M-POLYNOMIAL AND DEGREE-BASED TOPOLOGICAL INDICES OF GRAPHS

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ABSTRACT. For a graph $G$, the $M$-polynomial is defined as $M(G;x,y) = \sum_{i \leq j} m_{ij}(G)x^iy^j$, where $m_{ij}, (i, j \geq 1)$, is the number of edges $uv$ of $G$ such that $d_G(u) = i$ and $d_G(v) = j$. The topological indices play an important role in determining physico-chemical properties of chemical graphs, among them the degree-based topological indices can be easily driven from an algebraic expression corresponding to the chemical graphs called $M$-polynomial. In this note, we first compute $M-$polynomial of some special graphs. Further, we derive some degree-based topological indices of these graphs from their respective $M-$polynomial.

1. Introduction

Throughout this paper, by a graph $G = (V,E)$ we mean a simple, undirected, finite graph of order $n$ and size $m$. Let $V(G)$ and $E(G)$ denote the vertex set and an edge set, respectively. The degree $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to it in $G$. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of $G$ and let $d_{v_i} = d_G(v_i)$. The line graph $L(G)$ of a graph $G$ is a graph whose vertex set is one-to-one correspondence with the edge set of the graph $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in $G$. The subdivision graph $S(G)$ of a graph $G$ is the graph obtained by inserting a new vertex onto each edge of $G$. The product $G \times H$ of graphs $G$ and $H$ has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if and only if $[a = b$ and $xy \in E(H)]$ or $[x = y$ and $ab \in E(G)]$. The corona $G \circ H$ of graphs $G$ and $H$ is a graph obtained from $G$ and $H$ by taking one copy of $G$ and $|V(G)|$ copies of $H$ and then joining by an edge each vertex of the $i^{th}$ copy of $H$ is named $(H, i)$ with the $i^{th}$ vertex of $G$. The join $G_1 + G_2$ of two graphs $G_1$ and $G_2$ is the graph obtained from $G_1$ and $G_2$ by joining every vertex of $G_1$ to all vertices of $G_2$. For undefined graph theoretic terminologies and notions, refer to [13, 15, 23].

It is always interesting to find some properties of graphs which are invariant. Topological indices and polynomials are foremost among them. Over the last decade...
there are numerous research papers devoted to topological indices and polynomials. Several topological indices have been defined in the literature. For details of topological indices one can refer to [7, 16]. For different topological indices and their applications one can refer to [1, 2, 3, 10, 11, 12]. The general form of degree-based topological index of a graph is given by

$$TI(G) = \sum_{e=uv \in E(G)} f(d_G(u), d_G(v))$$

where $f = f(x, y)$ is a function appropriately chosen for the computation. Table 1 gives the standard topological indices defined by $f(x, y)$.

There are many graph polynomials too [4, 25]. The Hosoya polynomial is the most well-known polynomial which plays a vital role in determining distance-based topological indices such as Wiener index [24], hyper Wiener index [4] of graphs. The $M-polynomial$ [5] is one among other algebraic polynomials which was introduced in 2015 and useful in determining many degree-based topological indices (listed in Table 1) [7, 16]. Recently, the study of $M-polynomial$ are reported in [18, 19, 20].

**Definition 1.** [5] If $G$ is a graph, then $M-polynomial$ of $G$ is defined as

$$M(G; x, y) = \sum_{i \leq j} m_{ij}(G)x^iy^j,$$ (1.1)

where $m_{ij}, (i, j \geq 1)$, is the number of edges $uv$ in $G$ such that $d_G(u) = i$ and $d_G(v) = j$.

**Table 1.** Operations to Derive degree-based topological indices from $M-polynomial$ [5].

<table>
<thead>
<tr>
<th>Notation</th>
<th>Topological Index</th>
<th>$f(x, y)$</th>
<th>Derivation from $M(G; x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1(G)$</td>
<td>First Zagreb</td>
<td>$x + y$</td>
<td>$(D_x + D_y)(M(G; x, y))</td>
</tr>
<tr>
<td>$M_2(G)$</td>
<td>Second Zagreb</td>
<td>$xy$</td>
<td>$(D_xD_y)(M(G; x, y))</td>
</tr>
<tr>
<td>$^mM_2(G)$</td>
<td>Second modified Zagreb</td>
<td>$\frac{1}{xy}$</td>
<td>$(S_xS_y)(M(G; x, y))</td>
</tr>
<tr>
<td>$S_D(G)$</td>
<td>Symmetric division index</td>
<td>$\frac{x^2+y^2}{xy}$</td>
<td>$(D_xS_y + D_yS_x)(M(G; x, y))</td>
</tr>
<tr>
<td>$H(G)$</td>
<td>Harmonic</td>
<td>$\frac{2}{x+y}$</td>
<td>$2S_xJ(M(G; x, y))</td>
</tr>
<tr>
<td>$I_n(G)$</td>
<td>Inverse sum index</td>
<td>$\frac{xy}{x+y}$</td>
<td>$S_xJD_xD_y(M(G; x, y))</td>
</tr>
</tbody>
</table>

where $D_x = x\frac{\partial f(x,y)}{\partial x}$, $D_y = y\frac{\partial f(x,y)}{\partial y}$, $S_x = \int_0^x f(t,y)\,dt$, $S_y = \int_0^y f(x,t)\,dt$ and $J(f(x,y)) = f(x,x)$ are the operators.

2. M-polynomial of some special Graphs

**Proposition 2.1.** The $M-polynomial$ of a path, a cycle, a complete graph, a complete bipartite graph, a wheel, a star and a double star are as follows:
(i) For a path $P_n$ of order $n$, we have
$$M(P_n; x, y) = 2xy^2 + (n - 3)x^2y^2.$$ 

(ii) For a cycle $C_n$ of order $n$, we have
$$M(C_n; x, y) = nx^2y^2.$$ 

(iii) For a complete graph of order $n$, we have
$$M(K_n; x, y) = (\frac{n}{2})^nx^{n-1}y^{n-1}.$$ 

(iv) For a complete bipartite graph $K_{a,b}$ of order $a + b$, we have
$$M(K_{a,b}; x, y) = abx^ay^b.$$ 

(v) For a wheel $W_n$ of order $n + 1$, we have
$$M(W_n; x, y) = nx^3y^3 + nxy^n.$$ 

(vi) For a star $S_n$ of order $n + 1$, we have
$$M(S_n; x, y) = nxy^n.$$ 

(vii) For a double star $S_{a,b}$ of order $a + b + 2$, we have
$$M(S_{a,b}; x, y) = ax^ay^{a+1} + bxy^{b+1} + x^{a+1}y^{b+1}.$$ 

**Definition 2.** The vertex splitting graph $[22] S'(G)$ of a graph $G$ is obtained from $G$ by adding for each vertex $v$ of $G$ a new vertex $v'$ so that $v'$ is adjacent to every vertex that is adjacent to $v$.

