WEIGHTED SHARING OF Q-SHIFT
DIFFERENCE-DIFFERENTIAL POLYNOMIALS OF
MEROMORPHIC FUNCTIONS SHARING A SMALL FUNCTION

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Abstract. In this article, with the notion of weighted sharing we study the uniqueness problems of \( q \)-shift difference-differential polynomials of meromorphic functions sharing a small function \( a(z) \) with weight \( l \). Our result improves and generalizes a recent result of Renukadevi S. Dyavanal and Ashwini M. Hatikal.

1. Introduction and main results

Let \( f \) be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory: \( T(r, f), N(r, f), N(r, f, m(r, f)), \) (see [17]). The notation \( S(r, f) \) is defined to be any quantity satisfying \( S(r, f) = o(T(r, f)) \) as \( r \to \infty, r \not\in E \), where \( E \) is a set of positive real number of finite linear measure, not necessarily the same at each occurrence. A meromorphic function \( a(z) \) is called a small function with respect to \( f(z) \) provided that \( T(r, a) = S(r, f) \). Suppose that \( f(z) - a(z) \) and \( g(z) - a(z) \) have the same zeros with same counting multiplicities (ignoring multiplicities), then we say that \( f \) and \( g \) share \( a(z) \) CM(IM).

Definition 1.[13] Let \( k \) be a non-negative integer or infinity. For \( a \in C \cup \{\infty\} \), we denote by \( E_k(a, f) \) the set of all \( a \)-points of \( f \), where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k + 1 \) times if \( m > k \). If \( E_k(a, f) = E_k(a, g) \), then we say that \( f \) and \( g \) share the value \( a \) with weight \( k \).

The definition implies that if \( f, g \) share a value \( a \) with weight \( k \), then \( z_0 \) is a zero of \( f - a \) with multiplicity \( m(\leq k) \) if and only if it is a zero of \( g - a \) with multiplicity \( m(\leq k) \); and \( z_0 \) is a zero of \( f - a \) with multiplicity \( m(> k) \) if and only if it is a zero of \( g - a \) with multiplicity \( n(> k) \), where \( m \) is not necessarily equal to \( n \).

We write \( f, g \) share \( (a, l) \) to mean that \( f, g \) share the value ‘\( a \)’ with weight \( l \). Clearly if \( f, g \) share \( (a, l) \), then \( f, g \) share \( (a, p) \) for all integer \( p, 0 \leq p < k \). Also, we note that \( f, g \) share a value ‘\( a \)’ IM or CM, if and only if \( f, g \) share \( (a, 0) \) or

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(a, ∞), respectively.

**Definition 2.** [2] We denote and define order of \( f(z) \) by

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}
\]

If a non-constant meromorphic function \( f(z) \) is of zero order, then \( \rho(f) = 0 \).

Recently difference polynomials in the complex plane \( \mathbb{C} \) become a subject of great interest among the researcher around the world. With the development of difference analogue of Nevanlinna theory [see [3], [4], [5], [6]], a large number of papers have focused on value distribution and uniqueness of difference polynomials.

In 2014, X.M. Li, H.X. Yi and W.L. Li [7] proved the following theorem on uniqueness of difference polynomials of meromorphic functions sharing a small function.

**Theorem 1.** Let \( f \) and \( g \) be two transcendental meromorphic function of finite order, let \( \alpha \neq 0 \) be an entire function such that \( \rho(\alpha) < \rho(f) \), let \( \eta \) be a non-zero complex number and let \( n \) and \( m \) be two positive integers such that \( n \geq m + 12 \) and \( m \geq 2 \). Suppose \( f^n(z)(f^m(z) - 1)f(z + \eta) - \alpha(z) \) and \( g^n(z)(g^m(z) - 1)g(z + \eta) - \alpha(z) \) share \( 0, \infty \) CM. Then \( f(z) = tg(z) \), where \( t \) is a constant satisfying \( t^m = 1 \).

Further, K.Y. Zhang and H.X. Yi [19] extended the result of X.M. Li, H.X. Yi and W.L. Li [7] and proved the theorem on uniqueness of product of differential-difference polynomials of entire functions as in the following theorem.

**Theorem 2.** Let \( f(z) \) and \( g(z) \) be transcendental entire functions of finite order, \( \alpha(z) \neq 0 \) be a common small function with respect to both \( f \) and \( g \), let \( c_j \) \((j = 1, 2, \ldots, d)\) be distinct finite complex numbers and \( n \), \( m \), \( d \) and \( v_j \) \((j = 1, 2, \ldots, d)\) are non-negative integers. If \( n \geq 4k - m + \sigma + 9 \) and the differential-difference polynomial \( (f^n(z)(f(z) - 1)^m \prod_{j=1}^d f(z + c_j)^{v_j}(k)) \) and \( (g^n(z)(g(z) - 1)^m \prod_{j=1}^d g(z + c_j)^{v_j}(k)) \) share \( \alpha(z) \) CM, then \( f \equiv g \).

