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BIFURCATION OF BRANCHES OF SOLUTIONS FOR IMPULSIVE BOUNDARY VALUE PROBLEMS

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Abstract. This work is concerned with an impulsive boundary value problem for second order differential equations with real parameter. Our approach is based on the implicit function theorem to prove existence of a unique branches of solutions, moreover we use bifurcation Krasnosel'ski theorems to prove existence of multiple branches of solutions depending on the values of the real parameter.

1. Introduction

Recently, the theory of Impulsive differential equations was distinguishing as an important area of investigation among several theories, since such equations arise in many mathematical models of real processes and phenomena studied in applied sciences, see for instance [5], [8], [10], [13], [23], [29] and references therein. Many problems were investigated for impulsive differential equations and among of them are interested by the study of the existence of solutions for boundary value problems of second order impulsive differential equations by using different methods; upper and lower solutions ([14], [15]), the topological degree theory ([16], [28]) and variational methods ([27], [30]).

As we know, the bifurcation technique is of great importance in the qualitative theory of differential equations (see [6], [7], [21] − [24]). In the case of impulsive differential equations the works [2] − [4], [17] − [22] and [26] have studied the problem of bifurcation analysis.

Liu and O'Regan [17] established some important results, they applied the Rabinowitz-type global bifurcation theorems ([24], [25]) from the trivial solution and infinity to show the existence of multiple solutions for second order impulsive differential equation. In [26], Wang and Yan studied the existence of multiple solutions for the second order impulsive differential equation. They used the properties of the eigenvalues and eigenfunctions to prove two Rabinowitz-type global bifurcation theorems. In [18], Ma. *et al.* showed the existence of sign-changing solutions of the problem of [17] by global bifurcation techniques.

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A recent work on bifurcation for impulsive differential equations by Niu and Yan in [22] considered the following impulsive boundary value problem

$$(I)\begin{cases} -x^{''}(t)+f\left(t,x(t)\right)=\lambda a x(t), \quad t\in(0,1), \quad t\neq \frac{1}{2},\\ \Delta x|_{t=\frac{1}{2}}=\beta_1 x(\frac{1}{2}),\\ \Delta x^{'}|_{t=\frac{1}{2}}=-\beta_2 x^{'}(\frac{1}{2}),\\ x(0)=x(1)=0.\end{cases}$$

$$(II)\begin{cases} u^{''}(t)=\lambda f\left(t,u(t),u^{'}(t)\right), \quad t\in(0,1), \quad t\neq t_k,\\ \Delta u(t_k)=\eta_k\left(u(t_k),u^{'}(t_k),\lambda\right), \quad k=1,\ldots,r,\\ \Delta u^{'}(t_k)=\theta_k\left(u(t_k),u^{'}(t_k),\lambda\right),\\ u(0)=u(1)=0.\end{cases}$$

We have studied the existence of multiple solutions for the relative nonlinear second order impulsive differential equations by Krasnosel'ski bifurcation theory. In this work we investigate the existence of solutions of the following impulsive boundary value problem

$$u^{''}(t) = f\left(t,u(t),u^{'}(t),\lambda\right), \quad t\in(0,1), \quad t\neq t_k, \tag{1}$$

$$\Delta u(t_k) = \eta_k\left(u(t_k),u^{'}(t_k),\lambda\right), \quad k=1,\ldots,r, \tag{2}$$

$$\Delta u^{'}(t_k) = \theta_k\left(u(t_k),u^{'}(t_k),\lambda\right), \tag{3}$$

$$u(0) = u(1)=0, \tag{4}$$

where $r\in\mathbb{N}^*=\mathbb{N}\setminus\{0\}$, $\Delta u(t_k)=u(t_k^+)-u(t_k)$, $\Delta u^{'}(t_k)=u^{'}(t_k^+)-u'(t_k^-)$, $0=t_0<t_1<t_2<\ldots<t_r<t_{r+1}=1$, $\lambda\in\mathbb{R}$, the functions $f:I^{'}\times\mathbb{R}^3\longrightarrow\mathbb{R}$ is smooth enough, $\eta_k\in C^1(\mathbb{R}^3,\mathbb{R})$ and $\theta_k\in C^1(\mathbb{R}^3,\mathbb{R})$ with $I^{'}:=I-\{t_k\}_{k=1}^r$.

2. Preliminaries

In this section, we give some definitions and preliminary results to be used in the coming sections of this work. For $i\geq 0$, let

$$\begin{aligned} PC^i(I) \;:=\; & \{u\in C^i(I^{'},\mathbb{R})/u^{(j)} \text{ is left continuous at } t_k, \text{ and}\\ & u^{(j)}(t_k^+) \text{ exist for all } k,j;\ 0\leq k\leq r, 0\leq j\leq i\}.\end{aligned}$$

$(PC^i(I),\|.\|_i)$ is a Banach space with the norm $\|w\|_i=\max(\|w\|_0,\|w^{'}\|_0,...,\|w^{(i)}\|_0)$, where
$\|w\|_0=\sup\{|w(t)|,t\in I\}$ for $w\in PC^0(I)$.
Let $\mathfrak{L}(PC^i(I))$ be the Banach space of bounded linear operators on $PC^i(I)$ endowed by the norm
$\|L\|_{\mathfrak{L}(PC^i(I))}=\sup\limits_{\|x\|\leq 1}\|Lx\|_i$, where $x\in PC^i(I)$ and $L\in\mathfrak{L}(PC^i(I))$.

Definition 1 A pair (u,λ) is called a solution of $(1)-(4)$ if it satisfies $(1)-(4)$.

Remark 1 If $(0,\lambda)$ is a solution of $(1)-(4)$, it is called a trivial solution of $(1)-(4)$.

Lemma 1 ([3]) $(u,\lambda) \in PC^2(I) \times \mathbb{R}$ is a solution of $(1)-(4)$ if and only if $(u,\lambda) \in PC^1(I) \times \mathbb{R}$ and it satisfies the following equation

$$\begin{aligned} u(t) \quad &= \quad \int_0^1 G(t,s) f\left(s, u(s), u^{'}(s), \lambda\right) \mathrm{d}s \\ &+ \quad \sum_{0<t_k<t} \left[\eta_k\left(u(t_k), u^{'}(t_k), \lambda\right) + \theta_k\left(u(t_k), u^{'}(t_k), \lambda\right)(t-t_k)\right] \\ &- \quad t\sum_{k=1}^{r} \left[\eta_k\left(u(t_k), u^{'}(t_k), \lambda\right) + \theta_k\left(u(t_k), u^{'}(t_k), \lambda\right)(1-t_k)\right], \qquad \forall t \in I, \end{aligned}$$

where G is defined by

$$G(t,s) = ts - \min(t,s), \quad (t,s) \in [0,1]^2.$$

Let X be a Banach space normed by $\|.\|_X$ and consider the following equation

$$u - \lambda A u + N(u,\lambda) = 0, \tag{5}$$

where $u \in X$, $A : X \to X$ is a linear compact operator, and $N : X \times \mathbb{R} \to X$ is a continuous mapping satisfying
$(H1)$ $\qquad N(u,\lambda) = \circ(\|u\|_X).$
The trivial solution of $(1)-(4)$ is a solution of (5).

Remark 2 The bifurcation problem of (5) is to obtain a nontrivial solution $(u_\lambda, \lambda) \neq (0, \lambda^\star)$ of (5) from some point $(0,\lambda^\star)$ such that $u_\lambda \to 0 \quad \text{as} \quad \lambda \to \lambda^\star$.

The following theorems will be used to obtain bifurcation of branches of solutions of $(1)-(4)$.

Theorem 1 ([21, Krasnosel'ski Theorem]) Under hypothesis (H1), if $\mu \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$ is an eigenvalue of A with odd algebraic multiplicity $m \geq 1$, then $(0,\mu^{-1})$ is a bifurcation point of (5).

Theorem 2 ([21, Theorem 1.11]) If $\mu \in \mathbb{R}^*$ is a simple eigenvalue of A $(m=1)$, then it bifurcates from $(0,\mu^{-1})$ exactly two branches Γ_1 and Γ_2 of solutions of (5).

Define a linear operator L on $PC^2(I)$ by

$$D(L) := \{u \in PC^2(I); u(0) = u(1) = 0\}, \quad (Lv)(t) = v^{''}(t), \quad v \in D(L).$$

Then, we have.