**Theorem 2.2.** If $G$ is a graph of order $n$ and size $m$ with the $M$-polynomial $M(G; x, y) = \sum_{i,j} m_{ij}(G)x^iy^j$, then
$$M(S'(G); x, y) = \sum_{i,j} m_{ij}(S'(G))x^iy^j = \sum_{i,j} m_{ij}(G)x^{2i}y^{2j} + \sum_{a\leq b} m_{ab}(G)x^ay^b,$$
where $m_{ab}(G) = \begin{cases} m_{ij}(G) & \text{for } a = i, b = 2j \text{ and } i \neq j, \\ 2m_{ij}(G) & \text{for } a = i, b = 2j \text{ and } i = j. \end{cases}$

**Proof.** By definition of vertex splitting graph, we have the degree of the original vertices of $G$ in $S'(G)$ is twice the degree of that vertex in $G$ while the degree of the duplicates of those vertices are the same as the degree of corresponding vertices in $G$. Therefore, we have the following:

$$m_{2i,2j}(S'(G)) = m_{ij}(G) \quad \text{and} \quad m_{ab}(G) = \begin{cases} m_{ij}(G) & \text{for } a = i, b = 2j \text{ and } i \neq j, \\ 2m_{ij}(G) & \text{for } a = i, b = 2j \text{ and } i = j. \end{cases}$$

Thus, we get the desired result by substituting these values in Eq. (1.1). □

**Definition 3.** The edge splitting graph $[14] L_S(G)$ of a graph $G$ is a graph with vertex set $E(G) \cup E_1(G)$ with two vertices adjacent if they correspond to adjacent edges of $G$ or one corresponds to an element $e_i'$ of $E_1(G)$ and the other to an element $e_j$ of $E(G)$ where $e_j \in N(e_i)$.
Theorem 2.3. If $G$ is a graph of order $n$ and size $m$ with the $M$-polynomial $M(G; x, y) = \sum_{i \leq j} m_{ij}(G)x^iy^j$, then

$$M(L_S(G); x, y) = \sum_{i \leq j} m_{ij}(L_S(G))x^iy^j = \sum_{i \leq j} m_{ij}(L(G))x^{2i}y^{2j} + \sum_{a \leq b} m_{ab}(L(G))x^ay^b,$$

where $m_{ab}(L(G)) = \begin{cases} m_{ij}(L(G)) & \text{for } a = i, b = 2j \text{ and } i \neq j, \\ 2m_{ij}(L(G)) & \text{for } a = i, b = 2j \text{ and } i = j. \end{cases}$

Proof. By definition of edge splitting graph, we have the degree of the original vertex of $L(G)$ in $L_S(G)$ is twice the degree of that edge vertex in $L(G)$ while the degree of the duplicates of those vertices are the same as the degree of corresponding vertices in $L(G)$. Therefore, we have the following:

$$m_{2i2j}(L_S(G)) = m_{ij}(L(G)) \text{ and } m_{ab}(L(G)) = \begin{cases} m_{ij}(L(G)) & \text{for } a = i, b = 2j \text{ and } i \neq j, \\ 2m_{ij}(L(G)) & \text{for } a = i, b = 2j \text{ and } i = j. \end{cases}$$

Thus, we get the desired result by substituting these values in Eq. (1.1). □

Definition 4. The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$, say $G'$, $G''$ and joining each vertex $v'$ in $G'$ to the neighbors of the corresponding vertex $v''$ in $G''$.

Theorem 2.4. If $G$ is a graph of order $n$ and $D_2(G)$ is the shadow graph of $G$, then

$$M(D_2(G); x, y) = \sum_{i \leq j} 4m_{ij}(G)x^{2i}y^{2j}.$$ 

Proof. Let $D_2(G)$ be the shadow graph of a graph $G$ of order $n$ which has $2n$ vertices and $4m$ edges. Then we have by definition of shadow graph $d_{D_2(G)}(v') = 2d_G(v)$ for each $v' \in V(D_2(G))$ corresponds to $v \in V(G)$. Thus,

$$|E_{\{2i,2j\}}| = |uv \in E(D_2(G)) : d_u = 2i \text{ and } d_v = 2j| = 2|u'v' \in E(G') : d_{u'} = i \text{ and } d_{v'} = j| + 2|u''v'' \in E(G'') : d_{u''} = i \text{ and } d_{v''} = j| = 2m_{ij}(G) + 2m_{ij}(G) = 4m_{ij}(G).$$

Thus, the $M$-polynomial of $D_2(G)$ is

$$M(D_2(G); x, y) = \sum_{i \leq j} m_{ij}(D_2(G))x^iy^j = \sum_{i \leq j} 4m_{ij}(G)x^{2i}y^{2j}.\quad \square$$

Corollary 2.5. If $G$ is an $r$-regular graph of order $n$ and size $m$, then

$$M(D_2(G); x, y) = 4m x^{2r}y^{2r}.$$ 

Definition 5. For a graph $G = (V(G), E(G))$, the Mycielskian $\mu(G)$ of $G$ is a graph with vertex set consisting the disjoint union $V(G) \cup V'(G) \cup \{u\}$, where $V'(G) = \{x' : x \in V(G)\}$, and the edge set $E(G) \cup \{x'y : xy \in E(G)\} \cup \{x'u : x' \in V'(G)\}$.
Theorem 2.6. If $G$ is a graph of order $n$ and size $m$ with the $M$-polynomial $M(G; x, y) = \sum_{i \leq j} m_{ij}(G)x^iy^j$, then

$$M(\mu(G); x, y) = \sum_{i \leq j} m_{ij}(G)x^{2i}y^{2j} + \sum_{a' \leq b'} m_{a'b'}(G)x^{a'}y^{b'},$$

where $a' = \min\{a, b\}, b' = \max\{a, b\}$, and for $i' = \min\{i, j\}, j' = \max\{i, j\}$

$$m_{a'b'}(G) = \begin{cases} m_{i'j'}(G) & \text{if } a = i + 1, b = 2j \text{ and } i \neq j, \\ 2m_{i'j'}(G) & \text{if } a = i + 1, b = 2j \text{ and } i = j, \\ |\{v : d_G(v) = i\}| & \text{if } a = i + 1, b = n \text{ for } i = 1, 2, ..., n - 1. \end{cases}$$

Proof. By definition of mycielskian of a graph, we have the degree of the original vertices of $G$ in $\mu(G)$ is twice the degree of that vertex in $G$, the degree $d_{\mu(G)}(v_i') = d_G(v_i) + 1$ of the duplicates $v_i'$ of $v_i \in V(G)$ and the degree of the vertex $u \in \mu(G)$
is $n$. Therefore, we have the following:

$$m_{2i2j}(\mu(G)) = m_{ij}(G)$$

and

$$m_{a'\nu'}(G) = \begin{cases} 
  m_{a'\nu'}(G) & \text{if } a = i + 1, b = 2j \text{ and } i \neq j, \\
  2m_{a'\nu'}(G) & \text{if } a = i + 1, b = 2j \text{ and } i = j, \\
  |\{v : d_v = i\}| & \text{if } a = i + 1, b = n \text{ for } i = 1, 2, \ldots, n - 1.
\end{cases}$$

Thus, we get the desired result by substituting these values in Eq. (1.1). □

**Corollary 2.7.** If $M$-polynomial of Mycielskian of a graph $G$ is

$$M(\mu(G); x, y) = \sum_{i\leq j} m_{ij}(G)x^{2i}y^{2j} + \sum_{a'\leq b'} m_{a'\nu'}(G)x^{a'}y^{b'},$$

then

$$M_1(\mu(G)) = 2\sum_{i\leq j}(i + j)m_{ij}(G) + \sum_{a'\leq b'} (a' + b')m_{a'\nu'}(G),$$

$$M_2(\mu(G)) = 4\sum_{i\leq j} im_{ij}(G) + \sum_{a'\leq b'} a'b'm_{a'\nu'}(G),$$

$$m_{M_2}(\mu(G)) = \frac{1}{4}\sum_{i\leq j} m_{ij}(G) + \sum_{a'\leq b'} m_{a'\nu'}(G),$$

$$S_D(\mu(G)) = \sum_{i\leq j} (i^2 + j^2)m_{ij}(G) + \sum_{a'\leq b'} (a'^2 + b'^2)m_{a'\nu'}(G),$$

$$H(\mu(G)) = \sum_{i\leq j} ijm_{ij}(G) + 2\sum_{a'\leq b'} a'b'm_{a'\nu'}(G),$$

$$I_n(\mu(G)) = \sum_{i\leq j} ij(i + j)m_{ij}(G) + \sum_{a'\leq b'} a'(a' + b')m_{a'\nu'}(G).$$

**Proof.** We get the desired results by applying the appropriate operators to $M$-polynomial of $\mu(G)$. □

**Definition 6.** Let $P_3$ be the 3-vertex tree rooted at one its terminal vertices. See Fig. 2. For $k = 2, 3, \ldots$ construct the rooted tree $B_k$ by identifying the roots of $k$ copies of $P_3$. The vertex obtained by identifying the roots of $P_3$-trees is the root of $B_k$. The illustrative structure of the rooted tree $B_k$ is depicted in Fig. 2.