In 2015, F.H. Liu and H.X. Yi [9] improved the previous results by considering uniqueness problems on product of difference polynomials of meromorphic functions.

**Theorem 3.** Let \( f(z) \) and \( g(z) \) be non-constant meromorphic functions satisfying \( \rho(f) < \infty \), \( \rho(g) < \infty \). \( f(z) \) and \( g(z) \) share \( \infty \) IM. \( \alpha(z) \neq 0 \) is an entire function satisfying \( \rho(\alpha) < \rho(f) \), \( m \), \( n \), \( s \), \( \mu_j \) \((j = 1, 2, \ldots, s)\) are non-negative integers, \( \sigma = \sum_{j=1}^s \mu_j \), \( c_j \) \((j = 1, 2, \ldots, s)\) are non-zero complex constants. \( F(z) = f^n(f^m - 1) \prod_{j=1}^s f(z + c_j)^{\mu_j}, G(z) = g^n(g^m - 1) \prod_{j=1}^s g(z + c_j)^{\mu_j} \) share \( \alpha, \infty \) CM. If \( n \geq m + 2s + 3\sigma + 7 \) we get \( f(z) = tg(z) \), where \( t \) is a constant satisfying \( t^m = 1 \).

Recently, R. S. Dyavanal and A. M. Hattikal [2] investigated the uniqueness of difference polynomials of meromorphic functions sharing a small function \( a(z) \) with counting multiplicity.

**Theorem 4.** Let \( f \) and \( g \) be two non-constant meromorphic functions of zero order and \( a(z) \) is a small function with respect to both \( f \) and \( g \). Let \( n \geq m + 3\lambda + 2d + 7 \) be a positive integer, where \( m \), \( d \), \( \lambda = \sum_{j=1}^d s_j \) for \( j = 1, 2, \ldots, d \) are finite positive integers such that \( d < \lambda \). Let \( q_j, c_j \) \((j = 1, 2, \ldots, d)\) are distinct non-zero complex
constants. If
\[ f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(q_jz + c_j)^{s_j} \]
and
\[ g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(q_jz + c_j)^{s_j} \]
share \( a(z) \) CM, \( f \) and \( g \) share \( \infty \) IM, then

(1) if \( m \geq 2 \), then either \( f = tg \) for a constant \( t \) such that \( t^d = 1 \) where \( d = GCD(n + m + \lambda, n + m + \lambda - 1, \ldots, n + m + \lambda - i, \ldots, n + \lambda) \) or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) \equiv 0 \), where
\[
R(w_1, w_2) = w_1^n(w_1 - 1)^m \prod_{j=1}^{d} w_1(q_jz + c_j)^{s_j} - w_2^n(w_2 - 1)^m \prod_{j=1}^{d} w_2(q_jz + c_j)^{s_j}
\]
(2) if \( m = 1 \), then \( f = tg \) for a constant \( t \) such that \( t^d = 1 \) where \( d = GCD(n + \lambda, n + 1 + \lambda) \).

In this paper, we define a \( q \)-shift difference product of meromorphic function \( f(z) \) as follows.
\[
F(z) = (f^n(z)P(f) \prod_{j=1}^{d} f(q_jz + c_j)^{s_j})^{(k)} \tag{1}
\]
\[
F_1(z) = f^n(z)P(f) \prod_{j=1}^{d} f(q_jz + c_j)^{s_j} \tag{2}
\]
where \( q_j, c_j \ (j = 1, 2, \ldots, d) \) are distinct non-zero complex constants, \( n, d, k, \lambda, s_j \ (j = 1, 2, \ldots, d) \) be positive integers, \( \lambda = \sum_{j=1}^{d} s_j \). Let \( P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_0 \) is a non-zero polynomial of degree \( m \) and \( \Gamma_0 = m_1 + m_2 \), where \( m_1 \) is the number of the simple zero of \( P(z) \) and \( m_2 \) is the number of multiple zeros of \( P(z) \).

Here, we used the idea of weighted sharing values to extend the above results for meromorphic functions.

