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PARTIAL SUMS OF τ- CONFLUENT HYPERGEOMETRIC FUNCTION

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Abstract. In the present investigation, τ-confluent hypergeometric function with their normalization are considered. In this paper, we will study the ratio of a function of the form (4) to its sequence of partial sums $({}_1\Phi_1^\tau(b;c;z))_n = z + \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{n} \frac{\Gamma(b+k\tau)}{\Gamma(c+k\tau)} \frac{z^{k+1}}{k!}$. We will determine the lower bounds for $\Re\left\{\frac{{}_1\Phi_1^\tau(b;c;z)}{({}_1\Phi_1^\tau(b;c;z))_n}\right\}$, $\Re\left\{\frac{({}_1\Phi_1^\tau(b;c;z))_n}{{}_1\Phi_1^\tau(b;c;z))}\right\}$, $\Re\left\{\frac{({}_1\Phi_1^\tau(b;c;z))'}{({}_1\Phi_1^\tau(b;c;z))'_n}\right\}$ and $\Re\left\{\frac{({}_1\Phi_1^\tau(b;c;z))'_n}{({}_1\Phi_1^\tau(b;c;z))'}\right\}$.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions f defined in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the subclass of $\mathcal{H}$, which are normalized by the condition $f(0) = 0 = f'(0) - 1$ and have representation of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

It is well known that the series

$${}_1\phi_1(b;c;z) = \sum_{n=0}^{\infty} \frac{(b)_n z^n}{(c)_n \, n!} \tag{2}$$

in which c is neither zero nor a negative integer is convergent for all finite z. Here $(b)_n$ denotes the Pochhammer (or Appell) symbol which is defined by

$$(b)_n := \begin{cases} 1, & (n = 0) \\ b(b+1)...(b+n-1), & (n \in \mathbb{N}). \end{cases}$$

The Pochhammer symbol is related to the gamma functions by the relation

$$(b)_n = \frac{\Gamma(b+n)}{\Gamma(b)},$$

where b is neither zero nor a negative integer. The function ${}_1\phi_1(b;c;z)$ is known as a confluent hypergeometric function for more details one can refer [7]. In 1999

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Virchenko [11] introduced τ-confluent hypergeometric function which is defined by (see also [12]):

$$ {}_1\phi_1^{\tau}(b;c;z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(b+\tau n)z^n}{\Gamma(c+\tau n)\, n!}, \tag{3} $$

$$ (\tau > 0, \Re(c) > \Re(b) > 0) $$

For $\tau = 1$,

$$ {}_1\phi_1^{\tau}(b;c;z) = {}_1\phi_1(b;c;z). $$

As the function ${}_1\phi_1^{\tau}(b,c;z)$ does not belong to the family $\mathcal{A}$, thus it is natural to consider the following normalization of function ${}_1\phi_1^{\tau}(b,c;z)$ in $\mathbb{D}$:

$$ \begin{aligned} {}_1\Phi_1^{\tau}(b;c;z) &= z\,{}_1\phi_1^{\tau}(b,c;z) \\ &= z + \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=2}^{\infty} \frac{\Gamma(b+\tau(n-1))}{\Gamma(c+\tau(n-1))} \frac{z^n}{(n-1)!}. \end{aligned} \tag{4} $$

For the present investigation we will study ${}_1\Phi_1^{\tau}(b,c;z)$ for real values of b and c satisfying $c \geq b > 0$ only.

If f, g are analytic functions in $\mathbb{D}$, then f is said to be subordinate to g, written as $f(z) \prec g(z)$ $(z \in \mathbb{D})$, if there exists an analytic function w with $w(0) = 0$ and $|w(z)| \leq 1$ $(z \in \mathbb{D})$ such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$ f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}). $$

For more details one can refer [4]. In the present paper, we will study the ratio of a function of the form (4) to its sequence of partial sums

$$ ({}_1\Phi_1^{\tau}(b;c;z))_n = z + \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{n} \frac{\Gamma(b+k\tau)}{\Gamma(c+k\tau)} \frac{z^{k+1}}{k!} = z + \sum_{k=1}^{n} b_k z^{k+1}, \tag{5} $$

$$ ({}_1\Phi_1^{\tau}(b;c;z))'_n = 1 + \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=1}^{n} \frac{(k+1)\Gamma(b+k\tau)}{\Gamma(c+k\tau)} \frac{z^k}{k!} = 1 + \sum_{k=1}^{n} (k+1) b_k z^k, \tag{6} $$

$$ ({}_1\Phi_1^{\tau}(b;c;z))_0 = z \text{ and } ({}_1\Phi_1^{\tau}(b;c;z))'_0 = 1. \tag{7} $$