**Definition 7.** Let $d$ be an integer and $\beta_1, \beta_2, \ldots, \beta_d$ be rooted trees as specified in Definition 4, i.e., $\beta_1, \beta_2, \ldots, \beta_d \in \{B_2, B_3, \ldots\}$. A Kragujevac tree $T_k$ is a tree possessing a vertex of degree $d$, adjacent to the roots of $\beta_1, \beta_2, \ldots, \beta_d$. This vertex is said to be the central vertex of $T_k$. The subgraphs $\beta_1, \beta_2, \ldots, \beta_d$ are the branches of $T_k$. Note that, some (or all) branches of $T_k$ may be mutually isomorphic.

**Theorem 2.8.** If $T_k$ is a Kragujevac tree with $\beta_1, \beta_2, \ldots, \beta_d \in \{B_2, B_3, \ldots\}$ branches, then

$$M(T_k; x, y) = \sum_{i\geq 2} ik_i xy^2 + \sum_{i\geq 2} k_i x^2 y^{i+1} + \sum_{i\geq 2} k_i x^d y^{i+1},$$

where $k_i = |\{\beta_i : \beta_i \text{ is a branch of } T_k \text{ such that } \beta_i = B_i\}|$ for $i \geq 2$. 
Figure 2. The rooted trees $B_k$'s and the Kragujevac tree $T_k$.

**Proof.** By definition of Kragujevac tree $T_k$, we have $\sum_{i \geq 2} ik_i$ vertices of degree 1, $\sum_{i \geq 2} ik_i$ vertices of degree 2 and $k_i$ vertices of degree $i + 1$. Therefore, the edge partition of $T_k$ is given as follows:

$$|E_{(1,2)}| = |uv \in E(T_k) : d_u = 1 \text{ and } d_v = 2| = \sum_{i \geq 2} ik_i,$$

$$|E_{(2,i+1)}| = |uv \in E(T_k) : d_u = 2 \text{ and } d_v = i + 1| = ik_i,$$

$$|E_{(d,i+1)}| = |uv \in E(T_k) : d_u = d \text{ and } d_v = i + 1| = k_i.$$

Thus, the $M$–polynomial of $T_k$ is

$$M(T_k; x, y) = \sum_{i \leq j} m_{ij}(T_k)x^iy^j = \sum_{i \geq 2} ik_ixy^2 + \sum_{i \geq 2} ik_ix^2y^{i+1} + \sum_{i \geq 2} k_ix^dy^{i+1}.$$

□

**Corollary 2.9.** If $M$-polynomial of Kragujevac tree $T_k$ is

$$M(T_k; x, y) = \sum_{i \geq 2} ik_ixy^2 + \sum_{i \geq 2} ik_ix^2y^{i+1} + \sum_{i \geq 2} k_ix^dy^{i+1},$$
then

\[
M_1(T_k) = \sum_{i \geq 2} (i^2 + 7i + d + 1)k_i,
\]

\[
M_2(T_k) = \sum_{i \geq 2} (2i^2 + (4 + d)i + d)k_i,
\]

\[
mM_2(T_k) = \sum_{i \geq 2} \frac{(i^2 + 5i + 2d)}{2(i + 1)}k_i,
\]

\[
S_D(T_k) = \sum_{i \geq 2} \frac{(7i^2 + 13i + 2d + 2)}{2(i + 1)}k_i,
\]

\[
H(T_k) = \sum_{i \geq 2} \frac{(2i^3 + 2(d + 7)i^2 + 6(2d + 3)i + 18)}{3(i + 3)(d + i + 1)}k_i,
\]

\[
I_n(T_k) = \sum_{i \geq 2} \frac{(8i^3 + (11d + 20)i^2 + 12(2d + 1)i + 9d)}{3(i + 3)(d + i + 1)}k_i.
\]

**Proof.** We get the desired results by applying the appropriate operators on \(M\)-polynomial of \(T_k\). \(\Box\)

The definitions of the special graphs used in this paper can be found in [9]. In this section, we obtain \(M\)-polynomials of some special graphs. We also derive some topological indices (mentioned in section 1) of these graphs from the respective \(M\)-polynomials.

**Definition 8.** The book graph \(B_m = S_m \times P_2\) is a graph with \(2(m + 1)\) vertices and \((3m + 1)\) edges, where \(S_m\) is a star of order \((m + 1)\) and \(P_2\) is a path of length one.

**Theorem 2.10.** If \(B_m\) is a book graph of order \(2(m + 1)\) and size \((3m + 1)\), then

\[
M(B_m; x, y) = mx^2y^2 + 2mx^2y^{m+1} + x^{m+1}y^{m+1}.
\]

**Proof.** The book graph \(B_m\) has \(2(m + 1)\) vertices and \((3m + 1)\) edges. The edge set of \(B_m\) can be partitioned as,

\[
|E_{(2,2)}| = |uw \in E(B_m) : d_u = 2 \text{ and } d_v = 2| = m,
\]

\[
|E_{(2,m+1)}| = |uw \in E(B_m) : d_u = 2 \text{ and } d_v = m + 1| = 2m,
\]

\[
|E_{(m+1,m+1)}| = |uw \in E(B_m) : d_u = m + 1 \text{ and } d_v = m + 1| = |E(B_m) - |E_{(2,2)}| - |E_{(2,m+1)}| = 1.
\]

Thus, the \(M\)-polynomial of \(B_m\) is

\[
M(B_m; x, y) = \sum_{i \leq j} m_{ij}(B_m)x^iy^j = mx^2y^2 + 2mx^2y^{m+1} + x^{m+1}y^{m+1}.
\]

\(\Box\)
Corollary 2.11. If $M$-polynomial of the book graph $B_m$ is $M(B_m; x, y) = mx^2y^2 + 2mx^2y^{m+1} + x^{m+1}y^{m+1}$, then

\[
\begin{align*}
M_1(B_m) &= 2(m^2 + 6m + 1), \\
M_2(B_m) &= 5m^2 + 10m + 1, \\
mM_2(B_m) &= \frac{m^3 + 6m^2 + 5m + 4}{4(m^2 + 2m + 1)}, \\
S_D(B_m) &= \frac{m^3 + 4m^2 + 9m + 2}{m + 1}, \\
H(B_m) &= \frac{m^3 + 12m^2 + 13m + 6}{2(m^2 + 4m + 3)}, \\
I_n(B_m) &= \frac{11m^2 + 18m + 3}{2(m + 3)}.
\end{align*}
\]

Proof. We have, the $M$-polynomial of the book graph $B_m$ as

\[
M(B_m; x, y) = mx^2y^2 + 2mx^2y^{m+1} + x^{m+1}y^{m+1}.
\]

Therefore,

\[
\begin{align*}
D_x &= x \frac{\partial f(x, y)}{\partial x} = 2mx^2y^2 + 4mx^2y^{m+1} + (m + 1)x^{m+1}y^{m+1}, \\
D_y &= y \frac{\partial f(x, y)}{\partial y} = 2mx^2y^2 + 2m(m + 1)x^2y^{m+1} + (m + 1)x^{m+1}y^{m+1}, \\
S_x &= \int_0^x f(t, y) \frac{dt}{t} = \frac{m}{2}x^2y^2 + mx^2y^{m+1} + \frac{1}{(m + 1)}x^{m+1}y^{m+1}, \\
S_y &= \int_0^y f(x, t) \frac{dt}{t} = \frac{m}{2}x^2y^2 + \frac{2m}{(m + 1)}x^2y^{m+1} + \frac{1}{(m + 1)}x^{m+1}y^{m+1}, \\
J(f(x, y)) &= f(x, x) = mx^4 + 2mx^{m+3} + x^{2(m+1)}.
\end{align*}
\]