Proposition 1 ([3]) The operator L is invertible and $L^{-1} : PC^0(I) \to PC^2(I)$ is given by $(L^{-1}v)(t) = \int_0^1 G(t,s)v(s)ds.$

Let F be the Nemitskii operator corresponding to f, then $F : PC^1(I)\times\mathbb{R} \to PC^0(I)$ such that

$F(u,\lambda)(t) := f\Big(t, u(t), u^{'}(t), \lambda\Big), \quad t \in I.$

Let $\Phi : PC^2(I) \times \mathbb{R} \to PC^2(I)$ be defined by

$$\begin{aligned}\Phi(u,\lambda)(t) &= \sum_{0<t_k<t} \Big[\eta_k \Big(u(t_k), u^{'}(t_k), \lambda\Big) + \theta_k \Big(u(t_k), u^{'}(t_k), \lambda\Big)(t - t_k)\Big] \\ &- \; t\sum_{k=1}^{r} \Big[\eta_k \Big(u(t_k), u^{'}(t_k), \lambda\Big) + \theta_k \Big(u(t_k), u^{'}(t_k), \lambda\Big)(1 - t_k)\Big].\end{aligned}$$

and $H : PC^2(I) \times \mathbb{R} \to PC^2(I)$ such that

$$H(u,\lambda) := L^{-1}FJ(u,\lambda) + \Phi(u,\lambda),$$

where J is the compact imbedding defined by $J : PC^2(I) \times \mathbb{R} \longrightarrow PC^1(I) \times \mathbb{R}$ with $J(u,\lambda) = (u,\lambda)$.

Then $L^{-1}\left[F\left(J(u,\lambda)\right)\right](t) = \int\limits_0^1 G(t,s) f\Big(s, u(s), u^{'}(s), \lambda\Big)\, \mathrm{d}s.$

Lemma 2 ([3]) The operators Φ and H are compact.

Lemma 3 ([3]) $(u,\lambda) \in PC^2(I) \times \mathbb{R}$ is a solution of $(1) - (4)$ if and only if $H(u,\lambda) = u$.

For fixed $\lambda \in \mathbb{R}$, $\dfrac{\partial H}{\partial u}(.,\lambda) : PC^2(I) \to \mathfrak{L}(PC^2(I))$ and $\dfrac{\partial H}{\partial u}(u,\lambda).\varphi = \dfrac{\partial (L^{-1}FJ)}{\partial u}(u,\lambda).\varphi + \dfrac{\partial \Phi}{\partial u}(u,\lambda).\varphi$, where $\varphi \in PC^2(I)$.

We have

$$\frac{\partial (L^{-1}FJ)}{\partial u}(u,\lambda).\varphi = \int\limits_0^1 G(t,s)\left[\frac{\partial f}{\partial x}\Big(s, u(s), u^{'}(s), \lambda\Big)\varphi(s) + \frac{\partial f}{\partial y}\Big(s, u(s), u^{'}(s), \lambda\Big)\varphi^{'}(s)\right] \mathrm{d}s$$

and

$$\begin{aligned}\frac{\partial \Phi}{\partial u}(u,\lambda).\varphi &= \sum_{0<t_k<t} \Big[\Big(\frac{\partial \eta_k}{\partial x}\Big(u(t_k), u^{'}(t_k), \lambda\Big)\varphi(t_k) + \frac{\partial \eta_k}{\partial y}\Big(u(t_k), u^{'}(t_k), \lambda\Big)\varphi^{'}(t_k)\Big) \\ &+ \; \Big(\frac{\partial \theta_k}{\partial x}\Big(u(t_k), u^{'}(t_k), \lambda\Big)\varphi(t_k) + \frac{\partial \theta_k}{\partial y}\Big(u(t_k), u^{'}(t_k), \lambda\Big)\varphi^{'}(t_k)\Big)(t - t_k)\Big] \\ &- \; t\sum_{k=1}^{r} \Big[\Big(\frac{\partial \eta_k}{\partial x}\Big(u(t_k), u^{'}(t_k), \lambda\Big)\varphi(t_k) + \frac{\partial \eta_k}{\partial y}\Big(u(t_k), u^{'}(t_k), \lambda\Big)\varphi^{'}(t_k)\Big) \\ &+ \; \Big(\frac{\partial \theta_k}{\partial x}\Big(u(t_k), u^{'}(t_k), \lambda\Big)\varphi(t_k) + \frac{\partial \theta_k}{\partial y}\Big(u(t_k), u^{'}(t_k), \lambda\Big)\varphi^{'}(t_k)\Big)(1 - t_k)\Big].\end{aligned}$$

Moreover

$$\begin{aligned}\left\|\frac{\partial\Phi}{\partial u}(u,\lambda)\right\|_{\mathfrak{L}(PC^2(I))} &= \sup_{\|\varphi\|_2\le 1}\left\|\frac{\partial\Phi}{\partial u}(u,\lambda).\varphi\right\|_2\\ &\le 2\sum_{k=1}^{r}\left[\left|\frac{\partial\eta_k}{\partial x}\left(u(t_k),u^{'}(t_k),\lambda\right)\right|+\left|\frac{\partial\eta_k}{\partial y}\left(u(t_k),u^{'}(t_k),\lambda\right)\right|\right.\\ &\quad\left.+\left|\frac{\partial\theta_k}{\partial x}\left(u(t_k),u^{'}(t_k),\lambda\right)\right|+\left|\frac{\partial\theta_k}{\partial y}\left(u(t_k),u^{'}(t_k),\lambda\right)\right|\right]\end{aligned}$$

and

$$\begin{aligned}\left\|\frac{\partial(L^{-1}FJ)}{\partial u}(u,\lambda)\right\|_{\mathfrak{L}(PC^2(I))} &= \sup_{\|\varphi\|_2\le 1}\left\|\frac{\partial(L^{-1}FJ)}{\partial u}(u,\lambda)\right\|_2\\ &\le \int_0^1 \|G\|_{L_\infty}\left[\left|\frac{\partial f}{\partial x}\left(s,u(s),u^{'}(s),\lambda\right)\right|+\left|\frac{\partial f}{\partial y}\left(s,u(s),u^{'}(s),\lambda\right)\right|\right]\mathrm{d}s.\end{aligned}$$

Hence

$$\begin{aligned}\left\|\frac{\partial H}{\partial u}(u,\lambda)\right\|_{\mathfrak{L}(PC^2(I))} &\le \left\|\frac{\partial(L^{-1}FJ)}{\partial u}(u,\lambda)\right\|_{\mathfrak{L}(PC^2(I))}+\left\|\frac{\partial\Phi}{\partial u}(u,\lambda)\right\|_{\mathfrak{L}(PC^2(I))}\\ &\le \int_0^1 \|G\|_{L_\infty}\left[\left|\frac{\partial f}{\partial x}\left(s,u(s),u^{'}(s),\lambda\right)\right|+\left|\frac{\partial f}{\partial y}\left(s,u(s),u^{'}(s),\lambda\right)\right|\right]\mathrm{d}s\\ &\quad+2\sum_{k=1}^{r}\left[\left|\frac{\partial\eta_k}{\partial x}\left(u(t_k),u^{'}(t_k),\lambda\right)\right|+\left|\frac{\partial\eta_k}{\partial y}\left(u(t_k),u^{'}(t_k),\lambda\right)\right|\right.\\ &\quad\left.+\left|\frac{\partial\theta_k}{\partial x}\left(u(t_k),u^{'}(t_k),\lambda\right)\right|+\left|\frac{\partial\theta_k}{\partial y}\left(u(t_k),u^{'}(t_k),\lambda\right)\right|\right].\end{aligned}$$

3. Main results

Let the following hypotheses be satisfied

(H2) $f(t,0,0,\lambda^\star)=0,\quad \forall t\in I,$ for some $\lambda^\star\in\mathbb{R}$,
(H3) $\eta_k(0,0,\lambda^\star)=0,$ for some $\lambda^\star\in\mathbb{R}$,
(H4) $\theta_k(0,0,\lambda^\star)=0,$ for some $\lambda^\star\in\mathbb{R}$.

Let

$$\psi(u,\lambda)=u-H(u,\lambda). \tag{6}$$

From (H2), (H3) and (H4) we have $H(0,\lambda^\star)=0$, then $\psi(0,\lambda^\star)=0$ and $\dfrac{\partial\psi}{\partial u}(0,\lambda^\star)=I-\dfrac{\partial H}{\partial u}(0,\lambda^\star)$.
We have the following results.

Theorem 3 If $I-\dfrac{\partial H}{\partial u}(0,\lambda^\star)$ is invertible and the hypotheses (H2) $-$ (H4) are satisfied, then there exists $\delta>0$ such that for $|\lambda-\lambda^\star|<\delta$, the problem (1) $-$ (4) has a unique solution (u,λ).

Proof. The existence of a nontrivial solution of the problem $(1)-(4)$ is equivalent to the existence of $u \in PC^2(I)$ and $\lambda \in \mathbb{R}$ such that $\psi(u,\lambda)=0$. Since $\psi(0,\lambda^\star)=0$, and $\dfrac{\partial \psi}{\partial u}(0,\lambda^\star) = I - \dfrac{\partial H}{\partial u}(0,\lambda^\star)$ is invertible operator, then the implicit function theorem implies that there exists $\delta > 0$ such that for $|\lambda - \lambda^\star| < \delta$, the problem $(1)-(4)$ has a unique solution (u,λ).