We will determine lower bounds for $\Re\left\{\frac{{}_1\Phi_1^{\tau}(b;c;z)}{({}_1\Phi_1^{\tau}(b;c;z))_n}\right\}$, $\Re\left\{\frac{({}_1\Phi_1^{\tau}(b;c;z))_n}{{}_1\Phi_1^{\tau}(b;c;z))}\right\}$, $\Re\left\{\frac{({}_1\Phi_1^{\tau}(b;c;z))'}{({}_1\Phi_1^{\tau}(b;c;z))'_n}\right\}$ and $\Re\left\{\frac{({}_1\Phi_1^{\tau}(b;c;z))'_n}{({}_1\Phi_1^{\tau}(b;c;z))'}\right\}$. For various known results concerning with partial sums of analytic univalent functions one can refer the works of Bansal and Orhan [1], Çağlar and Deniz [2], Choi [3], Orhan and Yağmur [5], Owa et. al [6], Sheil-Small [8], Silverman [9] and Silvia [10].

To prove main results we need following Lemma:

Lemma 1. If $\tau > 0$ and $c \geq b > max\{2-\tau, 0\}$ then,

$$ |{}_1\Phi_1^{\tau}(b;c;z)| \leq 1 + \frac{2\Gamma(c)}{\Gamma(b)} \ (z \in \mathbb{D}) \tag{8} $$

and

$$ |{}_1\Phi_1^{\tau}(b;c;z)'| \leq 1 + \frac{11}{2} \frac{\Gamma(c)}{\Gamma(b)} \ (z \in \mathbb{D}). \tag{9} $$

Proof. To prove this lemma, we use the following inequalities

$$\frac{n}{(n-1)!} \leq \left(\frac{2}{3}\right)^{n-3} \quad \forall\, n \geq 4$$

$\Gamma(c+\tau(n-1)) \geq \Gamma(b+\tau(n-1))$ (for $\tau > 0,\ c \geq b > max\{2-\tau, 0\}$ and $n \in \{2, 3, 4...\}$) and $n! \geq 2^{n-1}$ for all $n \in \mathbb{N}$. Using (4), we have

$$\begin{aligned} |{}_1\Phi_1^\tau(b;c;z)| &\leq |z| + \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=2}^{\infty}\frac{1}{(n-1)!}\frac{\Gamma(b+\tau(n-1))}{\Gamma(c+\tau(n-1))}|z|^n \\ &\leq 1 + \frac{\Gamma(c)}{\Gamma(b)}\left[\sum_{n=2}^{\infty}\left(\frac{1}{2}\right)^{n-2}\right] \\ &= 1 + \frac{2\Gamma(c)}{\Gamma(b)}. \end{aligned}$$

Similarly

$$\begin{aligned} |{}_1\Phi_1^\tau(b;c;z)'| &\leq 1 + \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=2}^{\infty}\frac{n}{(n-1)!}\frac{\Gamma(b+\tau(n-1))}{\Gamma(c+\tau(n-1))}|z|^{n-1} \\ &< 1 + \frac{\Gamma(c)}{\Gamma(b)}\left[\frac{7}{2} + \sum_{n=4}^{\infty}\left(\frac{2}{3}\right)^{n-3}\right] \\ &= 1 + \frac{11}{2}\frac{\Gamma(c)}{\Gamma(b)}. \end{aligned}$$

□

2. Main results

Theorem 1. If $\tau > 0$, $c \geq b > max\{2-\tau, 0\}$ and $\Gamma(b) \geq 2\Gamma(c)$, then

$$\Re\left\{\frac{{}_1\Phi_1^\tau(b;c;z)}{({}_1\Phi_1^\tau(b;c;z))_n}\right\} \geq \left(1 - \frac{2\Gamma(c)}{\Gamma(b)}\right)\ (z \in \mathbb{D}) \tag{10}$$

and

$$\Re\left\{\frac{({}_1\Phi_1^\tau(b;c;z))_n}{{}_1\Phi_1^\tau(b;c;z)}\right\} \geq \frac{\Gamma(b)}{\Gamma(b)+2\Gamma(c)}\ (z \in \mathbb{D}). \tag{11}$$

Proof. It is easy to see from (8) of Lemma 1 that

$$1 + \sum_{k=1}^{\infty} b_k \leq \frac{\Gamma(b)+2\Gamma(c)}{\Gamma(b)}$$

which is equivalent to

$$\frac{\Gamma(b)}{2\Gamma(c)}\sum_{k=1}^{\infty} b_k \leq 1 \left(\text{where } b_k = \frac{\Gamma(c)\Gamma(b+\tau n)}{n!\Gamma(b)\Gamma(c+\tau n)}\right). \tag{12}$$