**Theorem 5.** Let \( f \) and \( g \) be two non-constant meromorphic functions of zero order and \( a(z) \) is a small function with respect to both \( f \) and \( g \). If \( F \) and \( G \) share \( (a(z), l) \), where \( l, n \) are positive integers; \( f \) and \( g \) share \( \infty \) IM with the conditions of \( n \) as below

(i) \( n > 3k + 2\Gamma_0 - m + kd + \lambda + 3d + 7 \), when \( l \geq 2 \)
(ii) \( n > 4k + \frac{5k\lambda}{2} - m + \frac{3kd}{2} + \frac{3\lambda}{2} + 7d + 8 \), when \( l = 1 \)
(iii) \( n > 9k + 5\Gamma_0 - m + 4kd + 4\lambda + 6d + 13 \), when \( l = 0 \) then one of the following cases hold:

1) \( f \equiv tg \) for a constant \( t \) such that \( t^l = 1 \), where \( l = GCD\{n + \lambda_0 + \lambda, n + \lambda_1 + \lambda, \ldots, n + \lambda_m + \lambda\} \) and
\[
\lambda_i = \begin{cases} i, & a_i \neq 0 \\ m, & a_i = 0 \end{cases} \quad i = 0, 1, \ldots, m.
\]
2) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^n P(w_1) \prod_{j=1}^{d} w_1(z + c_j)^{s_j} - w_2^n P(w_2) \prod_{j=1}^{d} w_2(z + c_j)^{s_j}.$$ 

Remark 1. When $k = 0$ and $\Gamma_0 = m_1 + m_2 = m$ in Theorem 5, then Theorem 5 improves and generalize Theorem 3 and Theorem 4.

Remark 2. When $k = 0$, $\Gamma_0 = m_1 + m_2 = m$, $\lambda = 1$ and $d = 1$ in Theorem 5, then Theorem 5 reduces to Theorem 1.

Corollary. Let $f$ and $g$ be two non-constant entire functions of zero order and $a(z)$ is a small function with respect to both $f$ and $g$. If $F$ and $G$ share $(a(z), l)$, where $l$, $n$ are positive integers; $f$ and $g$ share $\infty$ IM with the conditions of $n$ as below

(i) $n \geq 2k - m + 2\Gamma_0 + \lambda + 5$, when $l \geq 2$

(ii) $n \geq \frac{5k}{2} + \frac{5\lambda}{2} + \frac{3\lambda}{2} - m + \frac{11}{2}$, when $l = 1$

(iii) $n \geq 5k + 5\Gamma_0 + 4\lambda - m + 8$, when $l = 0$ then conclusion of Theorem 5 holds.

2. Some Lemmas

Lemma 1.[18] Let $f(z)$ be a non-constant meromorphic function, and $a_n(\neq 0), a_{n-1}, ..., a_0$ be small functions with respect to $f$. Then

$$T(r, a_nf^n + a_{n-1}f^{n-1} + ... + a_1f + a_0) = nT(r, f) + S(r, f)$$

Lemma 2.[16] Let $f(z)$ be a non-constant meromorphic function of zero order, and let $c$ and $q$ be two non-zero complex numbers. Then

$$T(r, f(qz + c)) = T(r, f(z)) + S(r, f),$$

on a set of logarithmic density 1.

Lemma 3.[8] Let $f$ be a meromorphic function with zero order and $c$ and $q$ be two non-zero complex numbers. Then

$$N \left( r, \frac{1}{f(qz + c)} \right) \leq N \left( r, \frac{1}{f(z)} \right) + S(r, f)$$

$$N( r, f(qz + c)) \leq N \left( r, \frac{1}{f(z)} \right) + S(r, f)$$

outside of a possible exceptional set $E$ with finite logarithmic measure.

Lemma 4.[10] Let $f(z)$ be a non-constant meromorphic function and $p$, $k$ be positive integers. Then

$$N_p \left( r, \frac{1}{f^{(k)}} \right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k} \left( r, \frac{1}{f} \right) + S(r, f), \quad (3)$$

$$N_p \left( r, \frac{1}{f^{(k)}} \right) \leq kN(r, f) + N_{p+k} \left( r, \frac{1}{f} \right) + S(r, f). \quad (4)$$

Lemma 5.[1] Let $F$, $G$ be two nonconstant meromorphic functions sharing $(1, 2), (\infty, 0)$ and $H \neq 0$. Then

(i) $T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + N_2(r, \infty; F, G)$
\[m(r, 1; G) - N_E^3(r, 1; F) - N_L(r, 1; G) + S(r, F) + S(r, G);\]

(ii) \[T(r, G) \leq N_2(r, 0; F) + N_2(r, 0; G) + \frac{3}{2} N(r, \infty; F) + N(r, \infty; G) + \frac{1}{2} N(r, 0; F) + \frac{1}{2} N(r, 0; G) + S(r, F) + S(r, G);\]

**Lemma 6.**[12] Let \( F, G \) be two nonconstant meromorphic functions sharing \((1, 1), (\infty, 0)\) and \( H \neq 0 \). Then

(i) \[T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + 3 N(r, \infty; F) + 2 N(r, \infty; G) + 2 N(r, 0; F) + N(r, 0; G) + S(r, F) + S(r, G);\]

(ii) \[T(r, G) \leq N_2(r, 0; F) + N_2(r, 0; G) + 3 N(r, \infty; F) + 2 N(r, \infty; G) + N(r, 0; F) + 2 N(r, 0; G) + S(r, F) + S(r, G).\]