Lemma 4 If $\left\| \dfrac{\partial H}{\partial u}(0,\lambda^\star) \right\|_{\mathfrak{L}(PC^2(I))} < 1$, then $\dfrac{\partial \psi}{\partial u}(0,\lambda^\star) = I - \dfrac{\partial H}{\partial u}(0,\lambda^\star)$ is invertible.

Corollary 1 If $\left\| \dfrac{\partial H}{\partial u}(0,\lambda^\star) \right\|_{\mathfrak{L}(PC^2(I))} < 1$ and hypotheses (H2) $-$ (H4) are satisfied, then there exists $\delta > 0$ such that for $|\lambda - \lambda^\star| < \delta$, the problem $(1)-(4)$ has a unique solution (u,λ).

Corollary 2 If

$$\int_0^1 \|G\|_{L_\infty} \left[\left| \frac{\partial f}{\partial x}(s,0,0,\lambda^\star) \right| + \left| \frac{\partial f}{\partial y}(s,0,0,\lambda^\star) \right| \right] \mathrm{d}s$$
$$+2\sum_{k=1}^{r} \left[\left| \frac{\partial \eta_k}{\partial x}(0,0,\lambda^\star) \right| + \left| \frac{\partial \eta_k}{\partial y}(0,0,\lambda^\star) \right| + \left| \frac{\partial \theta_k}{\partial x}(0,0,\lambda^\star) \right| + \left| \frac{\partial \theta_k}{\partial y}(0,0,\lambda^\star) \right| \right] < 1,$$

and the hypotheses (H2) $-$ (H4) are satisfied, then there exists $\delta > 0$ such that for $|\lambda - \lambda^\star| < \delta$, the problem $(1)-(4)$ has a unique solution (u,λ).

If the operator $I - \dfrac{\partial H}{\partial u}(0,\lambda)$ is not invertible, we investigate the existence of bifurcated solutions.
We have $N(u,\lambda) = \lambda A u - H(u,\lambda)$. If $\Big(D_u N(0,\lambda)\Big)\varphi(t) = 0$, we have

$$\begin{aligned}
A\varphi(t) &= \frac{1}{\lambda}\Big[\int_0^1 G(t,s) \left(\frac{\partial f}{\partial x}(s,0,0,\lambda)\varphi(s) + \frac{\partial f}{\partial y}(s,0,0,\lambda)\varphi'(s) \right) ds \\
&+ \sum_{0<t_k<t} \left(\frac{\partial \eta_k}{\partial x}(0,0,\lambda)\varphi(t_k) + \frac{\partial \eta_k}{\partial y}(0,0,\lambda)\varphi'(t_k) \right) \\
&+ \sum_{0<t_k<t} \left(\frac{\partial \theta_k}{\partial x}(0,0,\lambda)\varphi(t_k) + \frac{\partial \theta_k}{\partial y}(0,0,\lambda)\varphi'(t_k) \right)(t-t_k) \\
&- t\sum_{k=1}^{r} \left(\frac{\partial \eta_k}{\partial x}(0,0,\lambda)\varphi(t_k) + \frac{\partial \eta_k}{\partial y}(0,0,\lambda)\varphi'(t_k) \right) \\
&- t\sum_{k=1}^{r} \left(\frac{\partial \theta_k}{\partial x}(0,0,\lambda)\varphi(t_k) + \frac{\partial \theta_k}{\partial y}(0,0,\lambda)\varphi'(t_k) \right)(1-t_k) \Big] := A(\lambda)\varphi(t).
\end{aligned}$$

Then $\psi(u,\lambda) = u - \lambda A(\lambda)u + N(u,\lambda) = 0$. So, Krasnosel'ski theorem is not applicable, then we put additional hypotheses as follow

(H5) $\eta_k\Big(u(t_k), u^{'}(t_k), \lambda\Big) = \lambda\eta_k^1\Big(u(t_k), u^{'}(t_k)\Big)$,

(H6) $\theta_k\Big(u(t_k), u^{'}(t_k), \lambda\Big) = \lambda\theta_k^1\Big(u(t_k), u^{'}(t_k)\Big)$.

Let $A^1 : PC^2(I) \to PC^2(I)$ be the linear compact operator given by

$$\begin{aligned}
A^1\varphi(t) \quad &:= \sum_{0<t_k<t}\left(\frac{\partial\eta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial\eta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\right)\\
&+ \sum_{0<t_k<t}\left(\frac{\partial\theta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial\theta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\right)(t-t_k)\\
&- t\sum_{k=1}^{r}\left(\frac{\partial\eta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial\eta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\right)\\
&- t\sum_{k=1}^{r}\left(\frac{\partial\theta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial\theta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\right)(1-t_k).
\end{aligned}$$

We have $\psi(u,\lambda) = u - \lambda A^1 u + N(u,\lambda) = 0$, where $N(u,\lambda) := \lambda A^1 u - H(u,\lambda)$. Then

$$\begin{aligned}
\Big(D_u N(0,\lambda)\Big)\varphi(t) \quad &= \lambda A^1\varphi(t) - \frac{\partial H}{\partial u}(0,\lambda)\varphi(t)\\
&= \lambda\sum_{0<t_k<t}\left(\frac{\partial\eta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial\eta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\right)\\
&+ \lambda\sum_{0<t_k<t}\left(\frac{\partial\theta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial\theta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\right)(t-t_k)\\
&- \lambda t\sum_{k=1}^{r}\left(\frac{\partial\eta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial\eta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\right)\\
&- \lambda t\sum_{k=1}^{r}\left(\frac{\partial\theta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial\theta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\right)(1-t_k)\\
&- \int_0^1 G(t,s)\left(\frac{\partial f}{\partial x}(s,0,0,\lambda)\varphi(s) + \frac{\partial f}{\partial y}(s,0,0,\lambda)\varphi^{'}(s)\right)ds\\
&- \lambda\sum_{0<t_k<t}\left(\frac{\partial\eta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial\eta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\right)
\end{aligned}$$

$$
\begin{aligned}
&- \lambda \sum_{0<t_k<t} \left(\frac{\partial \theta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial \theta_k^1}{\partial y}(0,0)\varphi^{'}(t_k) \right)(t-t_k) \\
&+ \lambda t \sum_{k=1}^{r} \left(\frac{\partial \eta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial \eta_k^1}{\partial y}(0,0)\varphi^{'}(t_k) \right) \\
&+ \lambda t \sum_{k=1}^{r} \left(\frac{\partial \theta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial \theta_k^1}{\partial y}(0,0)\varphi^{'}(t_k) \right)(1-t_k)
\end{aligned}
$$

and

$$
\begin{aligned}
N(0,\lambda) &= -H(0,\lambda) \\
&= -\int_0^1 G(t,s)f(s,0,0,\lambda)ds - \lambda \sum_{0<t_k<t} \left[\eta_k^1(0,0) + \theta_k^1(0,0)(t-t_k) \right] \\
&+ \lambda t \sum_{k=1}^{r} \left[\eta_k^1(0,0) + \theta_k^1(0,0)(1-t_k) \right].
\end{aligned}
$$

Let the following hypotheses be satisfied

(H7) $\frac{\partial f}{\partial x}(t,0,0,\lambda) = 0 \ \ \forall t \in I, \ \forall \lambda \in \mathbb{R}$,

(H8) $\frac{\partial f}{\partial y}(t,0,0,\lambda) = 0 \ \ \forall t \in I, \ \forall \lambda \in \mathbb{R}$,

(H9) $\mu \in \mathbb{R}^*$ is an eigenvalue of A^1 with odd algebraic multiplicity,

(H10) $\mu \in \mathbb{R}^*$ is a simple eigenvalue of A^1.

From $(H2)-(H8)$, we have $D_u N(0,\lambda) = 0$ and $N(0,\lambda) = 0$, so $N(u,\lambda) = o(\|u\|_2)$. Then, from theorem 1 we have

Theorem 4 If the hypotheses (H2) − (H9) are satisfied, then $(u,\lambda) = (0,\mu^{-1})$ is a bifurcation point of $\psi(u,\lambda) = 0$ and (1) − (4) has a bifurcated branches of solutions.

And from theorem 2 we have

Theorem 5 If the hypotheses (H2) − (H8) and (H10) are satisfied, then (1) − (4) has exactly two bifurcated branches of solutions from $(0,\mu^{-1})$.

