the help of the second part of Lemma 6.

Theorem 17. Let f, g, k and h be any four entire functions such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty, 0 < \lambda_h(f) < \infty$ and $0 < \gamma < \rho_h(f) < \infty$. Then for any odd number $n (\neq 1)$,

$$\limsup_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_h^{-1}(\mu_f(\exp r^\gamma))} = \infty,$$

where δ satisfies the conditions of Lemma 6.

Theorem 18. Let f, g, k and h be any four entire functions such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty, 0 < \lambda_h(f) < \infty$ and $0 < \gamma < \rho_h(f) < \infty$. Then for any odd number $n (\neq 1)$,

$$\limsup_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r^\delta))}{\log \mu_k^{-1}(\mu_g(\exp r^\gamma))} = \infty,$$

where δ satisfies the conditions of Lemma 6.

Theorem 19. Let f, g, k and h be any four entire functions such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty, 0 < \rho_k(g) < \infty$ and $\lambda_k(g) < \gamma < \infty$. Then for any even number n ,

$$\liminf_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(\exp r^{\delta\gamma}))} = 0,$$

where δ satisfies the conditions of Lemma 5.

Proof. From the first part of Lemma 5, it follows for a sequence of values of r tending to infinity that

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) < (Ar^\delta)^{(\lambda_k(g)+\varepsilon)}. \tag{25}$$

Again for all sufficiently large values of r we get that

$$\log \mu_h^{-1}(\mu_f(\exp r^{\delta\gamma})) \geq (\lambda_h(f) - \varepsilon) r^{\delta\gamma}. \tag{26}$$

Now from (25) and (26), it follows for a sequence of values of r tending to infinity that

$$\frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(\exp r^{\delta\gamma}))} \leq \frac{(Ar^\delta)^{(\lambda_k(g)+\varepsilon)}}{(\lambda_h(f) - \varepsilon) r^{\delta\gamma}}. \tag{27}$$

As $\lambda_k(g) < \gamma$, we can choose $\varepsilon (> 0)$ in such a way that

$$\lambda_k(g) + \varepsilon < \gamma. \tag{28}$$

Thus the theorem follows from (27) and (28). □

In the line of Theorem 19, we may state the following theorem without its proof:

Theorem 20. Let f, g, k and h be any four entire functions such that $0 < \rho_h(f) < \infty, 0 < \rho_k(g) < \infty$ and $\lambda_k(g) < \gamma < \infty$. Then for any even number n ,

$$\liminf_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_k^{-1}(\mu_g(\exp r^{\delta\gamma}))} = 0,$$

where δ satisfies the conditions of Lemma 5.

Theorem 21. Let f, g, k and h be any four entire functions such that $0 < \lambda_k(g) \leq \rho_k(g) < \infty, 0 < \rho_h(f) < \infty$ and $\lambda_h(f) < \gamma < \infty$. Then for any odd number $n (\neq 1)$,

$$\liminf_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_k^{-1}(\mu_g(\exp r^{\delta\gamma}))} = 0,$$

where δ satisfies the conditions of Lemma 5.

Theorem 22. Let f, g, k and h be any four entire functions such that $0 < \rho_k(g) < \infty, 0 < \rho_h(f) < \infty$ and $\lambda_h(f) < \gamma < \infty$. Then for any odd number $n (\neq 1)$,

$$\liminf_{r \rightarrow +\infty} \frac{\mu_{h_n}^{-1}(\mu_{f_n}(r))}{\log \mu_h^{-1}(\mu_f(\exp r^{\delta\gamma}))} = 0,$$

where δ satisfies the conditions of Lemma 5.

We omit the proofs of Theorem 21 and Theorem 22 as those can be carried out in the line of Theorem 19 and Theorem 20 respectively and with the help of the second part of Lemma 5.

We omit the proofs of Theorem 21 and Theorem 22 as those can be carried out in the line of Theorem 19 and Theorem 20 respectively and with the help of the second part of Lemma 5.

Theorem 23. Let F, G, H, K, f, g, h and k be any eight entire functions such that $0 < \lambda_H(F) < \infty, 0 < \lambda_K(G) < \infty, 0 < \rho_h(f) < \infty$ and $0 < \rho_k(g) < \infty$. Then for any two integers $m (\neq 1)$ and $n (\neq 1)$

$$(i) \lim_{r \rightarrow +\infty} \frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r))} = \infty$$

and

$$(ii) \lim_{r \rightarrow +\infty} \frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_k^{-1}(\mu_g(r))} = \infty,$$

when

$$\left\{ \begin{array}{l} \delta^2 \rho_k(g) < \lambda_K(G) \text{ for } m, n \text{ both even} \\ \delta^2 \rho_h(f) < \lambda_H(F) \text{ for } m(\neq 1), n(\neq 1) \text{ both odd} \\ \delta^2 \rho_h(f) < \lambda_K(G) \text{ for } m \text{ even and } n(\neq 1) \text{ odd} \\ \delta^2 \rho_k(g) < \lambda_H(F) \text{ for } m(\neq 1) \text{ odd and } n \text{ even,} \end{array} \right. \tag{29}$$

where δ satisfies the conditions of Lemma 5.

Proof. We have from the definition of relative order and for all sufficiently large values of r that

$$\log \mu_h^{-1}(\mu_f(r)) \leq (\rho_h(f) + \varepsilon) \log r. \tag{30}$$

Case I. Let m and n are any two even numbers.