Thus, we get

\[
\begin{align*}
M_1(B_m) &= (D_x + D_y)(M(B_m; x, y))|_{x=y=1} = 2(m^2 + 6m + 1), \\
M_2(B_m) &= (D_x D_y)(M(B_m; x, y))|_{x=y=1} = 5m^2 + 10m + 1, \\
mM_2(B_m) &= (S_xS_y)(M(B_m; x, y))|_{x=y=1} = \frac{m^3 + 6m^2 + 5m + 4}{4(m^2 + 2m + 1)}, \\
S_D(B_m) &= (D_xS_y + D_yS_x)(M(B_m; x, y))|_{x=y=1} = \frac{m^3 + 4m^2 + 9m + 2}{m + 1}, \\
H(B_m) &= 2S_xJ(M(B_m; x, y))|_{x=1} = \frac{m^3 + 12m^2 + 13m + 6}{2(m^2 + 4m + 3)}, \\
I_n(B_m) &= S_xJD_xD_y(M(B_m; x, y))|_{x=1} = \frac{11m^2 + 18m + 3}{2(m + 3)}.
\end{align*}
\]

\[\square\]

Definition 9. The Ladder $L_n = P_n \times P_2$ is a graph of order $2n$ and size $(3n - 2)$, where $P_n$ and $P_2$ are two paths of length $(n - 1)$ and $1$, respectively.

Theorem 2.12. If $L_n$ is a ladder, then

\[
M(L_n; x, y) = 2x^2y^2 + 4x^2y^3 + (3n - 8)x^3y^3.
\]
Figure 3. Plot of $M$-polynomial of the book graph $B_{10}$

Proof. The ladder $L_n$ has $2n$ vertices and $(3n - 2)$ edges. The edge set of $L_n$ can be partitioned as,

\[
|E_{(2,2)}| = |uv \in E(L_n) : d_u = 2 \text{ and } d_v = 2| = 2,
\]

\[
|E_{(2,3)}| = |uv \in E(L_n) : d_u = 2 \text{ and } d_v = 3| = 4,
\]

\[
|E_{(3,3)}| = |uv \in E(L_n) : d_u = 3 \text{ and } d_v = 3|
\]

\[
= |E(L_n) - |E_{(2,2)}| - |E_{(2,3)}| = 3n - 8.
\]

Thus, the $M$-polynomial of $L_n$ is

\[
M(L_n; x, y) = \sum_{i \leq j} m_{ij}(L_n) x^i y^j = 2x^2y^2 + 4x^2y^3 + (3n - 8)x^3y^3.
\]

Corollary 2.13. If the $M$-polynomial of the ladder $L_n$ is $M(L_n; x, y) = 2x^2y^2 + 4x^2y^3 + (3n - 8)x^3y^3$, then

\[
M_1(L_n) = 2(9n - 10),
\]

\[
M_2(L_n) = 27n - 40,
\]

\[
mM_2(L_n) = \frac{6n + 5}{18},
\]

\[
S_D(L_n) = \frac{2(9n - 5)}{3},
\]

\[
H(L_n) = \frac{15n - 1}{15},
\]

\[
I_n(L_n) = \frac{45n - 52}{10}.
\]

Proof. We have, the $M$-polynomial of the ladder $L_n$ as

\[
M(L_n; x, y) = 2x^2y^2 + 4x^2y^3 + (3n - 8)x^3y^3.
\]
Therefore,

\[
D_x = x \frac{\partial f(x, y)}{\partial x} = 4x^2y^2 + 8x^2y^3 + 3(3n - 8)x^3y^3,
\]

\[
D_y = y \frac{\partial f(x, y)}{\partial y} = 4x^2y^2 + 12x^2y^3 + 3(3n - 8)x^3y^3,
\]

\[
S_x = \int_0^x f(t, y) \frac{dt}{t} = x^2y^2 + 2x^2y^3 + \frac{(3n - 8)}{3}x^3y^3,
\]

\[
S_y = \int_0^y f(x, t) \frac{dt}{t} = x^2y^2 + \frac{4}{3}x^2y^3 + \frac{(3n - 8)}{3}x^3y^3,
\]

\[
J(f(x, y)) = f(x, x) = 2x^4 + 4x^5 + (3n - 8)x^6.
\]

Thus, we get

\[
M_1(L_n) = (D_x + D_y)(M(L_n; x, y))|_{x=y=1} = 2(9n - 10),
\]

\[
M_2(L_n) = (D_xD_y)(M(L_n; x, y))|_{x=y=1} = 27n - 40,
\]

\[
mM_2(L_n) = (S_xS_y)(M(L_n; x, y))|_{x=y=1} = \frac{6n + 5}{18},
\]

\[
S_D(L_n) = (D_xS_y + D_yS_x)(M(L_n; x, y))|_{x=y=1} = \frac{2(9n - 5)}{3},
\]

\[
H(L_n) = 2S_xJ(M(L_n; x, y))|_{x=1} = \frac{15n - 1}{15},
\]

\[
I_n(L_n) = S_xJD_xD_y(M(L_n; x, y))|_{x=1} = \frac{45n - 52}{10}.
\]

The surfaces in Figures 3 and 4 are plotted by using Mathematica. These surfaces are obtained by M-polynomial of the respective graph which shows different behaviours for different parameters m, n, x and y.
**Definition 10.** A planar grid $P_m \times P_n$ is a graph obtained by the product of two paths $P_m$ and $P_n$ of lengths $(m-1)$ and $(n-1)$, respectively.

**Theorem 2.14.** If $P_m \times P_n$ is a planar grid, then

$$M(P_m \times P_n; x, y) = 8x^2y^3 + 2(m+n-6)x^3y^3 + 2(m+n-4)x^3y^4 + (2mn-5m-5n+12)x^4y^4.$$ 

**Proof.** The planar grid $P_m \times P_n$ has $mn$ vertices and $(2mn - m - n)$ edges. The edge set of $P_m \times P_n$ can be partitioned as,

$$|E_{(2,3)}| = |uv \in E(P_m \times P_n) : d_u = 2 \text{ and } d_v = 3| = 8,$$

$$|E_{(3,3)}| = |uv \in E(P_m \times P_n) : d_u = 3 \text{ and } d_v = 3| = 2(m + n - 6),$$

$$|E_{(3,4)}| = |uv \in E(P_m \times P_n) : d_u = 3 \text{ and } d_v = 4| = 2(m + n - 4),$$

$$|E_{(4,4)}| = |uv \in E(P_m \times P_n) : d_u = 4 \text{ and } d_v = 4|$$

$$= |E(P_m \times P_n)| - |E_{(2,3)}| - |E_{(3,3)}| - |E_{(3,4)}| = 2mn - 5m - 5n + 12.$$ 

Thus, the $M-$ polynomial of $P_m \times P_n$ is

$$M(P_m \times P_n; x, y) = \sum_{i \leq j} m_{ij}(P_m \times P_n)x^iy^j$$

$$= 8x^2y^3 + 2(m+n-6)x^3y^3 + 2(m+n-4)x^3y^4 + (2mn-5m-5n+12)x^4y^4.$$ 


**Definition 11.** The prism $\Pi_n = C_n \times P_2$ is a 3-regular graph of order $2n$ and size $3n$, where $C_n$ is cycle of order $n$ and $P_2$ is a path of length one.