In the following we study the multiplicity of the eigenvalues of A^1 to determine the number of branches of solutions.
To do that let $a_k := \frac{\partial \eta_k^1}{\partial x}(0,0)$, $b_k := \frac{\partial \eta_k^1}{\partial y}(0,0)$, $c_k := \frac{\partial \theta_k^1}{\partial x}(0,0)$ and $d_k := \frac{\partial \theta_k^1}{\partial y}(0,0)$, and put
$A_k := -c_k t_k^2 + (a_k - d_k)t_k + b_k$, $B_k := -c_k t_k^2 + (a_k + c_k - d_k)t_k + b_k + d_k = A_k + c_k t_k + d_k$, $C_k := -c_k t_k + a_k + c_k$, $D_k := -c_k t_k + a_k$.
Let $g_k(t) = h_k(t).t$ with

$$
h_k(t) = \begin{cases} 1 & \text{if } t \in]t_k, t_{k+1}[, \\ 0 & \text{otherwise,} \end{cases}
$$

$k = 0, 1, 2, ..., r$.

Proposition 2 Let $\mathbb{E} = \{\varphi \in PC^2(I)/\varphi(t) = \sum_{k=0}^{r}[\alpha_k g_k(t) + \beta_k h_k(t)],\ t \neq t_k\}$.
Then $\mathbb{E}$ be a Banach space with $\dim \mathbb{E} = 2r + 2$, moreover $\forall \varphi \in PC^2(I),\ A^1\varphi \in \mathbb{E}$.

Remark 3 Let μ be an eigenvalue of A^1 and φ_μ an eigenvector of A^1 associated to μ. Then

$$\varphi_\mu(t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ \sum_{k=1}^{r}[\alpha_k(\varphi_\mu)g_k(t) + \beta_k(\varphi_\mu)h_k(t)] & \text{if} \quad t \neq t_k, \\ \alpha_{k-1}(\varphi_\mu)t_k + \beta_{k-1}(\varphi_\mu) & \text{if} \quad t = t_k, \\ 0 & \text{if} \quad t = 1. \end{cases}$$

We denote $\alpha_k(\varphi_\mu) := \alpha_k$ and $\beta_k(\varphi_\mu) := \beta_k$.

Proposition 3 Let $\mu \in \mathbb{R}^*$. Then μ is an eigenvalue of A^1 if and only if there exist $\alpha_0, ..., \alpha_r, \beta_0, ..., \beta_r \in \mathbb{R}$ such that μ satisfies the following system with $(2r+2)$ equations

$$(III) \begin{cases} \mu\beta_0 = 0, \\ \Big(\mu + B_1\Big)\alpha_0 + \sum_{i=2}^{r} B_i\alpha_{i-1} + \sum_{i=1}^{r} C_i\beta_{i-1} = 0, \\ \\ A_1\alpha_0 + \Big(\mu + B_2\Big)\alpha_1 + \sum_{i=3}^{r} B_i\alpha_{i-1} + D_1\beta_0 + \sum_{i=2}^{r} C_i\beta_{i-1} = 0, \\ -A_1\alpha_0 - D_1\beta_0 + \mu\beta_1 = 0, \\ \vdots \\ A_1\alpha_0 + ... + A_k\alpha_{k-1} + (\mu + B_{k+1})\alpha_k + \sum_{i=k+2}^{r} B_i\alpha_{i-1} \\ +D_1\beta_0 + ... + D_k\beta_{k-1} + \sum_{i=k+1}^{r} C_i\beta_{i-1} = 0, \\ -A_1\alpha_0 - ... - A_k\alpha_{k-1} - D_1\beta_0 - ... - D_k\beta_{k-1} + \mu\beta_k = 0, \\ \vdots \\ A_1\alpha_0 + ... + A_{r-1}\alpha_{r-2} + (\mu + B_r)\alpha_{r-1} + D_1\beta_0 + ... + D_{r-1}\beta_{r-2} + C_r\beta_{r-1} = 0, \\ -A_1\alpha_0 - ... - A_{r-1}\alpha_{r-2} - D_1\beta_0 - ... - D_{r-1}\beta_{r-2} + \mu\beta_{r-1} = 0, \\ A_1\alpha_0 + ... + A_r\alpha_{r-1} + \mu\alpha_r + D_1\beta_0 + ... + D_r\beta_{r-1} = 0, \\ -A_1\alpha_0 - ... - A_r\alpha_{r-1} - D_1\beta_0 - ... - D_r\beta_{r-1} + \mu\beta_r = 0. \end{cases}$$

Moreover the eigenvector associated to μ is given by

$$\begin{aligned} \varphi_\mu(t) &= \sum_{k=1}^{r}[\alpha_k g_k(t) + \beta_k h_k(t)] \\ &= \sum_{k=1}^{r} h_k(t)\Big(\alpha_k t + \beta_k\Big),\ t \neq t_k,\ t \in [0, 1]. \end{aligned}$$

Proof. If $t \in]0, t_1[$, $A^1\varphi(t) = \mu\varphi(t)$ is equivalent to

$$\sum_{0<t_k<t} \Big[a_k\varphi(t_k) + b_k\varphi^{'}(t_k) + \Big(c_k\varphi(t_k) + d_k\varphi^{'}(t_k)\Big)(t-t_k)\Big]$$
$$-t\sum_{k=1}^{r} \Big[a_k\varphi(t_k) + b_k\varphi^{'}(t_k) + \Big(c_k\varphi(t_k) + d_k\varphi^{'}(t_k)\Big)(1-t_k)\Big] = \mu(\alpha_0 t + \beta_0).$$

Then

$$-t\sum_{k=1}^{r} a_k\Big[\varphi(t_k) + b_k\varphi^{'}(t_k) + \Big(c_k\varphi(t_k) + d_k\varphi^{'}(t_k)\Big)(1-t_k)\Big] = \mu(\alpha_0 t + \beta_0),\ \forall t \in]0, t_1[,$$

we obtain

$$t\Big[\mu\alpha_0 + a_1(\alpha_0 t_1 + \beta_0) + b_1\alpha_0 + \Big(c_1(\alpha_0 t_1 + \beta_0) + d_1\alpha_0\Big)(1-t_1)\Big]$$
$$+t\sum_{i=2}^{r} \Big[a_i(\alpha_{i-1}t_i + \beta_{i-1}) + b_i\alpha_{i-1} + \Big(c_i(\alpha_{i-1}t_i + \beta_{i-1}) + d_i\alpha_{i-1}\Big)(1-t_i)\Big] + \mu\beta_0 = 0,$$

$\forall t \in]0, t_1[$. Then

$$\begin{cases} \mu\beta_0 = 0 \\ \beta_0\Big[a_1 + c_1(1-t_1)\Big] + \displaystyle\sum_{i=2}^{r}\beta_{i-1}\Big[a_i + c_i(1-t_i)\Big] \\ +\alpha_0\Big[\mu + a_1t_1 + b_1 + c_1t_1(1-t_1) + d_1(1-t_1)\Big] \\ +\displaystyle\sum_{i=2}^{r}\alpha_{i-1}\Big[a_it_i + b_i + c_it_i(1-t_i) + d_i(1-t_i)\Big] = 0. \end{cases}$$

Finally, we have

$$\begin{cases} \mu\beta_0 = 0, \\ \Big(\mu + B_1\Big)\alpha_0 + \displaystyle\sum_{i=2}^{r} B_i\alpha_{i-1} + \sum_{i=1}^{r} C_i\beta_{i-1} = 0. \end{cases}$$

Similarly, for $t \in]t_k, t_{k+1}[$ with $k = 1, ..., r-1$, we obtain the following result

$$\begin{cases} A_1\alpha_0 + A_2\alpha_1 + ... + A_k\alpha_{k-1} + (\mu + B_{k+1})\alpha_k + \displaystyle\sum_{i=k+2}^{r} B_i\alpha_{i-1} + \\ D_1\beta_0 + D_2\beta_1 + ... + D_k\beta_{k-1} + \displaystyle\sum_{i=k+1}^{r} C_i\beta_{i-1} = 0, \\ -A_1\alpha_0 - A_2\alpha_1 - ... - A_k\alpha_{k-1} - D_1\beta_0 - D_2\beta_1 - ... - D_k\beta_{k-1} + \mu\beta_k = 0. \end{cases}$$

For $t \in]t_r, 1[$, $A^1\varphi(t) = \mu\varphi(t)$ is equivalent to

$$\begin{aligned} \mu(\alpha_r t + \beta_r) &= \sum_{k=1}^{r} a_k\Big[\varphi(t_k) + b_k\varphi^{'}(t_k) + \Big(c_k\varphi(t_k) + d_k\varphi^{'}(t_k)\Big)(t-t_k)\Big] \\ &- t\Big[\sum_{k=1}^{r} a_k\varphi(t_k) + b_k\varphi^{'}(t_k) + \Big(c_k\varphi(t_k) + d_k\varphi^{'}(t_k)\Big)(1-t_k)\Big], \end{aligned}$$

then we have

$$\begin{cases} A_1\alpha_0 + A_2\alpha_1 + ... + A_r\alpha_{r-1} + \mu\alpha_r + D_1\beta_0 + D_2\beta_1 + ... + D_r\beta_{r-1} = 0, \\ \\ -A_1\alpha_0 - A_2\alpha_1 - ... - A_r\alpha_{r-1} - D_1\beta_0 - D_2\beta_1 - ... - D_r\beta_{r-1} + \mu\beta_r = 0. \end{cases}$$