**Lemma 7.**[12] Let \( F, G \) be two nonconstant meromorphic functions sharing \((1, 0), (\infty, 0)\) and \( H \neq 0 \). Then

(i) \[T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + 3 N(r, \infty; F) + 2 N(r, \infty; G) + 2 N(r, 0; F) + N(r, 0; G) + S(r, F) + S(r, G);\]

(ii) \[T(r, G) \leq N_2(r, 0; F) + N_2(r, 0; G) + 3 N(r, \infty; F) + 2 N(r, \infty; G) + N(r, 0; F) + 2 N(r, 0; G) + S(r, F) + S(r, G).\]

**Lemma 8.** Let \( f(z) \) and \( g(z) \) be two nonconstant meromorphic functions, let \( n, k \) be two positive integers with \( n > k + m + 2 \lambda + d + 2 \) and \( a(z) (\neq 0, \infty) \) be a small function with respect to \( f \) and \( g \) and let \( P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0 \), where \( a_0, a_1, \ldots, a_{m-1}, a_m \) are complex constants. If

\[(f^n(z)P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k)}(g^n(z)P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j})^{(k)} \equiv a^2,\]

\( f \) and \( g \) share \( \infty \) IM, then \( P(z) \) is reduced to a nonzero monomial, that is \( P(z) = a_i z^i \neq 0 \) for some \( i = 0, 1, 2, \ldots, m \).

**Proof.** If \( P(z) \) is not reduced to a nonzero monomial, then without loss of generality, we assume that \( P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0 \), where \( a_0 (\neq 0), a_1, \ldots, a_{m-1}, a_m (\neq 0) \) are complex constants. By hypothesis of Lemma 8, we know that either both \( f \) and \( g \) are transcendental meromorphic functions or they are both rational functions. Since \( f \) and \( g \) share \( \infty \) IM, the poles of \( f \) and \( g \) are finite. Similarly \( f \) and \( g \) has finitely many zeros.

**Case 1.** If \( f \) and \( g \) are transcendental meromorphic functions. Let \( f = h e^{\beta} \), where \( \beta \) is a non-constant entire function and \( h(z) \) is a nonzero rational function. Thus, by induction on \( k \), we get

\[
(a_f f^{i+n} \prod_{j=1}^{d} f(q_j z + c_j)^{s_j})^{(k)} = P_i(\beta', \beta'', \ldots, \beta^{(k)}), \sum s_j \beta'(q_j z + c_j), \ldots, \]

\[
\sum s_j \beta^{(k)}(q_j z + c_j), h, h', \ldots, h^{(k)}, \sum s_j h(q_j z + c_j), \sum s_j h'(q_j z + c_j), \ldots, \]

\[
\sum s_j h^{(k)}(q_j z + c_j)e^{(i+n)\beta(z)+\sum_{j=1}^{d} s_j \beta(q_j z + c_j)}
\]

where, \( P_i (i = 1, 2, \ldots, m) \) are difference-differential polynomials with coefficients as rational functions in \( h(z) \) and \( \sum s_j h(z + c_j) \) or its derivatives.
Notice that
\[ P_0(\beta', \beta'', \ldots, \beta^{(k)}), \sum s_j \beta'(q_j z + c_j), \ldots, \sum s_j \beta^{(k)}(q_j z + c_j), h, h', \ldots, h^{(k)}, \]
\[ \sum s_j h(q_j z + c_j), \sum s_j h'(q_j z + c_j), \ldots, \sum s_j h^{(k)}(q_j z + c_j), \ldots, P_m(\beta', \beta'', \ldots, \beta^{(k)}), \]
\[ \sum s_j \beta'(q_j z + c_j), \ldots, \sum s_j \beta^{(k)}(q_j z + c_j), h, h', \ldots, h^{(k)}, \sum s_j h(q_j z + c_j), \]
\[ \sum s_j h'(q_j z + c_j), \ldots, \sum s_j h^{(k)}(q_j z + c_j) \neq 0. \]
Since \( \beta(z) \) is an entire function,
\[ T(r, \beta'(z)) = m(r, \beta'(z)) = m \left( r, \left( \frac{e^{\beta(z)}}{\beta'(z)} \right) \right) = S(r, f). \]
Thus, we obtain
\[ T(r, \beta^{(k)}(z)) \leq T(r, \beta') + S(r, f) = S(r, f) \text{ for } j = 1, 2, \ldots, k, \]
and
\[ T(r, \sum s_j \beta'(q_j z + c_j)) = m(r, \sum s_j \beta'(q_j z + c_j)) + N(r, \sum s_j \beta'(q_j z + c_j)) \]
\[ = m(r, \sum s_j \beta'(q_j z + c_j)) \]
\[ = m \left( r, \left( \frac{e^{\sum s_j \beta(q_j z + c_j)}}{e^{\beta'(z)}} \right) \right) = S(r, f). \]
Therefore
\[ T(r, \sum s_j \beta^{(k)}(q_j z + c_j)) \leq T(r, \sum s_j \beta'(q_j z + c_j)) + S(r, f) = S(r, f) \text{ for } j = 1, 2, \ldots, k, \]
which is a contradiction.