**Theorem 2.15.** If $\Pi_n$ is a prism, then

$$M(\Pi_n; x, y) = 3nx^3y^3.$$ 

**Proof.** Let prism $\Pi_n$ be a 3-regular graph having $2n$ vertices and $3n$ edges. The edge partition of $\Pi_n$ is given by,

$$|E_{(3,3)}| = |uv \in E(\Pi_n) : d_u = 3 \text{ and } d_v = 3| = 3n.$$ 

Thus, the $M-$ polynomial of the prism $\Pi_n$ is

$$M(\Pi_n; x, y) = \sum_{i \leq j} m_{ij}(\Pi_n)x^iy^j = 3nx^3y^3.$$ 


**Definition 12.** The book graph with triangular pages $B_m^t = P_2 + mK_1$ is a graph with $(n+2)$ vertices and $(2n+1)$ edges, where $P_2$ is a path of length one and $mK_1$ are the $m$ isolated vertices.

**Theorem 2.16.** If $B_m^t$ is a book graph with triangular pages having $(n+2)$ vertices and $(2n+1)$ edges, then

$$M(B_m^t; x, y) = 2mx^2y^{m+1} + x^{m+1}y^{m+1}.$$ 

**Proof.** Let $B_m^t$ is a book graph with triangular pages having $(n+2)$ vertices and $(2n+1)$ edges. The edge partition of $B_m^t$ is given by,

$$|E_{(2,m+1)}| = |uv \in E(B_m^t) : d_u = 2 \text{ and } d_v = m + 1| = 2m,$$

$$|E_{(m+1,m+1)}| = |uv \in E(B_m^t) : d_u = m + 1 \text{ and } d_v = m + 1|$$

$$= |E(B_m^t)| - |E_{(2,m+1)}| = 1.$$ 

Thus, $M(B_m^t; x, y) = \sum_{i \leq j} m_{ij}(B_m^t)x^iy^j = 2mx^2y^{m+1} + x^{m+1}y^{m+1}.$ 

$\square$
Definition 13. The corona $P_n \circ K_1$ of a path $P_n$ of length $(n-1)$ with an isolated vertex $K_1$ is called a comb graph and the corona $P_n \circ 2K_1$ of a path $P_n$ of length $(n-1)$ with two isolated vertices $2K_1$ is called a double comb graph.

Theorem 2.17. If $P_n \circ K_1$ is a comb graph, then

$$M(P_n \circ K_1; x, y) = 2xy^2 + (n-2)xy^3 + 2x^2y^3 + (n-3)x^3y^3.$$  

Proof. The comb graph $P_n \circ K_1$ has $2n$ vertices and $(2n-1)$ edges. The edge set of $P_n \circ K_1$ can be partitioned as,

$$|E_{1,2}| = |uv \in E(P_n \circ K_1) : d_u = 1 \text{ and } d_v = 2| = 2,$$

$$|E_{1,3}| = |uv \in E(P_n \circ K_1) : d_u = 1 \text{ and } d_v = 3| = (n-2),$$

$$|E_{2,3}| = |uv \in E(P_n \circ K_1) : d_u = 2 \text{ and } d_v = 3| = 2,$$

$$|E_{3,3}| = |uv \in E(P_n \circ K_1) : d_u = 3 \text{ and } d_v = 3| = |E(P_n \circ K_1) - |E_{1,2}| - |E_{1,3}| - |E_{2,3}|| = n-3.$$

Thus, the $M$-polynomial of $P_n \circ K_1$ is

$$M(P_n \circ K_1; x, y) = \sum_{i \leq j} m_{ij}(P_n \circ K_1)x^iy^j = 2xy^2 + (n-2)xy^3 + 2x^2y^3 + (n-3)x^3y^3.$$  

□

Theorem 2.18. If $P_n \circ 2K_1$ is a double comb graph, then

$$M(P_n \circ 2K_1; x, y) = 4xy^3 + 2(n-2)xy^4 + 2x^3y^4 + (n-3)x^4y^4.$$  

Proof. The double comb graph $P_n \circ 2K_1$ has $3n$ vertices and $(3n-1)$ edges. The edge set of $P_n \circ 2K_1$ can be partitioned as,

$$|E_{1,3}| = |uv \in E(P_n \circ 2K_1) : d_u = 1 \text{ and } d_v = 3| = 4,$$

$$|E_{1,4}| = |uv \in E(P_n \circ 2K_1) : d_u = 1 \text{ and } d_v = 4| = 2(n-2),$$

$$|E_{3,4}| = |uv \in E(P_n \circ 2K_1) : d_u = 3 \text{ and } d_v = 4| = 2,$$

$$|E_{4,4}| = |uv \in E(P_n \circ 2K_1) : d_u = 4 \text{ and } d_v = 4| = |E(P_n \circ 2K_1) - |E_{1,3}| - |E_{1,4}| - |E_{3,4}|| = n-3.$$

Thus, the $M$-polynomial of $P_n \circ 2K_1$ is

$$M(P_n \circ 2K_1; x, y) = \sum_{i \leq j} m_{ij}(P_n \circ 2K_1)x^iy^j = 4xy^3 + 2(n-2)xy^4 + 2x^3y^4 + (n-3)x^4y^4.$$  

□

Definition 14. A jelly fish $J(m, n)$ is a graph obtained from a cycle $C_4 : uxvyu$ by joining $x$ and $y$ with an edge and appending $m$ pendant edges to $u$ and $n$ pendant edges to $v$.

Theorem 2.19. If $J(m, n)$ is a jelly fish graph, then

$$M(J(m, n); x, y) = mxy^{m+2} + nxy^{n+2} + 2x^3y^{m+2} + 2x^3y^{n+2} + x^3y^3.$$
Thus, the edge set of $J$ can be partitioned as,

$$
|E_{1,m+2}| = |uv \in E(J(m,n)) : d_u = 1 \text{ and } d_v = m + 2| = m,
$$

$$
|E_{1,n+2}| = |uv \in E(J(m,n)) : d_u = 1 \text{ and } d_v = n + 2| = n,
$$

$$
|E_{3,m+2}| = |uv \in E(J(m,n)) : d_u = 3 \text{ and } d_v = m + 2| = 2,
$$

$$
|E_{3,n+2}| = |uv \in E(J(m,n)) : d_u = 3 \text{ and } d_v = n + 2| = 2,
$$

$$
|E_{3,3}| = |uv \in E(J(m,n)) : d_u = 3 \text{ and } d_v = 3| = |E(J(m,n))| - |E_{1,m+2}| - |E_{1,n+2}| - |E_{3,m+2}| - |E_{3,n+2}| = 1.
$$

Thus, the $M$–polynomial of $J(m,n)$ is

$$
M(J(m,n); x, y) = \sum_{i \leq j} m_{ij}(J(m,n))x^iy^j
$$

$$
= mxy^{m+2} + nxy^{n+2} + 2x^3y^{m+2} + 2x^3y^{n+2} + x^3y^3.
$$

\[ \square \]

**Definition 15.** A butterfly graph $B_{m,n}$ is obtained from two even cycles of the same order $n$ for $n \geq 3$, sharing a common vertex with $m$ pendant edges attached at the common vertex.

**Theorem 2.20.** If $B_{m,n}$ is a butterfly graph, then

$$
M(B_{m,n}; x, y) = mxy^{m+4} + 4x^2y^{m+4} + (2n - 4)x^2y^2.
$$

**Proof.** The butterfly graph $B_{m,n}$ has $(2n + m - 1)$ vertices and $(2n + m)$ edges. The edge set of $B_{m,n}$ can be partitioned as,

$$
|E_{1,m+4}| = |uv \in E(B_{m,n}) : d_u = 1 \text{ and } d_v = m + 4| = m,
$$

$$
|E_{2,m+4}| = |uv \in E(B_{m,n}) : d_u = 2 \text{ and } d_v = m + 4| = 4,
$$

$$
|E_{2,2}| = |uv \in E(B_{m,n}) : d_u = 2 \text{ and } d_v = 2| = |E(B_{m,n})| - |E_{1,m+4}| - |E_{2,m+4}| = 2n - 4.
$$

Thus, the $M$–polynomial of $B_{m,n}$ is

$$
M(B_{m,n}; x, y) = \sum_{i \leq j} m_{ij}(B_{m,n})x^iy^j
$$

$$
= mxy^{m+4} + 4x^2y^{m+4} + (2n - 4)x^2y^2.
$$

\[ \square \]

**Definition 16.** The triangular snake $T_n$ is a graph obtained from the path $P_n$ of length $(n - 1)$, by replacing each edge of the path by a triangle $C_3$.