To prove 10, we have to show that

$$\frac{\Gamma(b)}{2\Gamma(c)}\left[\frac{{}_1\Phi_1^\tau(b;c;z)}{({}_1\Phi_1^\tau(b;c;z))_n} - \left(\frac{\Gamma(b)-2\Gamma(c)}{\Gamma(b)}\right)\right] \prec \frac{1+z}{1-z}. \tag{13}$$

Using definition of subordination, and putting the values of ${}_1\Phi_1^{\tau}(b;c;z)$ and $({}_1\Phi_1^{\tau}(b;c;z))_n$, we have

$$\frac{1+\sum\limits_{k=1}^{n} b_k z^k + \frac{\Gamma(b)}{2\Gamma(c)} \sum\limits_{k=n+1}^{\infty} b_k z^k}{1+\sum\limits_{k=1}^{n} b_k z^k} = \frac{1+w(z)}{1-w(z)}.$$

Our assertion 10 is true if we show that $w(0)=0$ and $|w(z)|<1$ provided $z \in \mathbb{D}$. Simplifying for $w(z)$, we get

$$w(z) = \frac{\frac{\Gamma(b)}{2\Gamma(c)} \sum\limits_{k=n+1}^{\infty} b_k z^k}{2+2\sum\limits_{k=1}^{n} b_k z^k + \frac{\Gamma(b)}{2\Gamma(c)} \sum\limits_{k=n+1}^{\infty} b_k z^k}.$$

Obviously $w(0)=0$ and

$$|w(z)| \le \frac{\frac{\Gamma(b)}{2\Gamma(c)} \sum\limits_{k=n+1}^{\infty} b_k}{2-2\sum\limits_{k=1}^{n} b_k - \frac{\Gamma(b)}{2\Gamma(c)} \sum\limits_{k=n+1}^{\infty} b_k} \le 1$$

provided

$$\sum_{k=1}^{n} b_k + \frac{\Gamma(b)}{2\Gamma(c)} \sum_{k=n+1}^{\infty} b_k \le 1. \tag{14}$$

It suffices to show that the left hand side of (14) is bounded above by left hand side of (12), which is equivalent to

$$\left(\frac{\Gamma(b)}{2\Gamma(c)} - 1\right) \sum_{k=1}^{n} b_k \ge 0.$$

This is true as $\Gamma(b) \ge 2\Gamma(c)$.

To prove the result (11), we write

$$\frac{\Gamma(b)+2\Gamma(c)}{2\Gamma(c)} \left[\frac{({}_1\Phi_1^{\tau}(b;c;z))_n}{{}_1\Phi_1^{\tau}(b;c;z)} - \frac{\Gamma(b)}{\Gamma(b)+2\Gamma(c)}\right] = \frac{1+w(z)}{1-w(z)}.$$

Substituting the values of ${}_1\Phi_1^{\tau}(b;c;z)$ and $({}_1\Phi_1^{\tau}(b;c;z))_n$ and simplifying for $w(z)$, we have

$$w(z) = \frac{-\frac{\Gamma(b)+2\Gamma(c)}{2\Gamma(c)} \sum\limits_{k=n+1}^{\infty} b_k z^k}{2+2\sum\limits_{k=1}^{n} b_k z^k - \frac{\Gamma(b)-2\Gamma(c)}{2\Gamma(c)} \sum\limits_{k=n+1}^{\infty} b_k z^k}.$$

Obviously $w(0)=0$ and

$$|w(z)| \le \frac{\frac{\Gamma(b)+2\Gamma(c)}{2\Gamma(c)} \sum\limits_{k=n+1}^{\infty} b_k}{2-2\sum\limits_{k=1}^{n} b_k - \frac{\Gamma(b)-2\Gamma(c)}{2\Gamma(c)} \sum\limits_{k=n+1}^{\infty} b_k} \le 1 \tag{15}$$

as (14) is true for $\Gamma(b) \ge 2\Gamma(c)$. □

Theorem 2. If $\tau > 0$, $c \geq b > max\{2-\tau, 0\}$ and $2\Gamma(b) \geq 11\Gamma(c)$, then

$$\Re\left\{\frac{({}_1\Phi_1^\tau(b;c;z))'}{({}_1\Phi_1^\tau(b;c;z))_n'}\right\} \geq \frac{2\Gamma(b)-11\Gamma(c)}{2\Gamma(b)} \quad (z \in \mathbb{D}) \tag{16}$$

and

$$\Re\left\{\frac{({}_1\Phi_1^\tau(b;c;z))_n'}{({}_1\Phi_1^\tau(b;c;z))'}\right\} \geq \frac{2\Gamma(b)}{2\Gamma(b)+11\Gamma(c)} \quad (z \in \mathbb{D}). \tag{17}$$