\textbf{Case 2.} If \( f \) and \( g \) are rational functions, then \( a \) is a nonzero constant, thus \( f \) and \( g \) have no zeros and no poles, which is impossible. Since \( f \) and \( g \) are not constants.

The above two Cases imply that \( P(z) \) is reduced to a nonzero monomial, namely, \( P(z) = a_i z^i \neq 0 \) for some \( i \in \{0, 1, \ldots, m\} \).

\subsection*{3. Proof of Theorem 5.}
Let \( F^* = \frac{F}{\alpha(z)} \) and \( G^* = \frac{G}{\alpha(z)} \). From the hypothesis we have \( F(z) \) and \( G(z) \) share \((a(z), l)\) and \( f, g \) share \( \infty \) IM. It follows that \( F^* \) and \( G^* \) share 1CM and \( \infty \) IM. We now discuss the following two cases separately.

\textbf{Case 1.} We assume that \( H \neq 0 \). Now we consider the following three subcases.

\textbf{Subcase 1.} Suppose that \( l \geq 2 \). Then using Lemma 5 we obtain
\[ T(r, F) \leq T(r, F^*) + S(r, F) \]
\[ \leq N_2(r, 0; F^*) + N_2(r, 0; G^*) + N(r, \infty; F^*) + N(r, \infty; G^*) + N_*(r, \infty; F^*, G^*) \]
\[ - m(r, 1; G^*) - N_2^*(r, 1; F^*) - N_2^*(r, 1; G^*) + S(r, F^*) + S(r, G^*) \]
\[ \leq N_2(r, 0; F) + N_2(r, 0; G) + N(r, \infty; F) + N(r, \infty; G) + N_*(r, \infty; F, G) \]
\[ + S(r, F) + S(r, G). \]
Let a contradiction with the fact that \( n > \) from (6). Similarly, we have for
\[
\text{Equation not clearly visible.}
\]
By using (3) and (4), we have
\[
T(r, F) \leq T(r, F_1) + N_{k+2}(r, 0; F_1) + kN(r, \infty; G_1) + N_{k+2}(r, 0; G_1) + 2N(r, \infty; F) + N(r, \infty; G) + S(r, F) + S(r, G).
\]
Similarly, we have for \( T(r, g) \),
\[
(n + m + \lambda)T(r, f) \leq (k + \Gamma_0 + 2d + \lambda + 4)T(r, f) + (2k + \Gamma_0 + \lambda + kd + d + 3)T(r, g) + S(r, f) + S(r, g).
\]
Similarly, we have for \( T(r, g) \),
\[
(n + m + \lambda)T(r, g) \leq (k + \Gamma_0 + 2d + \lambda + 4)T(r, g) + (2k + \Gamma_0 + \lambda + kd + d + 3)T(r, f) + S(r, f) + S(r, g)
\]
from (8) and (9), we have
\[
(n + m + \lambda)[T(r, f) + T(r, g)] \leq (3k + 2\Gamma_0 + 2\lambda + kd + 3d + 7)[T(r, f) + T(r, g)] + S(r, f) + S(r, g),
\]
a contradiction with the fact that \( n > 3k + 2\Gamma_0 - m + kd + \lambda + 3d + 7 \).