Lemma 5 Let $\mu \in \mathbb{R}^\star$. Then μ is an eigenvalue of A^1 if and only if there exist $\alpha_0, ..., \alpha_r, \beta_0, ..., \beta_r \in \mathbb{R}$ such that

$$(IV) \qquad M(\mu)\begin{pmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_r\\ \beta_0\\ \beta_1\\ \vdots\\ \beta_r\end{pmatrix} = 0,$$

where $M(\mu)$ is the $(2r+2)$ square matrix such that

$$M(\mu) = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \\ \tilde{E} & \tilde{F} \end{pmatrix}$$

where $\tilde{A}$ is a $2\times(r+1)$ matrix, $\tilde{B}$ is a $2\times(r+1)$ matrix, $\tilde{C}$ is a $(2r-2)\times(r+1)$ matrix, $\tilde{D}$ is a $(2r-2)\times(r+1)$ matrix, $\tilde{E}$ is a $2\times(r+1)$ matrix and $\tilde{F}$ is a $2\times(r+1)$ matrix such that:

1) $\tilde{A} = (a_{ij})$ with

$$\begin{cases} a_{1j} = 0 \quad \text{for} \quad j = \overline{1, r+1}, \\ \\ a_{21} = \mu + B_1, a_{2j} = B_j \quad \text{for} \quad j = \overline{2, r},\ a_{2(r+1)} = 0. \end{cases}$$

Then

$$\tilde{A} = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ \mu + B_1 & B_2 & ... & B_r & 0 \end{pmatrix}.$$

2) $\tilde{B} = (a_{ij})$ with

$$\begin{cases} a_{1(r+2)} = \mu,\ a_{1j} = 0 \quad \text{for} \quad j = \overline{(r+3), (2r+2)}, \\ \\ a_{2j} = C_{j-(r+1)} \quad \text{for} \quad j = \overline{(r+2), (2r+1)}, \quad a_{2(2r+2)} = 0. \end{cases}$$

Then

$$\tilde{B} = \begin{pmatrix} \mu & 0 & ... & 0 & 0 \\ C_1 & C_2 & ... & C_r & 0 \end{pmatrix}.$$

3) $\tilde{C} = (a_{ij})$ with

$$\begin{cases} a_{(2i+1)j} = A_j \quad \text{for} \quad i = \overline{1,(r-1)} \quad \text{and} \quad 1 \le j \le i, \\ a_{(2i+1)j} = \mu + B_j \quad \text{for} \quad i = \overline{1,(r-1)} \quad \text{and} \quad j = i+1, \\ a_{(2i+1)j} = B_j \quad \text{for} \quad i = \overline{1,(r-1)} \quad \text{and} \quad i+2 \le j \le r+1, \\ a_{(2i+1)(r+1)} = 0 \quad \text{for} \quad i = \overline{1,(r-1)}, \\ a_{(2i)j} = -A_j \quad \text{for} \quad i = \overline{2,r} \quad \text{and} \quad 1 \le j < i, \\ a_{(2i)j} = 0 \quad \text{for} \quad i = \overline{2,r} \quad \text{and} \quad i \le j \le r+1. \end{cases}$$

Then

$$\tilde{C} = \begin{pmatrix} A_1 & \mu + B_2 & B_3 & B_4 & ... & B_{r-1} & B_r & 0 \\ -A_1 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ A_1 & A_2 & \mu + B_3 & B_4 & ... & B_{r-1} & B_r & 0 \\ -A_1 & -A_2 & 0 & 0 & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots & \vdots & \vdots \\ A_1 & A_2 & A_3 & A_4 & ... & A_{r-1} & \mu + B_r & 0 \\ -A_1 & -A_2 & -A_3 & -A_4 & ... & -A_{r-1} & 0 & 0 \end{pmatrix}.$$

4) $\tilde{D} = (a_{ij})$ with

$$\begin{cases} a_{(2i+1)j} = D_{j-(r+1)} \quad \text{for} \quad i = \overline{1,(r-1)} \quad \text{and} \quad 1 \le j-(r+1) \le i, \\ a_{(2i+1)j} = C_{j-(r+1)} \quad \text{for} \quad i = \overline{1,(r-1)} \quad \text{and} \quad i+1 \le j-(r+1) \le r, \\ a_{(2i+1)(2r+2)} = 0 \quad \text{for} \quad i = \overline{1,(r-1)}, \\ a_{(2i)j} = -D_{j-(r+1)} \quad \text{for} \quad i = \overline{2,r} \quad \text{and} \quad 1 \le j-(r+1) < i, \\ a_{(2i)j} = \mu \quad \text{for} \quad i = \overline{2,r} \quad \text{and} \quad j-(r+1) = i, \\ a_{(2i)j} = 0 \quad \text{for} \quad i = \overline{2,r} \quad \text{and} \quad i < j-(r+1) \le r+1. \end{cases}$$

Then

$$\tilde{D} = \begin{pmatrix} D_1 & C_2 & C_3 & C_4 & ... & C_{r-1} & C_r & 0 \\ -D_1 & \mu & 0 & 0 & ... & 0 & 0 & 0 \\ D_1 & D_2 & C_3 & C_4 & ... & C_{r-1} & C_r & 0 \\ -D_1 & -D_2 & \mu & 0 & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & ... & \vdots & \vdots & \vdots \\ D_1 & D_2 & D_3 & D_4 & ... & D_{r-1} & C_r & 0 \\ -D_1 & -D_2 & -D_3 & -D_4 & ... & -D_{r-1} & \mu & 0 \end{pmatrix}.$$

5) $\tilde{E} = (a_{ij})$ with

$$\begin{cases} a_{(2r+1)j} = A_j \quad \text{for} \quad 1 \le j \le r, \; a_{(2r+1)(r+1)} = \mu, \\ a_{(2r+2)j} = -A_j \quad \text{for} \quad 1 \le j \le r, \; a_{(2r+2)(r+1)} = 0. \end{cases}$$

Then

$$\tilde{E} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & ... & A_r & \mu \\ -A_1 & -A_2 & -A_3 & -A_4 & ... & -A_r & 0 \end{pmatrix}.$$

6) $\tilde{F} = (a_{ij})$ with

$$\begin{cases} a_{(2r+1)j} = D_{j-(r+1)} \quad \text{for} \quad 1 \le j-(r+1) \le r \quad \text{and} \quad j = \overline{(r+2),(2r+1)}, \\ a_{(2r+1)(2r+2)} = 0, \\ a_{(2r+2)j} = -D_{j-(r+1)} \quad \text{for} \quad 1 \le j-(r+1) \le r \quad \text{and} \quad j = \overline{(r+2),(2r+1)}, \\ a_{(2r+2)(2r+2)} = \mu. \end{cases}$$

Then

$$\tilde{F} = \begin{pmatrix} D_1 & D_2 & ... & D_r & 0 \\ -D_1 & -D_2 & ... & -D_r & \mu \end{pmatrix}.$$

Proof. From the proposition 3, the system (III) is equivalent to (IV).

Put $P(\mu) = detM(\mu)$, then $\mu \in \mathbb{R}^*$ is an eigenvalue of A^1 if and only if $M(\mu)$ is not invertible, i.e. $P(\mu) = 0$.

Remark 4 Let μ be a real eigenvalue of A^1. If μ satisfies
$(H11)$ $\quad P(\mu) = P^{'}(\mu) = P^{''}(\mu) = ... = P^{2q}(\mu) = 0$ and $P^{2q+1}(\mu) \neq 0,\ q \in \mathbb{N}$,
then it is an eigenvalue with odd algebraic multiplicity $2q+1$.
If μ is a simple eigenvalue of A^1, i.e $q = 0$, then
$(H12)$ $\quad P(\mu) = 0$ and $P^{'}(\mu) \neq 0$.

From theorem 4 we have

Corollary 3 If (H2) – (H8) and (H11) are satisfied with $\mu \in \mathbb{R}^*$, then $(1)-(4)$ has a bifurcated branches of solutions from $(0, \mu^{-1})$.

From theorem 5 we have

Corollary 4 If (H2) – (H8) and (H12) are satisfied with $\mu \in \mathbb{R}^*$, then $(1)-(4)$ has exactly two bifurcated branches of solutions Γ_1 and Γ_2 from $(0, \mu^{-1})$.

Proposition 4 Let $A_k = 0$ for $k = 1, ..., r-1$. We have

- $P(\mu) = \mu^{r+2} \prod_{k=1}^{r} (\mu + B_k)$, moreover the eigenvalues of A^1 are 0 and $-B_k$.
- If there exists $k_0 \in \{1, ..., r\}$ such that $B_{k_0} \neq B_k \quad \forall k \in \{1, ..., r\}/\{k_0\}$ and $B_{k_0} \neq 0$, then $-B_{k_0}$ is a simple eigenvalue of A^1.