Proof. It is easy to see from (9) of Lemma 1 that

$$1+\sum_{k=1}^{\infty} b_k(k+1) \leq \frac{2\Gamma(b)+11\Gamma(c)}{2\Gamma(b)}$$

which is equivalent to

$$\frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}\sum_{k=1}^{\infty} b_k(k+1) \ \leq 1 \ \left(\text{where } b_k = \frac{\Gamma(c)\Gamma(b+\tau n)}{\Gamma(b)n!\Gamma(c+\tau n)}\right). \tag{18}$$

To prove (16), we have to show that

$$\frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}\left[\frac{({}_1\Phi_1^\tau(b;c;z))'}{({}_1\Phi_1^\tau(b;c;z))_n'} - \left(\frac{2\Gamma(b)-11\Gamma(c)}{2\Gamma(b)}\right)\right] \prec \frac{1+z}{1-z}. \tag{19}$$

Using definition of subordination, and putting the values of ${}_1\Phi_1^\tau(b;c;z)$ and $({}_1\Phi_1^\tau(b;c;z))_n$, we have

$$\frac{1+\sum\limits_{k=1}^{n} b_k(k+1)z^k + \frac{2\Gamma(b)}{11\Gamma(c)}\sum\limits_{k=n+1}^{\infty}(k+1)b_k z^k}{1+\sum\limits_{k=1}^{n}(k+1)b_k z^k} = \frac{1+w(z)}{1-w(z)}.$$

Our assertion (10) is ture if we show that $w(0)=0$ and $|w(z)|<1$ provided $z \in \mathbb{D}$. Simplifying for $w(z)$, we get

$$w(z) = \frac{\frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}\sum\limits_{k=n+1}^{\infty}(k+1)b_k z^k}{2+2\sum\limits_{k=1}^{n}(k+1)b_k z^k + \frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}\sum\limits_{k=n+1}^{\infty}(k+1)b_k z^k}$$

Obviously $w(0)=0$ and

$$|w(z)| \leq \frac{\frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}\sum\limits_{k=n+1}^{\infty}(k+1)b_k}{2-2\sum\limits_{k=1}^{n}(k+1)b_k - \frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}\sum\limits_{k=n+1}^{\infty}(k+1)b_k} \leq 1$$

provided

$$\sum_{k=1}^{n}(k+1)b_k + \frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}\sum_{k=n+1}^{\infty}(k+1)b_k \leq 1. \tag{20}$$

It suffices to show that the left hand side of (20) is bounded above by left hand side of (18), which is equivalent to

$$\left(\frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}-1\right)\sum_{k=1}^{n} b_k \geq 0.$$

This is true in view of hypothesis.

To prove the result (17), we write

$$\frac{2\Gamma(b)+11\Gamma(c)}{11\Gamma(c)}\left[\frac{({}_1\Phi_1^{\tau}(b;c;z))'_n}{({}_1\Phi_1^{\tau}(b;c;z))'}-\left(\frac{2\Gamma(b)}{2\Gamma(b)+11\Gamma(c)}\right)\right]=\frac{1+w(z)}{1-w(z)}.$$

Substituting the values of $({}_1\Phi_1^{\tau}(b;c;z))'$ and $({}_1\Phi_1^{\tau}(b;c;z))'_n$ and simplifying for $w(z)$, we have

$$w(z)=\frac{-(1+\frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)})\sum\limits_{k=n+1}^{\infty}(k+1)b_kz^k}{2+2\sum\limits_{k=1}^{n}(k+1)b_kz^k+\left(1-\frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}\right)\sum\limits_{k=n+1}^{\infty}(k+1)b_kz^k}.$$

Obviously $w(0)=0$ and

$$|w(z)|\leq\frac{(1+\frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)})\sum\limits_{k=n+1}^{\infty}(k+1)b_kz^k}{2-2\sum\limits_{k=1}^{n}(k+1)b_kz^k-\left(\frac{2}{11}\frac{\Gamma(b)}{\Gamma(c)}-1\right)\sum\limits_{k=n+1}^{\infty}(k+1)b_kz^k}\leq 1 \qquad (21)$$

as (20) is true under the hypothesis.

□

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