**Subcase 2.** Let \( l = 1 \). Then using (6) and Lemma 6 we obtain
\[
T(r, F) \leq T(r, F^*) + S(r, F)
\]
\[
\leq N_2(r, 0; F^*) + N_2(r, 0; G^*) + \frac{3}{2}N(r, \infty; F^*) + N(r, \infty; G^*) + N^*(r, \infty; F^*, G^*) + \frac{1}{2}N(r, 0; F) + S(r, F^*) + S(r, G^*)
\]
\[
\leq N_2(r, 0; F) + N_2(r, 0; G) + \frac{3}{2}N(r, \infty; F) + \frac{5}{2}N(r, \infty; F) + \frac{5}{2}N(r, 0; F) + \frac{1}{2}N(r, 0; F) + \frac{1}{2}N(r, 0; F) + S(r, F) + S(r, G).
\]
Using (10), (3) and (4), we have
\[ T(r, F) \leq T(r, F) - T(r, F_1) + N_{k+2}(r, 0; F_1) + kN(r, \infty; G_1) + N_{k+2}(r, 0; G_1) + S(r, f) + S(r, g) \]
\[ + \frac{5}{2}N(r, \infty; F_1) + \overline{N}(r, \infty; G_1) + \frac{1}{2}[k\overline{N}(r, \infty; F_1) + N_{k+1}(r, 0; F_1)] \]
\[ + T(r, F_1) \leq N_{k+2}(r, 0; f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{n_j}) + k\overline{N}(r, \infty; g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{n_j}) 
+ N_{k+2}(r, 0; g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{n_j}) + \frac{5}{2}N(r, \infty; f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{n_j}) 
+ \overline{N}(r, \infty; g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{n_j}) + \frac{1}{2}[k\overline{N}(r, \infty; f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{n_j}) 
+ N_{k+1}(r, 0; f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{n_j})] + S(r, f) + S(r, g) \]
\[ (n + m + \lambda)T(r, f) \leq (2k + \frac{3\Gamma_0}{2} + \frac{3\lambda}{2} + \frac{kd}{2} + \frac{5d}{2} + 5)T(r, f) + (2k + \Gamma_0 + d + \lambda) \]
\[ + kd + 3)T(r, g) + S(r, f) + S(r, g). \] (11)
Similarly, we have for \( T(r, g) \),
\[ (n + m + \lambda)T(r, g) \leq (2k + \frac{3\Gamma_0}{2} + \frac{3\lambda}{2} + \frac{kd}{2} + \frac{5d}{2} + 5)T(r, g) + (2k + \Gamma_0 + d + \lambda) \]
\[ + kd + 3)T(r, f) + S(r, f) + S(r, g) \] (12)
from (11) and (12), we have
\[ (n + m + \lambda)[T(r, f) + T(r, g)] \leq (4k + \frac{5\Gamma_0}{2} + \frac{5\lambda}{2} + \frac{3kd}{2} + \frac{7d}{2} + 8)[T(r, f) + T(r, g)] 
+ S(r, f) + S(r, g), \]
a contradiction with the fact that \( n > 4k + \frac{5\Gamma_0}{2} - m + \frac{3kd}{2} + \frac{3\lambda}{2} + \frac{7d}{2} + 8. \)

**Subcase 3.** Let \( l = 0 \). Then using (6) and Lemma 7 we obtain
\[ T(r, F) \leq T(r, F^*) + S(r, F) \]
\[ \leq N_2(r, 0; F^*) + N_2(r, 0; G^*) + 3\overline{N}(r, \infty; F^*) + 2\overline{N}(r, \infty; G^*) + \overline{N}_*(r, \infty; F^*, G^*) 
+ 2\overline{N}(r, 0; F^*) + \overline{N}(r, 0; G^*) + S(r, F^*) + S(r, G^*) \leq N_2(r, 0; F) + N_2(r, 0; G) + 3\overline{N}(r, \infty; F) + 2\overline{N}(r, \infty; G) + \overline{N}_*(r, \infty; F, G) 
+ 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + S(r, F) + S(r, G). \] (13)
We now assume that Case 2. 

\[ T(r, F) \leq T(r, F) - T(r, F_1) + N_{k+2}(r, 0; F_1) + kN(r, \infty; G_1) + N_{k+2}(r, 0; G_1) + 4N(r, \infty; F_1) \]
\[ + 2N(r, \infty; G_1) + 2[kN(r, \infty; F_1) + N_{k+1}(r, 0; F_1)] + kN(r, \infty; G_1) + N_{k+1}(r, 0; G_1) \]
\[ + S(r, F) + S(r, G) \]

\[ T(r, F_1) \leq N_{k+2}(r, 0; f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j}) + kN(r, \infty; g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}) \]
\[ + N_{k+2}(r, 0; g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}) + 4N(r, \infty; f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j}) \]
\[ + 2N(r, \infty; g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}) + 2[kN(r, \infty; f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j}) \]
\[ + N_{k+1}(r, 0; f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{s_j})] + kN(r, \infty; g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}) \]
\[ + N_{k+1}(r, 0; g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{s_j}) + S(r, f) + S(r, g) \]

\[(n + m + \lambda)T(r, f) \leq (5k + 3\Gamma_0 + 3\lambda + 2kd + 4d + 8)T(r, f) + (4k + 2\Gamma_0 + 2kd) \]
\[ + 2\lambda + 2d + 5)T(r, g) + S(r, f) + S(r, g). \]

Similarly, we have for \( T(r, g) \),

\[(n + m + \lambda)T(r, g) \leq (5k + 3\Gamma_0 + 3\lambda + 2kd + 4d + 8)T(r, g) + (4k + 2\Gamma_0 + 2kd) \]
\[ + 2\lambda + 2d + 5)T(r, f) + S(r, f) + S(r, g) \]

from (14) and (15), we have

\[(n + m + \lambda)[T(r, f) + T(r, g)] \leq (9k + 5\Gamma_0 + 4kd + 5\lambda + 6d + 13)[T(r, f) + T(r, g)] \]
\[ + S(r, f) + S(r, g), \]

a contradiction with the fact that \( n > 9k + 5\Gamma_0 - m + 4kd + 4\lambda + 6d + 13 \).