**Theorem 2.21.** If $T_n$ is a triangular snake, then

$$
M(T_n; x, y) = 2x^2y^2 + 2(n - 1)x^2y^2 + (n - 3)x^2y^4.
$$

**Proof.** Let triangular snake $T_n$ be a graph having $(2n - 1)$ vertices and $3(n - 1)$ edges. The edge partition of $T_n$ is given by,

$$
|E_{2,2}| = |uv \in E(T_n) : d_u = 2 \text{ and } d_v = 2| = 2,
$$

$$
|E_{2,4}| = |uv \in E(T_n) : d_u = 2 \text{ and } d_v = 4| = 2(n - 1),
$$

$$
|E_{4,4}| = |uv \in E(T_n) : d_u = 4 \text{ and } d_v = 4| = |E(T_n)| - |E_{2,2}| - |E_{2,4}| = n - 3.
$$
Thus, the $M$–polynomial of $T_n$ is
$$M(T_n; x, y) = \sum_{i \leq j} m_{ij}(T_n)x^iy^j = 2x^2y^2 + 2(n - 1)x^4y^4 + (n - 3)x^4y^4.$$ \hfill \Box

**Definition 17.** The double triangular snake $DT_n$ is a graph consisting of two triangular snakes that have a common path. i.e., a double triangular snake is obtained from the path $P_n : u_1u_2...u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$, $(1 \leq i \leq n - 1)$ and to a new vertex $w_i$, $(1 \leq i \leq n - 1)$.

**Theorem 2.22.** If $DT_n$ is a double triangular snake, then
$$M(DT_n; x, y) = 4x^2y^3 + 4(n - 2)x^2y^6 + 2x^3y^6 + (n - 3)x^6y^6.$$ Proof. Let double triangular snake $DT_n$ be a graph having $(3n - 2)$ vertices and $5(n - 1)$ edges. The edge partition of $DT_n$ is given by,
\begin{align*}
|E_{\{2,3\}}| &= |uv \in E(DT_n) : d_u = 2 \text{ and } d_v = 3| = 4, \\
|E_{\{2,6\}}| &= |uv \in E(DT_n) : d_u = 2 \text{ and } d_v = 6| = 4(n - 2), \\
|E_{\{3,6\}}| &= |uv \in E(DT_n) : d_u = 3 \text{ and } d_v = 6| = 2, \\
|E_{\{6,6\}}| &= |uv \in E(DT_n) : d_u = 6 \text{ and } d_v = 6| = |E(DT_n)| - |E_{\{2,3\}}| - |E_{\{2,6\}}| - |E_{\{3,6\}}| = n - 3.
\end{align*}
Thus, the $M$–polynomial of $DT_n$ is
$$M(DT_n; x, y) = \sum_{i \leq j} m_{ij}(DT_n)x^iy^j = 4x^2y^3 + 4(n - 2)x^2y^6 + 2x^3y^6 + (n - 3)x^6y^6.$$ \hfill \Box

**Definition 18.** An irregular triangular snake $IT_n$ is a graph obtained from the path $P_n : u_1u_2...u_n$ with vertex set $V(IT_n) = V(P_n) \cup \{v_i : 1 \leq i \leq n - 2\}$ and the edge set $E(IT_n) = E(P_n) \cup \{u_iv_i, v_{i+1}u_{i+2} : 1 \leq i \leq n - 2\}$.

**Theorem 2.23.** If $IT_n$ is an irregular triangular snake, then
$$M(IT_n; x, y) = 2x^2y^2 + 4x^2y^3 + 2x^3y^4 + 2(n - 4)x^2y^4 + (n - 5)x^4y^4.$$ Proof. Let an irregular triangular snake $IT_n$ be a graph having $2(n - 1)$ vertices and $(3n - 5)$ edges. The edge partition of $IT_n$ is given by,
\begin{align*}
|E_{\{2,2\}}| &= |uv \in E(IT_n) : d_u = 2 \text{ and } d_v = 2| = 2, \\
|E_{\{2,3\}}| &= |uv \in E(IT_n) : d_u = 2 \text{ and } d_v = 3| = 4, \\
|E_{\{2,4\}}| &= |uv \in E(IT_n) : d_u = 2 \text{ and } d_v = 4| = 2(n - 4), \\
|E_{\{3,3\}}| &= |uv \in E(IT_n) : d_u = 3 \text{ and } d_v = 4| = 2, \\
|E_{\{4,4\}}| &= |uv \in E(IT_n) : d_u = 4 \text{ and } d_v = 4| = |E(IT_n)| - |E_{\{2,2\}}| - |E_{\{2,3\}}| - |E_{\{2,4\}}| - |E_{\{3,3\}}| = n - 5.
\end{align*}
Thus, the $M$–polynomial of $IT_n$ is
$$M(IT_n; x, y) = \sum_{i \leq j} m_{ij}(IT_n)x^iy^j = 2x^2y^2 + 4x^2y^3 + 2x^3y^4 + 2(n - 4)x^2y^4 + (n - 5)x^4y^4.$$ \hfill \Box
Definition 19. The alternate triangular snake $A(T_n)$ is obtained from a path $v_1v_2...v_n$ by joining $v_i$ and $v_{i+1}$ (alternatively) to new vertex $v_i$, that is, every alternate edge of a path is replaced by $C_5$.

Theorem 2.24. If $A(T_n)$ is an alternate triangular snake, then

$$M(A(T_n); x, y) = \begin{cases} 
2x^2y^2 + nx^2y^3 + (n - 3)x^3y^3 & \text{if } n \text{ is even}, \\
xy^3 + x^2y^3 + (n - 1)x^2y^3 + (n - 3)x^3y^3 & \text{if } n \text{ is odd}.
\end{cases}$$

Proof. Let an alternate triangular snake $A(T_n)$ be a graph having $(n + \lfloor \frac{n}{2} \rfloor)$ vertices and $(n - 1 + \lfloor \frac{n}{2} \rfloor)$ edges. The edge partition of $A(T_n)$ is given as follows:

If $n$ is even, then there will be no pendant edge in $A(T_n)$. Therefore, we have

$$|E_{(2,2)}| = |uw \in E(A(T_n)) : d_u = 2 \text{ and } d_v = 2| = 2,$$
$$|E_{(2,3)}| = |uw \in E(A(T_n)) : d_u = 2 \text{ and } d_v = 3| = n,$$
$$|E_{(3,3)}| = |uw \in E(A(T_n)) : d_u = 3 \text{ and } d_v = 3| \quad = |E(A(T_n))| - |E_{(2,2)}| - |E_{(2,3)}| - n = 3.$$ 

If $n$ is odd, then there will be a pendant edge in $A(T_n)$. Therefore, we have

$$|E_{(1,3)}| = |uw \in E(A(T_n)) : d_u = 1 \text{ and } d_v = 3| = 1,$$
$$|E_{(2,2)}| = |uw \in E(A(T_n)) : d_u = 2 \text{ and } d_v = 2| = 1,$$
$$|E_{(2,3)}| = |uw \in E(A(T_n)) : d_u = 2 \text{ and } d_v = 3| = n - 1,$$
$$|E_{(3,3)}| = |uw \in E(A(T_n)) : d_u = 3 \text{ and } d_v = 3| \quad = |E(A(T_n))| - |E_{(1,3)}| - |E_{(2,2)}| - |E_{(2,3)}| = n - 3.$$ 

Thus, the $M-$ polynomial of $A(T_n)$ is

$$M(A(T_n); x, y) = \sum_{i,j} m_{ij}(A(T_n))x^iy^j = \begin{cases} 
2x^2y^2 + nx^2y^3 + (n - 3)x^3y^3 & \text{if } n \text{ is even}, \\
xy^3 + x^2y^3 + (n - 1)x^2y^3 + (n - 3)x^3y^3 & \text{if } n \text{ is odd}.
\end{cases}$$

Definition 20. A double alternate triangular snake $DA(T_n)$ consists of two alternate triangular snakes that have a common path.