Let $b_k = t_k^2 c_k + t_k(d_k - a_k)$ with $k = 1, ..., r-1$, we have
(H13) $\quad k_0 \in \{1, ..., r-1\}$ such that $d_{k_0} \neq -t_{k_0} c_{k_0}$, $d_{k_0} + t_{k_0} c_{k_0} \neq d_k + t_k c_k \quad \forall k \in \{1, ..., r-1\}/k_0$ and $d_{k_0} + t_{k_0} c_{k_0} \neq A_r + c_r t_r + d_r$.
(H14) $\quad b_r \neq c_r t_r^2 + (d_r - a_r - c_r) t_r - d_r$ and $d_k + t_k c_k \neq -c_r t_r^2 + (a_r + c_r - d_r) t_r + d_r + b_r \ \forall k \in \{1, ..., r-1\}$.

Remark 5 If (H13) is satisfied then $\mu = -B_{k_0}$ is a real simple eigenvalue of A^1 and $B_{k_0} \neq 0$.
If (H14) is satisfied then $\mu = -B_r$ is a real simple eigenvalue of A^1 and $B_r \neq 0$.

From theorem 5, we have

Corollary 5 If (H2) – (H8) and (H13) are satisfied then $(1)-(4)$ has exactly two bifurcated branches of solutions Γ_1 and Γ_2 from $\left(0, -B_{k_0}^{-1}\right)$ with $k_0 \in \{1, ..., r-1\}$.
If (H2) – (H8) and (H14) are satisfied, then $(1)-(4)$ has exactly two bifurcated branches of solutions Γ_1 and Γ_2 from $\left(0, -B_r^{-1}\right)$.

4. Examples

In this section we give some examples to illustrate the applications of our results.

Example 1 Consider the following homogeneous boundary value problem of $(1)-(4)$

$$\begin{cases} u^{''}(t) = \lambda f\left(t, u(t), u^{'}(t)\right), \quad t \in (0,1), \quad t \neq t_k, \\ \Delta u(t_k) = \lambda \eta_k\left(u(t_k), u^{'}(t_k)\right), \quad k = 1, \ldots, r, \\ \Delta u^{'}(t_k) = \lambda \theta_k\left(u(t_k), u^{'}(t_k)\right), \\ u(0) = u(1) = 0, \end{cases} \tag{7}$$

where $f(t,0,0) = 0$, $\eta_k(0,0) = 0$ and $\theta_k(0,0) = 0$. We have

$$\begin{aligned} H(u,\lambda) \quad &= \quad \lambda \int_0^1 G(t,s) f\left(s, u(s), u^{'}(s)\right) \mathrm{d}s \\ &+ \quad \lambda \sum_{0<t_k<t} \left[\eta_k\left(u(t_k), u^{'}(t_k)\right) + \theta_k\left(u(t_k), u^{'}(t_k)\right)(t - t_k)\right] \\ &- \quad \lambda t \sum_{k=1}^{r} \left[\eta_k\left(u(t_k), u^{'}(t_k)\right) + \theta_k\left(u(t_k), u^{'}(t_k)\right)(1 - t_k)\right], \qquad \forall t \in I, \end{aligned}$$

On the one hand, for $\dfrac{\partial f}{\partial x}(t,0,0) = \dfrac{\partial f}{\partial y}(t,0,0) = 0 \quad \forall t \in I$, we have
If $\left\|\dfrac{\partial H}{\partial u}(0,\lambda^\star)\right\|_{\mathfrak{L}(PC^2(I))} < 1$. So from corollary 1, there exists $\delta > 0$ such that for $|\lambda - \lambda^\star| < \delta$, the problem (7) has a unique solution (u,λ).
If $\left\|\dfrac{\partial H}{\partial u}(0,\lambda^\star)\right\|_{\mathfrak{L}(PC^2(I))} > 1$, we investigate the existence of bifurcated branches of solutions. Then

$$\begin{aligned} A^1\varphi(t) \quad &:= \quad \sum_{0<t_k<t} \Big[\frac{\partial \eta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial \eta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\Big] \\ &+ \quad \sum_{0<t_k<t} \Big[\frac{\partial \theta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial \theta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\Big](t - t_k) \\ &- \quad t\sum_{k=1}^{r} \Big[\frac{\partial \eta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial \eta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\Big] \\ &- \quad t\sum_{k=1}^{r} \Big[\frac{\partial \theta_k^1}{\partial x}(0,0)\varphi(t_k) + \frac{\partial \theta_k^1}{\partial y}(0,0)\varphi^{'}(t_k)\Big](1 - t_k). \end{aligned}$$

So, the existence of bifurcated solutions of (7) is equivalent to $(1)-(4)$. Hence, we can apply corollaries 3-5 to prove existence of bifurcated solutions of (7).

For $\dfrac{\partial \eta_k}{\partial x}(0,0) = \dfrac{\partial \eta_k}{\partial y}(0,0) = 0$ and $\dfrac{\partial \theta_k}{\partial x}(0,0) = \dfrac{\partial \theta_k}{\partial y}(0,0) = 0$, we have

$$A\varphi(t) := \int_0^1 G(t,s)\Big(\frac{\partial f}{\partial x}(s,0,0).\varphi(s) + \frac{\partial f}{\partial y}(s,0,0).\varphi^{'}(s)\Big)\mathrm{d}s.$$

So, the existence of bifurcated solutions of (7) is equivalent to (II) (see [3]).

Example 2 Consider the following two point boundary value problem for second order impulsive differential equation

$$\begin{cases} u^{''}(t) = f\Big(t, u(t), u^{'}(t), \lambda\Big) \quad t \neq t_1 \\ \Delta u(t_1) = \lambda\gamma u(t_1) \\ \Delta u^{'}(t_1) = \lambda\gamma^{'} u^{'}(t_1) \\ u(0) = u(1) = 0 \end{cases} \tag{8}$$

where $f(t,0,0,\lambda) = 0$, $\dfrac{\partial f}{\partial x}(t,0,0,\lambda) = 0$, $\dfrac{\partial f}{\partial y}(t,0,0,\lambda) = 0 \quad \forall t \in I$ and $\gamma, \gamma^{'} \in \mathbb{R}^*$.
We have

$$\begin{aligned} H(u,\lambda) = &\int_0^1 G(t,s) f\Big(s, u(s), u^{'}(s), \lambda\Big)\,\mathrm{d}s - \lambda t\Big[\gamma u(t_1) + \gamma^{'} u^{'}(t_1)(1-t_1)\Big] \\ &+\lambda \sum_{0<t_1<t}\Big[\gamma u(t_1) + \gamma^{'} u^{'}(t_1)(t-t_1)\Big]. \end{aligned}$$

If $|\lambda^\star| < \dfrac{1}{2(|\gamma| + |\gamma^{'}|)}$, then $\left\| \dfrac{\partial H}{\partial u}(0,\lambda^\star)\right\|_{\mathfrak{L}(PC^2(I))} < 1$. So from corollary 1, there exists $\delta > 0$ such that for $|\lambda - \lambda^\star| < \delta$, the problem (8) has a unique solution (u,λ). For $|\lambda^\star| \geq \dfrac{1}{2(|\gamma| + |\gamma^{'}|)}$, we investigate the existence of bifurcated branches of solutions. Then

$$A^1\varphi(t) = -t\Big[\gamma\varphi(t_1) + \gamma^{'}\varphi^{'}(t_1)(1-t_1)\Big] + \sum_{0<t_1<t}\Big[\gamma\varphi(t_1) + \gamma^{'}\varphi^{'}(t_1)(t-t_1)\Big].$$

$\mu \in \mathbb{R}^*$ is an eigenvalue of A^1 if and only if there exist $\alpha_0, \alpha_1, \beta_0, \beta_1$ such that μ satisfies the following system with four equations

$$\begin{cases} \mu\beta_0 = 0, \\ \Big(\mu + B_1\Big)\alpha_0 + C_1\beta_0 = 0, \\ A_1\alpha_0 + \mu\alpha_1 + D_1\beta_0 = 0, \\ -A_1\alpha_0 - D_1\beta_0 + \mu\beta_1 = 0, \end{cases}$$

where $A_1 = (\gamma - \gamma^{'})t_1$, $B_1 = (\gamma - \gamma^{'})t_1 + \gamma^{'}$ and $C_1 = D_1 = \gamma$.
The matrix $M(\mu)$ is given by

$$M(\mu) = \begin{pmatrix} 0 & 0 & \mu & 0 \\ \mu + (\gamma - \gamma^{'})t_1 + \gamma^{'} & 0 & \gamma & 0 \\ (\gamma - \gamma^{'})t_1 & \mu & \gamma & 0 \\ -(\gamma - \gamma^{'})t_1 & 0 & -\gamma & \mu \end{pmatrix}.$$

Let $P(\mu) := \det M(\mu) = \mu^3\Big(\mu + (\gamma - \gamma^{'})t_1 + \gamma^{'}\Big)$.
If $\gamma = \gamma'$ we have exactly two bifurcated branches of solutions Γ_1 and Γ_2 from $\big(0, (-\gamma)^{-1}\big)$.
If $\gamma^{'} \neq \gamma$ and $t_1 \neq \dfrac{\gamma^{'}}{\gamma^{'} - \gamma}$, then $\mu = -(\gamma - \gamma^{'})t_1 - \gamma^{'}$ is a real simple eigenvalue of A^1. From corollary 4, (8) has exactly two bifurcated branches of solutions Γ_1 and Γ_2 from $\left(0, \Big(-(\gamma - \gamma^{'})t_1 - \gamma^{'}\Big)^{-1}\right)$.