**Case 2.** We now assume that \( H \equiv 0 \). Then

\[ \left( \frac{F^{*\prime\prime}}{F^{*\prime}} - \frac{2F^{*\prime}}{F^{*} - 1} \right) - \left( \frac{G^{*\prime\prime}}{G^{*\prime}} - \frac{2G^{*\prime}}{G^{*} - 1} \right) = 0. \]

Integrating both sides of the above equality twice we get

\[ \frac{1}{F^{*} - 1} = \frac{A}{G^{*} - 1} + B, \]

where \( A(\neq 0) \) and \( B \) are constants. From (16) it is obvious that \( F^{*}, G^{*} \) share the value 1CM and hence they share the value 1 with weight 2, and therefore, \( n > 3k + 2\Gamma_0 - m + kd + \lambda + 3d + 7 \). We now discuss the following three subcases separately.
Subcase 4. Suppose that $B \neq 0$ and $A = B$. Then from (16) we obtain

$$\frac{1}{F^* - 1} = \frac{BG^*}{G^* - 1}. \quad (17)$$

If $B = -1$, then from (17) we obtain

$$F^* G^* = 1,$$

i.e.,

$$(f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{(k)}(g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{(k)}) = a^2(z),$$

which is a contradiction by Lemma 8.

If $B \neq -1$, from (17), we have $\frac{1}{F^*} = \frac{BG^*}{(1+B)G^* - 1}$ and so $\mathcal{N}\left(r, \frac{1}{1+B}; G^*\right) = \mathcal{N}(r, 0; F^*)$.

Using (3), (4) and the Second fundamental theorem of Nevanlinna, we deduce that

$$T(r, G) \leq T(r, G^*) + S(r, G)$$

$$\leq \mathcal{N}(r, 0; G^*) + \mathcal{N}\left(r, \frac{1}{1+B}; G^*\right) + \mathcal{N}(r, \infty; G^*) + S(r, G)$$

$$\leq \mathcal{N}(r, 0; G^*) + \mathcal{N}(r, 0; G^*) + \mathcal{N}(r, \infty; G^*) + S(r, G)$$

$$\leq \mathcal{N}(r, 0; F) + \mathcal{N}(r, 0; G) + \mathcal{N}(r, \infty; G) + S(r, G). \quad (18)$$

Using (18), Lemma 4 we have

$$T(r, G) \leq k\mathcal{N}(r, \infty; F_1) + N_{k+1}(r, 0; F_1) + T(r, G) - T(r, G_1) + N_{k+1}(r, 0; G_1)$$

$$+ \mathcal{N}(r, \infty; G_1) + S(r, g)$$

$$(n + m + \lambda)T(r, g) \leq (2k + \Gamma_0 + kd + d + \lambda - 3)T(r, f) + (k + \Gamma_0 + d + \lambda + 2)T(r, g)$$

$$+ S(r, f) + S(r, g).$$

Similarly,

$$(n + m + \lambda)T(r, f) \leq (2k + \Gamma_0 + kd + d + \lambda + 1)T(r, g) + (k + \Gamma_0 + d + \lambda + 2)T(r, f)$$

$$+ S(r, f) + S(r, g).$$

Thus we obtain

$$(n + m - 3k - 2\Gamma_0 - kd - d + \lambda - 3)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g),$$

a contradiction with $n > 3k + 2\Gamma_0 - m + kd + \lambda + 3d + 7$.

Subcase 5. Let $B \neq 0$ and $A \neq B$. Then from (17) we get $F^* = \frac{(B+1)G^* - (B-A+1)}{BG^* + (A-B)}$ and so $\mathcal{N}\left(r, \frac{B-A+1}{B}; G^*\right) = \mathcal{N}(r, 0; F^*)$. Proceeding in a manner similar to Subcase 4 we can arrive at a contradiction.

Subcase 6. Let $B = 0$ and $A \neq 0$. Then from (17) we get $F^* = \frac{G + A - 1}{A}$ and $G = AF - (A - 1)$. If $A \neq 1$, it follows that $\mathcal{N}\left(r, \frac{A-1}{A}; F^*\right) = \mathcal{N}(r, 0; G^*)$ and $\mathcal{N}(r, 1 - A; G^*) = \mathcal{N}(r, 0; F^*)$. Using the similar arguments as in Subcase 4 we obtain a contradiction. Thus $A = 1$ which implies $F^* = G^*$, and therefore,

$$(f^n P(f) \prod_{j=1}^{d} f(q_j z + c_j)^{(k)}) \equiv (g^n P(g) \prod_{j=1}^{d} g(q_j z + c_j)^{(k)}$$
Let $n > 0$. Similarly, we get
\[
\frac{f^n(z)P(f) \prod_{j=1}^d f(q_j z + c_j)^{s_j}}{R(z)} = \frac{g^n(z)P(g) \prod_{j=1}^d g(q_j z + c_j)^{s_j}}{R(z)} + 1.
\]