Theorem 2.25. Let $DA(T_n)$ be a double alternate triangular snake. Then

$$M(DA(T_n); x, y) = \begin{cases} 
4x^2y^2 + (4 \lfloor \frac{n}{2} \rfloor - 4)x^2y^3 + 2x^3y^4 + (n - 3)x^4y^4 & \text{if } n \text{ is even}, \\
x^2y^3 + 2x^2y^3 + (4 \lfloor \frac{n}{2} \rfloor - 2)x^2y^4 + x^3y^4 + (n - 3)x^4y^4 & \text{if } n \text{ is odd}.
\end{cases}$$

Proof. Let a double alternate triangular snake $DA(T_n)$ be a graph having $(n + 2\lfloor \frac{n}{2} \rfloor)$ vertices and $(n - 1 + 4\lfloor \frac{n}{2} \rfloor)$ edges. The edge partition of $DA(T_n)$ is given as follows:

If $n$ is even, then there will be no pendant edge in $DA(T_n)$. Therefore, we have

$$|E_{(2,3)}| = |uw \in E(DA(T_n)) : d_u = 2 \text{ and } d_v = 3| = 4,$$
$$|E_{(4,4)}| = |uw \in E(DA(T_n)) : d_u = 4 \text{ and } d_v = 4| \quad = |E(DA(T_n))| - |E_{(2,3)}| - |E_{(2,4)}| - |E_{(3,4)}| = n - 3.$$
If $n$ is odd, then there will be a pendant edge in $DA(T_n)$. Therefore, we have

$$
|E_{(1,4)}| = |uv \in E(DA(T_n)) : d_u = 1 \text{ and } d_v = 4| = 1,
$$

$$
|E_{(2,3)}| = |uv \in E(DA(T_n)) : d_u = 2 \text{ and } d_v = 3| = 2,
$$

$$
|E_{(2,4)}| = |uv \in E(DA(T_n)) : d_u = 2 \text{ and } d_v = 4| = 4 \left\lfloor \frac{n}{2} \right\rfloor - 2,
$$

$$
|E_{(3,4)}| = |uv \in E(DA(T_n)) : d_u = 3 \text{ and } d_v = 4| = 1,
$$

$$
|E_{(4,4)}| = |uv \in E(DA(T_n)) : d_u = 4 \text{ and } d_v = 4|
= |E(DA(T_n))| - |E_{(1,4)}| - |E_{(2,3)}| - |E_{(2,4)}| - |E_{(3,4)}| = n - 3.
$$

Thus, the $M$–polynomial of $DA(T_n)$ is

$$
M(DA(T_n); x, y) = \sum_{i\leq j} m_{ij}(DA(T_n))x^iy^j
= \begin{cases} 
4x^2y^3 + (4 \left\lfloor \frac{n}{2} \right\rfloor - 4)x^2y^4 + 2x^3y^4 + (n - 3)x^4y^4 & \text{if } n \text{ is even}, \\
x^4y^2 + 2x^2y^3 + (4 \left\lfloor \frac{n}{2} \right\rfloor - 2)x^2y^4 + x^3y^4 + (n - 3)x^4y^4 & \text{if } n \text{ is odd}.
\end{cases}
$$

$\square$

**Definition 21.** The quadrilateral snake $Q_n$ is obtained from the path $P_n$ by replacing each edge of the path by a quadrilateral $C_4$.

**Theorem 2.26.** If $Q_n$ is a quadrilateral snake, then

$$
M(Q_n; x, y) = 4x^2y^2 + 4(n - 2)x^2y^4.
$$

**Proof.** Let quadrilateral snake $Q_n$ be a graph having $(3n - 2)$ vertices and $4(n - 1)$ edges. The edge partition of $Q_n$ is given by,

$$
|E_{(2,2)}| = |uv \in E(Q_n) : d_u = 2 \text{ and } d_v = 2| = 4,
$$

$$
|E_{(2,4)}| = |uv \in E(Q_n) : d_u = 2 \text{ and } d_v = 4|
= |E(Q_n)| - |E_{(2,2)}| = 4(n - 2).
$$

Thus, the $M$–polynomial of $Q_n$ is

$$
M(Q_n; x, y) = \sum_{i\leq j} m_{ij}(Q_n)x^iy^j = 4x^2y^2 + 4(n - 2)x^2y^4.
$$

$\square$

**Definition 22.** A double quadrilateral snake $DQ_n$ is a graph consisting two quadrilateral snakes that have a common path.

**Theorem 2.27.** If $DQ_n$ is a double quadrilateral snake, then

$$
M(DQ_n; x, y) = 2(n - 1)x^2y^2 + 4x^2y^3 + 4(n - 2)x^2y^6 + 2x^3y^6 + (n - 3)x^6y^6.
$$
Thus, the vertices and (3

\begin{align*}
|E_{(2,2)}| &= |w \in E(DQ_n) : d_u = 2 \text{ and } d_v = 2| = 2(n-1), \\
|E_{(2,3)}| &= |w \in E(DQ_n) : d_u = 2 \text{ and } d_v = 3| = 4, \\
|E_{(2,6)}| &= |w \in E(DQ_n) : d_u = 2 \text{ and } d_v = 6| = 4(n-2), \\
|E_{(3,6)}| &= |w \in E(DQ_n) : d_u = 3 \text{ and } d_v = 6| = 2, \\
|E_{(6,6)}| &= |w \in E(DQ_n) : d_u = 6 \text{ and } d_v = 6| \\
&= |E(DQ_n)| - |E_{(2,2)}| - |E_{(2,3)}| - |E_{(2,6)}| - |E_{(3,6)}| = n - 3.
\end{align*}

Thus, the $M$ - polynomial of $DQ_n$ is

\[
M(DQ_n; x, y) = \sum_{i \leq j} m_{ij}(DQ_n)x^iy^j = 2(n-1)x^2y^2 + 4x^2y^3 + 4(n-2)x^2y^6 + 2x^3y^6 + (n-3)x^6y^6.
\]

\[\square\]

**Definition 23.** The alternate quadrilateral snake $A(Q_n)$ is obtained from a path $v_1v_2\ldots v_n$ by joining $v_i, v_{i+1}$ (alternatively) to new vertices $v_i, w_i$ respectively and then joining $v_i$ and $w_i$. i.e., every alternate edge of a path is replaced by a cycle $C_4$.