Remark 6 In [26], the authors consider the problem (8) with $\gamma^{'} = -\gamma$ and $t_1 = \dfrac{1}{2}$. But in our case we must have $t_1 \neq \dfrac{\gamma^{'}}{\gamma^{'} - \gamma} = \dfrac{1}{2}$. So , for this case we need to use an other approach.

If $\gamma^{'} = -\gamma$ and $t_1 \neq \dfrac{1}{2}$, then (8) becomes

$$\begin{cases} u^{''}(t) = f\Big(t, u(t), u^{'}(t), \lambda\Big), \quad t \neq t_1, \\ \Delta u(t_1) = \lambda\gamma u(t_1), \\ \Delta u^{'}(t_1) = -\lambda\gamma u^{'}(t_1), \\ u(0) = u(1) = 0. \end{cases} \tag{9}$$

If $|\lambda^\star| < \dfrac{1}{4|\gamma|}$, then $\|\dfrac{\partial H}{\partial u}(0, \lambda^\star)\|_{\mathfrak{L}(PC^2(I))} < 1$. So from corollary 1, there exists $\delta > 0$ such that for $|\lambda - \lambda^\star| < \delta$, the problem (9) has a unique solution (u, λ).
For $|\lambda^\star| \geq \dfrac{1}{4|\gamma|}$, we investigate the existence of bifurcated branches of solutions. Then

$$A^1\varphi(t) = -t\Big[\gamma\varphi(t_1) - \gamma\varphi^{'}(t_1)(1 - t_1)\Big] + \sum_{0<t_1<t}\Big[\gamma\varphi(t_1) - \gamma\varphi^{'}(t_1)(t - t_1)\Big].$$

and $P(\mu) = \det M(\mu) = \mu^3\Big(\mu + \gamma(2t_1 - 1)\Big)$. From corollary 4, (9) has exactly two bifurcated branches of solutions Γ_1 and Γ_2 from $\left(0, \Big(-\gamma(2t_1 - 1)\Big)^{-1}\right)$.

Example 3 Consider the following two point boundary value problem for second order impulsive differential equation

$$\begin{cases} u^{''}(t) = f\Big(t, u(t), u^{'}(t), \lambda\Big), \quad t \neq t_k, \\ \Delta u(t_k) = \lambda\gamma_k u(t_k), \quad k = 1, 2, \\ \Delta u^{'}(t_1) = \lambda\zeta_k u^{'}(t_k), \\ u(0) = u(1) = 0. \end{cases} \tag{10}$$

If $|\lambda^\star| < \dfrac{1}{2(|\gamma_1| + |\gamma_2| + |\zeta_1| + |\zeta_2|)}$, then $\|\dfrac{\partial H}{\partial u}(0, \lambda^\star)\|_{\mathfrak{L}(PC^2(I))} < 1$. So from corollary 1, there exists $\delta > 0$ such that for $|\lambda - \lambda^\star| < \delta$, the problem (10) has a unique solution (u, λ).
For $|\lambda^\star| \geq \dfrac{1}{2(|\gamma_1| + |\gamma_2| + |\zeta_1| + |\zeta_2|)}$, we investigate the existence of bifurcated

branches of solutions. Then

$$A^1\varphi(t)=\sum_{0<t_k<t}\Big[\gamma_k\varphi(t_k)+\zeta_k\varphi^{'}(t_k)(t-t_k)\Big]-t\sum_{k=1}^{2}\Big[\gamma_k\varphi(t_k)+\zeta_k\varphi^{'}(t_k)(1-t_k)\Big].$$

$\mu\in\mathbb{R}^*$ is an eigenvalue of A^1 if and only if there exist $\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2\in\mathbb{R}$ such that μ satisfies the following system with 6 equations

$$\begin{cases}\mu\beta_0=0,\\ \Big(\mu+B_1\Big)\alpha_0+B_2\alpha_1+C_1\beta_0+C_2\beta_1=0,\\ A_1\alpha_0+\Big(\mu+B_2\Big)\alpha_1+D_1\beta_0+C_2\beta_1=0,\\ -A_1\alpha_0-D_1\beta_0+\mu\beta_1=0,\\ A_1\alpha_0+A_2\alpha_1+\mu\alpha_2+D_1\beta_0+D_2\beta_1=0,\\ -A_1\alpha_0-A_2\alpha_1-D_1\beta_0-D_2\beta_1+\mu\beta_2=0,\end{cases}$$

where $A_k=(\gamma_k-\zeta_k)t_k,\ \ B_k=(\gamma_k-\zeta_k)t_k+\zeta_k$ and $C_k=D_k=\gamma_k$.
The matrix $M(\mu)$ is given by

$$M(\mu)=\begin{pmatrix}0&0&0&\mu&0&0\\ \mu+B_1&B_2&0&C_1&C_2&0\\ A_1&\mu+B_2&0&D_1&C_2&0\\ -A_1&0&0&-D_1&\mu&0\\ A_1&A_2&\mu&D_1&D_2&0\\ -A_1&-A_2&0&-D_1&D_2&\mu\end{pmatrix},$$

$P(\mu)=\det M(\mu)=\mu^4\left[\mu^2+\mu(B_1+B_2)+A_1(C_2-B_2)+B_1B_2\right].$
μ is an eigenvalue of A^1 if it is equal to zero or it is a solution of the following equation

$$P_1(\mu)=\mu^2+\mu(B_1+B_2)+B_1B_2+A_1C_2-A_1B_2=0.$$

Case1: For $\gamma_1=\zeta_1$ we have $A_1=0$, then $P_1(\mu)=\mu^2+\mu(B_1+B_2)+B_1B_2$ and $\Delta_\mu=(B_1+B_2)^2-4B_1B_2=(B_1-B_2)^2$. So $P_1(\mu)=(\mu+B_1)(\mu+B_2)$ and $P(\mu)=\mu^4(\mu+B_1)(\mu+B_2)$. Then
$-B_1$ (resp. $-B_2$) is a simple eigenvalue if $B_2\neq B_1\neq 0$ (resp. $B_1\neq B_2\neq 0$),
$-B_1$ and $-B_2$ are simple eigenvalues if $B_1\neq B_2$ and $B_1B_2\neq 0$.
We deduce from corollary 5 that the problem (10) has exactly two bifurcated branches of solutions from $\Big(0,(-B_1)^{-1}\Big)$ and two bifurcated branches of solutions from $\Big(0,(-B_2)^{-1}\Big)$ if $-B_1$ and $-B_2$ are simple eigenvalues *i.e.* $B_1B_2\neq 0$ and $B_1\neq B_2$.
Case2: For $\gamma_1\neq\zeta_1$ and
$[(\gamma_1-\zeta_1)t_1+\zeta_1]^2+[(\gamma_2-\zeta_2)t_2+\zeta_2]^2>2[(\gamma_2-\zeta_2)\zeta_1t_2+\zeta_1\zeta_2-(\gamma_1-\zeta_1)(\gamma_2-\zeta_2)t_1t_2-(\gamma_1-\zeta_1)(\zeta_2-2\gamma_2)t_1]$, we have $A_1(C_2-B_2)+B_1B_2\neq 0$. Then $\mu\in\mathbb{R}^\star$ is an eigenvalue of A^1 if μ is a solution of $P_1(\mu)=0$.
The discriminant of $P_1(\mu)$ is

$$\begin{aligned}\Delta_\mu &= [(\gamma_1-\zeta_1)t_1+\zeta_1]^2+[(\gamma_2-\zeta_2)t_2+\zeta_2]^2\\ &+ 2[(\gamma_1-\zeta_1)(\gamma_2-\zeta_2)t_1t_2-(\gamma_1-\zeta_1)(\zeta_2-2\gamma_2)t_1-(\gamma_2-\zeta_2)\zeta_1t_2-\zeta_1\zeta_2]>0.\end{aligned}$$

So

$$\mu_1^2=\frac{-(\gamma_1-\zeta_1)t_1-(\gamma_2-\zeta_2)t_2-\zeta_1-\zeta_2-\sqrt{\Delta_\mu}}{2}$$

and

$$\mu_2^2 = \frac{-(\gamma_1-\zeta_1)t_1-(\gamma_2-\zeta_2)t_2-\zeta_1-\zeta_2+\sqrt{\Delta_\mu}}{2}$$

are simple eigenvalues of A^1. Then, from corollary 4 we have exactly two bifurcated branches of solutions from $\left(0,(\mu_1^2)^{-1}\right)$ and two bifurcated branches of solutions from $\left(0,(\mu_2^2)^{-1}\right)$.