By the second fundamental theorem, we have
\[
T \left( r, \frac{f^n(z)P(f) \prod_{j=1}^d f(q_j z + c_j)^{s_j}}{R} \right) \leq \mathcal{N} \left( r, \frac{f^n(z)P(f) \prod_{j=1}^d f(q_j z + c_j)^{s_j}}{R(z)} \right) \\
+ \mathcal{N} \left( r, \frac{R(z)}{f^n(z)P(f) \prod_{j=1}^d f(q_j z + c_j)^{s_j}} \right) + \mathcal{N} \left( r, \frac{R(z)}{g^n(z)P(g) \prod_{j=1}^d g(q_j z + c_j)^{s_j}} \right) + S(r, f)
\]

\[(n + m + \lambda)T(r, f) \leq (2 + \Gamma_0 + 2d)T(r, f) + (\Gamma_0 + d + 1)T(r, g) + S(r, f) + S(r, g).
\]

Similarly, we get
\[(n + m + \lambda)T(r, g) \leq (2 + \Gamma_0 + 2d)T(r, g) + (\Gamma_0 + d + 1)T(r, f) + S(r, f) + S(r, g).
\]

So
\[(n + m + \lambda)[T(r, f) + T(r, g)] \leq (2\Gamma_0 + 3d + 3)[T(r, f) + T(r, g)] + S(r, f) + S(r, g).
\]

Which is contradiction to $n > 2\Gamma_0 - m + 3d - \lambda + 3$. Thus, we get $R(z) \equiv 0$ and hence
\[
f^n(z)P(f) \prod_{j=1}^d f(q_j z + c_j)^{s_j} = g^n(z)P(g) \prod_{j=1}^d g(q_j z + c_j)^{s_j}.
\]

\[
f^n(a_m f^m + a_{m-1} f^{m-1} + \ldots + a_1 f + a_0) \prod_{j=1}^d f(q_j z + c_j) = g^n(a_m g^m + a_{m-1} g^{m-1} + \ldots + a_1 g + a_0) \prod_{j=1}^d g(q_j z + c_j)
\]

Let $h = \frac{f}{g}$. If $h$ is constant then substituting $f = gh$ and
\[
\prod_{j=1}^d f(q_j z + c_j)^{s_j} = \prod_{j=1}^d g(q_j z + c_j)^{s_j} h(q_j z + c_j)^{s_j}
\]

in above equation, we deduce
\[
\prod_{j=1}^d g(z + c_j)^{s_j} [a_m g^{n+m}(h^{n+m+\lambda} - 1) + a_{m-1} g^{n+m-1}(h^{n+m+\lambda-1} - 1) + \ldots + a_0 g^n(h^{n+\lambda} - 1)] \equiv 0,
\]

(19)
where $a_m$ is a non-zero complex constant and $\prod_{j=1}^d g(q_j z + c_j)^{s_j} \neq 0$. Since $g$ is nonconstant meromorphic function. Then, from (19), we have

$$a_m g \gamma^{m} (h^{n+m+\lambda}-1)+a_{m-1}g^{n+m-1}(h^{n+m+\lambda-1})+\ldots+a_0g^n(h^{n+\lambda-1}) \equiv 0. \quad (20)$$

If $a_m(\neq 0)$ and $a_{m-1} = a_{m-2} = \ldots = a_0 = 0$, then from (20) and $g$ is nonconstant meromorphic function, we get $h^{n+m+\lambda} = 1$.

If $a_m(\neq 0)$ and there exists $a_j \neq 0 (i \in \{0, 1, 2, \ldots, m-1\})$. Suppose that $h^{n+m+\lambda} \neq 1$, from (20), we have $T(r, g) = S(r, g)$ which is contradiction with transcendental function $g$. Then $h^{n+m+\lambda} = 1$. Similar to this discussion, we can see that $h^{n+m+\lambda} = 1$ when $a_j \neq 0$ for some $j = 0, 1, 2, \ldots, m$.

Thus, we have $f \equiv tg$ for a constant $t$ such that $t^l = 1$, where $l = \gcd\{n + \lambda_0 + \lambda, n + \lambda_1 + \lambda, \ldots, n + \lambda_m + \lambda\}$ and $\lambda_i (i = 0, 1, 2, \ldots, m)$ is stated as in Theorem 5.

Set $h = \frac{f}{g}$. If $h$ is not a constant, from (20), we find that $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^d P(w_1) \prod_{j=1}^d w_1(z + c_j)^{s_j} - w_2^d P(w_2) \prod_{j=1}^d w_2(z + c_j)^{s_j}.$$

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