Therefore in view of first part of Lemma 5, we get for all sufficiently large values of r that

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) < (Ar^\delta)^{(\rho_k(g)+\varepsilon)}. \tag{31}$$

So from (30) and (31) it follows for all sufficiently large values of r that

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r)) < (Ar^\delta)^{(\rho_k(g)+\varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r. \tag{32}$$

Also from first part of Lemma 6, we obtain for all sufficiently large values of r that

$$\mu_{H_m}^{-1}(\mu_{F_m}(r)) > \left(Ar^{\frac{1}{\delta}} \right)^{(\lambda_K(G) - \varepsilon)} . \tag{33}$$

Hence combining (32) and (33) we get for all sufficiently large values of r that,

$$\frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r))} > \frac{\left(Ar^{\frac{1}{\delta}} \right)^{(\lambda_K(G) - \varepsilon)}}{(Ar^\delta)^{(\rho_k(g) + \varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r} . \tag{34}$$

Since $\delta^2 \rho_k(g) < \lambda_K(G)$, we can choose $\varepsilon (> 0)$ in such a manner that

$$\delta^2 (\rho_k(g) + \varepsilon) \leq (\lambda_K(G) - \varepsilon) . \tag{35}$$

Thus from (34) and (35) we obtain that

$$\lim_{r \rightarrow +\infty} \frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r))} = \infty . \tag{36}$$

Case II. Let $m (\neq 1)$ and $n (\neq 1)$ are any two odd numbers .

Now in view of second part of Lemma 5, we get for all sufficiently large values of r that

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) < (Ar^\delta)^{(\rho_h(f) + \varepsilon)} . \tag{37}$$

So from (30) and (37) it follows for all sufficiently large values of r that

$$\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r)) < (Ar^\delta)^{(\rho_h(f) + \varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r . \tag{38}$$

Also from second part of Lemma 6, we obtain for all sufficiently large values of r that

$$\mu_{H_m}^{-1}(\mu_{F_m}(r)) > \left(Ar^{\frac{1}{\delta}} \right)^{(\lambda_H(F) - \varepsilon)} . \tag{39}$$

Hence combining (38) and (39) we get for all sufficiently large values of r that,

$$\frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r))} > \frac{\left(Ar^{\frac{1}{\delta}} \right)^{(\lambda_H(F) - \varepsilon)}}{(Ar^\delta)^{(\rho_h(f) + \varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r} . \tag{40}$$

As $\delta^2 \rho_h(f) < \lambda_H(F)$, we can choose $\varepsilon (> 0)$ in such a manner that

$$\delta^2 (\rho_h(f) + \varepsilon) \leq (\lambda_H(F) - \varepsilon) . \tag{41}$$

Therefore from (40) and (41) it follows that

$$\lim_{r \rightarrow +\infty} \frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r))} = \infty . \tag{42}$$

Case III. Let m be any even number and $n (\neq 1)$ be any odd number.

Then combining (33) and (38) we get for all sufficiently large values of r that

$$\frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r))} > \frac{\left(Ar^{\frac{1}{\delta}} \right)^{(\lambda_K(G) - \varepsilon)}}{(Ar^\delta)^{(\rho_h(f) + \varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r} . \tag{43}$$

Since $\delta^2 \rho_h(f) < \lambda_K(G)$, we can choose $\varepsilon (> 0)$ in such a manner that

$$\delta^2 (\rho_h(f) + \varepsilon) \leq (\lambda_K(G) - \varepsilon) . \tag{44}$$

So from (43) and (44) we get that

$$\lim_{r \rightarrow +\infty} \frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r))} = \infty. \quad (45)$$

Case IV. Let $m(\neq 1)$ be any odd number and n be any even number .

Therefore combining (32) and (39) we obtain for all sufficiently large values of r that

$$\frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r))} \geq \frac{(Ar^{\frac{1}{\delta}})^{(\lambda_H(F)-\varepsilon)}}{(Ar^\delta)^{(\rho_k(g)+\varepsilon)} \cdot (\rho_h(f) + \varepsilon) \log r}. \quad (46)$$

As $\delta^2 \rho_k(g) < \lambda_H(F)$, we can choose $\varepsilon(> 0)$ in such a manner that

$$\delta^2 (\rho_k(g) + \varepsilon) \leq (\lambda_H(F) - \varepsilon). \quad (47)$$

Hence from (46) and (47) we have

$$\lim_{r \rightarrow +\infty} \frac{\mu_{H_m}^{-1}(\mu_{F_m}(r))}{\mu_{h_n}^{-1}(\mu_{f_n}(r)) \cdot \log \mu_h^{-1}(\mu_f(r))} = \infty. \quad (48)$$

Thus the first part of the theorem follows from (36), (42), (45) and (48).

Similarly, from the definition of $\rho_k(g)$ one can easily derive the conclusion of the second part of the theorem.

Hence the theorem follows. \square

Remark 7. If we consider $\rho_K(G)$, $\rho_H(F)$, $\rho_K(G)$ and $\rho_H(F)$ instead of $\lambda_K(G)$, $\lambda_H(F)$, $\lambda_K(G)$ and $\lambda_H(F)$ respectively in (29) and the other conditions remain the same, the conclusion of Theorem 23 is remains valid with “limit superior” replaced by “limit”.

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TANMAY BISWAS, RAJBARI, RABINDRAPALLI, R. N. TAGORE ROAD, P.O.- KRISHNAGAR, DIST-
NADIA, PIN- 741101, WEST BENGAL, INDIA

E-mail address: tanmaybiswas@rediffmail.com