Remark 7 In [17], the authors consider the problem (10) with $\zeta_1=\zeta_2=0$, in this case (10) becomes

$$\begin{cases} u^{''}(t)=f\left(t,u(t),u^{'}(t),\lambda\right), \quad t\neq t_k, \\ \Delta u(t_k)=\lambda\gamma_k u(t_k), \quad k=1,2, \\ \Delta u^{'}(t_k)=0, \\ u(0)=u(1)=0. \end{cases} \tag{11}$$

If $|\lambda^\star|<\dfrac{1}{2(|\gamma_1|+|\gamma_2|)}$, then $\left\|\dfrac{\partial H}{\partial u}(0,\lambda^\star)\right\|_{\mathfrak{L}(PC^2(I))}<1$. From corollary 1, there exists $\delta>0$ such that for $|\lambda-\lambda^\star|<\delta$, the problem (11) has a unique solution (u,λ).

For $|\lambda^\star|\geq\dfrac{1}{2(|\gamma_1|+|\gamma_2|)}$, we investigate the existence of bifurcated branches of solutions. We have

$$A^1\varphi(t)=\sum_{0<t_k<t}[\gamma_k\varphi(t_k)]-t\sum_{k=1}^{2}[\gamma_k\varphi(t_k)]$$

and $P(\mu)=\det M(\mu)=\mu^4\left[\mu^2+\mu(\gamma_1t_1+\gamma_2t_2)+\gamma_1\gamma_2t_1\right]$.

If $\gamma_1=0$ and $\gamma_2\neq0$, then $P(\mu)=\mu^5(\mu+\gamma_2t_2)$. From corollary 5, the problem (11) has exactly two bifurcated branches of solutions Γ_1^1 and Γ_2^1 from $\left(0,(-\gamma_2t_2)^{-1}\right)$.

If $\gamma_1\neq0$ and $(\gamma_1t_1+\gamma_2t_2)^2>4\gamma_1\gamma_2t_1$, then

$$\mu_1^2=\frac{-\gamma_1t_1-\gamma_2t_2-\sqrt{(\gamma_1t_1+\gamma_2t_2)^2-4\gamma_1\gamma_2t_1}}{2}$$

and

$$\mu_2^2=\frac{-\gamma_1t_1-\gamma_2t_2+\sqrt{(\gamma_1t_1+\gamma_2t_2)^2-4\gamma_1\gamma_2t_1}}{2}$$

are simple eigenvalues of A^1. Corollary 5 implies that the problem (11) has exactly two bifurcated branches of solutions Γ_1^2 and Γ_2^2 from $\left(0,(\mu_i^2)^{-1}\right)$ with $i=1,2$.

5. Concluding remarks

In this work, we have considered the existence and multiplicity of branches of solutions of second order impulsive differential equation with real parameter λ. We have used a different approach then that used in [3] since the non linear term in the differential equation is depending implicitly on the parameter λ. It will be very interesting to consider the case with both nonlinear term and impulse functions depending implicitly on the real parameter. The two approaches used in this work and in [3] can not be applied to such case.

References

[1] R.P. Agarwal and D. O'Regan, Multiple nonnegative solutions for second-second impulsive differential equations, Appl. Math. Comput, Vol. 114, 51-59, 2000.
[2] M.U. Akhhmet and A. Kashkynbayev, Non-Autonomous Bifurcation in impulsive systems, Electronic Journal of Qualitative Theory of Differential Equations, Vol. 74, 1-23, 2014.
[3] Z. Belattar and A. Lakmeche, Impulsive boundary value problem with parameter, Georgian Mathematical Journal, Vol. 22, 331-339, 2015.
[4] A.A. Boichuk and S.M. Chuiko, Bifurcation of solutions of an impulsive boundary-value problem, J. Nonlinear Oscillations, Springer Link, Vol. 1, 18-28, 2008.
[5] M. Choisy, J. Guegan and P. Rohani, Dynamics of infectious diseases and pulse vaccination: teasing apart the embedded resonance effects, Physica, Vol. 22, 26-35, 2006.
[6] S. Chow and J. Hale, Methods of bifurcation theory, Springer-Verlag, 1982.
[7] M. Crandall and P. Rabinowitz, Bifurcation from simple eigenvalues, J. of Func. Anal., Vol. 8, 321-340, 1971.
[8] X. Fu, J. Qi and Y. Liu, General comparison principal for impulsive variable time differential equations with application, Nonlinear Anal., Vol. 42, 1421-1429, 2000.
[9] X. Fu, B. Yan and Y. Liu, Introduction to impulsive differential system, Beijing: China Science Publisher, 2005.
[10] S. Gao, L. Chen, J. J. Nieto and A. Torres, Analysis of a delayed epidemic model with pulse vaccination and saturation incidence, Vaccine, Vol. 24, 6037-6045, 2006.
[11] D. Guo, Positive solutions of an infinite boundary value problem for n^th-order nonlinear impulsive singular integro-differential equations in Banach spaces, Nonlinear Anal., Vol. 70, 2078-2090, 2003.
[12] Z. Hu and M. Han, Periodic solutions and bifurcations of first order periodic impulsive differential equations, International Journal of Bifurcation and Chaos, Vol. 19, 2515-2030, 2009.
[13] V. Lakshmikantham, D. Bainov and P. Simeonov, Theory of impulsive differential equations, Singapore: World Scientific, 1989.
[14] E.K. Lee and Y.H. Lee, Multiple position solutions of singular two point boundary value problems for second order impulsive differential equations, Appl. Math. Comut, Vol. 40, 745-759, 2004.
[15] Y. Lee and X. Liu, Study of singular boundary value problems for second order impulsive differential equations, J. Math. Appl, Vol. 331, 159-176, 2007.
[16] X. Lin and D. Jiang, Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, J. Anal. Appl., Vol. 321, 501-514, 2006.
[17] Y. Liu and D. O'Regan, Multiplicity results using bifurcation techniques for a class of boundary value problem of impulsive differential equations, Commun. Nonlinear Sci. Numer. Simul., Vol. 16, 1769-1775, 2011.
[18] R. Ma, J. Sun and M. Elsanosi, Sign-changing solutions of second order dirichlet problem with impulse effects, D. C. D. I. Syst., Vol. 20, 241-251, 2013.
[19] R. Ma, B. Yang and Z. Wang, Bifurcation of positive periodic solutions of first-order impulsive differential equations, Boundary value problems, Springer, Vol. 83, 2012.
[20] R. Ma, B. Yang and Z. Wang, Positive periodic solutions of first-order delay differential equations with impulses, App. Math. Comput., Vol. 219, 6074-6083, 2013.
[21] T. Ma and S. Wang, Bifurcation theory and applications, World Scientific, Vol. 10, 2005.
[22] Y. Niu and B. Yan, Global structure of solutions to boundary value problems of impulsive differential equations, Electronic Journal of Differential Equations, Vol.55, 1-23, 2016.
[23] A. d'Onofrio, On pulse vaccination strategy in the SIR epidemic model with vertical transmission, Appl. Math. Lett., Vol. 18, 729-732, 2005.
[24] P. Rabinowitz, On bifurcation from infinity, J. Differ. Equations, Vol. 14, 462-475, 1973.
[25] P. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal., Vol. 7, 487-513, 1971.
[26] J. Wang and B. Yan, Global properties and multiple solutions for boundary value problems of impulsive differential equtions, Electronic Journal of Differntial Equations, Vol. 171, 1-14, 2013.
[27] J. Xiao, J.J. Nieto and Z. Luo, Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods, Communication in Nonlinear Science and Numerical Simulation, Vol. 17, 426-423, 2012.

[28] M. Yao, A. Zhao and J. Yun, Periodic boundary value problems of second order impulsive differential equations, Nonlinear Anal.T.M.A., Vol. 70, 262-273, 2009.

[29] S. Zavalishchin and A. Sesekin, Dynamic impulse systems: Theory and applications, Dordrecht: Kluwer Academic Publishers Group, 1997.

[30] D. Zhang, Multiple solutions of nonlinear impulsive differential equations with Dirichlet boundary conditions via variational method, Results in Mathematics, Vol. 63, 611-628, 2013.

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