**Theorem 2.28.** If $A(Q_n)$ is an alternate quadrilateral snake, then

\[
M(A(Q_n); x, y) = \begin{cases} \\
\left(\frac{n}{2} + 2\right) x^2y^2 + nx^2y^3 + (n - 3)x^3y^3 & \text{if } n \text{ is even}, \\
x^3y^3 + \left(\frac{n}{2} + 1\right) x^2y^2 + 2\left(\frac{n}{2}\right)x^2y^3 + (n - 3)x^3y^3 & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proof.** Let an alternate quadrilateral snake $A(Q_n)$ be a graph having $(n + 2\lfloor n/2 \rfloor)$ vertices and $(3\lfloor n/2 \rfloor + n - 1)$ edges. The edge partition of $A(Q_n)$ is given as follows:

If $n$ is even, then there will be no pendant edge in $A(T_n)$. Therefore, we have

\[
|E_{(2,2)}| = |w \in E(A(Q_n)) : d_u = 2 \text{ and } d_v = 2| = \frac{n}{2} + 2,
\]

\[
|E_{(2,3)}| = |w \in E(A(Q_n)) : d_u = 2 \text{ and } d_v = 3| = n,
\]

\[
|E_{(3,3)}| = |w \in E(A(Q_n)) : d_u = 3 \text{ and } d_v = 3| \\
= |E(A(Q_n))| - |E_{(2,2)}| - |E_{(2,3)}| = n - 3.
\]

If $n$ is odd, then there will be a pendant edge in $A(T_n)$. Therefore, we have

\[
|E_{(1,3)}| = |w \in E(A(Q_n)) : d_u = 1 \text{ and } d_v = 3| = 1,
\]

\[
|E_{(2,2)}| = |w \in E(A(Q_n)) : d_u = 2 \text{ and } d_v = 2| = \left[\frac{n}{2}\right] + 1,
\]

\[
|E_{(2,3)}| = |w \in E(A(Q_n)) : d_u = 2 \text{ and } d_v = 3| = 2\left[\frac{n}{2}\right],
\]

\[
|E_{(3,3)}| = |w \in E(A(Q_n)) : d_u = 3 \text{ and } d_v = 3| \\
= |E(A(Q_n))| - |E_{(1,3)}| - |E_{(2,2)}| - |E_{(2,3)}| = n - 3.
\]

Thus, the $M$ - polynomial of $A(Q_n)$ is

\[
M(A(Q_n); x, y) = \sum_{i \leq j} m_{ij}(A(Q_n))x^iy^j
\]

\[
= \begin{cases} \\
\left(\frac{n}{2} + 2\right) x^2y^2 + nx^2y^3 + (n - 3)x^3y^3 & \text{if } n \text{ is even}, \\
x^3y^3 + \left(\frac{n}{2} + 1\right) x^2y^2 + 2\left(\frac{n}{2}\right)x^2y^3 + (n - 3)x^3y^3 & \text{if } n \text{ is odd}.
\end{cases}
\]
Definition 24. An irregular quadrilateral snake $IQ_n$ is a graph obtained from the path $P_n : u_1u_2 \ldots u_n$ with vertex set $V(IQ_n) = V(P_n) \cup \{v_i, w_i : 1 \leq i \leq n - 2\}$ and the edge set $E(IQ_n) = E(P_n) \cup \{u_ii_{i+1}, w_iw_{i+2} : 1 \leq i \leq n - 2\}$.

Theorem 2.29. If $IQ_n$ is an irregular quadrilateral snake, then

$M(IQ_n; x, y) = nx^2y^2 + 4x^2y^3 + 2(n-4)x^2y^4 + 2x^3y^4 + (n-5)x^4y^4$.

Proof. Let an irregular quadrilateral snake $IQ_n$ be a graph having $(3n-4)$ vertices and $(4n-7)$ edges. The edge partition of $IQ_n$ is given by,

$$|E_{(2,2)}| = |uv \in E(IQ_n) : d_u = 2 \text{ and } d_v = 2| = n,$$

$$|E_{(2,3)}| = |uv \in E(IQ_n) : d_u = 2 \text{ and } d_v = 3| = 4,$$

$$|E_{(2,4)}| = |uv \in E(IQ_n) : d_u = 2 \text{ and } d_v = 4| = 2(n-4),$$

$$|E_{(3,4)}| = |uv \in E(IQ_n) : d_u = 3 \text{ and } d_v = 4| = 2,$$

$$|E_{(4,4)}| = |uv \in E(IQ_n) : d_u = 4 \text{ and } d_v = 4| = n-5.$$ 

Thus, the $M$–polynomial of $IQ_n$ is

$M(IQ_n; x, y) = \sum_{i \leq j} m_{ij}(IQ_n)x^iy^j = nx^2y^2 + 4x^2y^3 + 2(n-4)x^2y^4 + 2x^3y^4 + (n-5)x^4y^4$.

Definition 25. A double alternate quadrilateral snake $DA(Q_n)$ consists of two alternate quadrilateral snakes that have a common path.

Theorem 2.30. If $DA(Q_n)$ is a double alternate quadrilateral snake, then

$M(DA(Q_n); x, y) = \begin{cases} 
\frac{nx^2y^2 + 4x^2y^3 + 2(n-2)x^2y^4 + 2x^3y^4 + (n-3)x^4y^4}{x^4y^2 + 2|\frac{n}{2}|x^2y^2 + 2x^2y^3 + 2(n-2)x^2y^4 + x^3y^4 + (n-3)x^4y^4} & \text{if } n \text{ is even,} \\
|E(DA(Q_n))| - |E_{(2,2)}| - |E_{(2,3)}| - |E_{(2,4)}| - |E_{(3,4)}| = n - 3. & \text{if } n \text{ is odd.} 
\end{cases}$

Proof. Let a double alternate quadrilateral snake $DA(Q_n)$ be a graph having $(n + 4\lfloor \frac{n}{2} \rfloor)$ vertices and $(6\lfloor \frac{n}{2} \rfloor + n - 1)$ edges. The edge partition of $DA(Q_n)$ is given as follows:

If $n$ is even, then there will be no pendant edge in $DA(T_n)$. Therefore, we have

$$|E_{(2,2)}| = |uv \in E(DA(Q_n)) : d_u = 2 \text{ and } d_v = 2| = n,$$

$$|E_{(2,3)}| = |uv \in E(DA(Q_n)) : d_u = 2 \text{ and } d_v = 3| = 4,$$

$$|E_{(2,4)}| = |uv \in E(DA(Q_n)) : d_u = 2 \text{ and } d_v = 4| = 2(n-2),$$

$$|E_{(3,4)}| = |uv \in E(DA(Q_n)) : d_u = 3 \text{ and } d_v = 4| = 2,$$

$$|E_{(4,4)}| = |uv \in E(DA(Q_n)) : d_u = 4 \text{ and } d_v = 4| = n-5.$$
If \( n \) is odd, then there will be a pendant edge in \( DA(T_n) \). Therefore, we have

\[
|E_{\{1,4\}}| = |uv \in E(DA(Q_n)) : d_u = 1 \text{ and } d_v = 4| = 1,
\]

\[
|E_{\{2,2\}}| = |uv \in E(DA(Q_n)) : d_u = 2 \text{ and } d_v = 2| = 2 \left\lfloor \frac{n}{2} \right\rfloor,
\]

\[
|E_{\{2,3\}}| = |uv \in E(DA(Q_n)) : d_u = 2 \text{ and } d_v = 3| = 2,
\]

\[
|E_{\{2,4\}}| = |uv \in E(DA(Q_n)) : d_u = 2 \text{ and } d_v = 4| = 2(n-2),
\]

\[
|E_{\{3,4\}}| = |uv \in E(DA(Q_n)) : d_u = 3 \text{ and } d_v = 4| = 1,
\]

\[
|E_{\{4,4\}}| = |uv \in E(DA(Q_n)) : d_u = 4 \text{ and } d_v = 4|
\]

Thus, the \( M - \) polynomial of \( DA(Q_n) \) is

\[
M(DA(Q_n); x, y) = \sum_{i \leq j} m_{ij}(DA(Q_n))x^iy^j
\]

\[
= \begin{cases} 
  nx^2y^2 + 4x^2y^3 + 2(n-2)x^2y^4 + 2x^3y^4 + (n-3)x^4y^4 \\
  xy^4 + 2[\frac{3}{2}]x^2y^2 + 2x^2y^3 + 2(n-2)x^2y^4 + x^3y^4 + (n-3)x^4y^4 & \text{if } n \text{ is even}, \\
  2(n+1)x^2y^3 + 2nx^3y^{2n} & \text{if } n \text{ is odd}.
\end{cases}
\]

\[
\Box
\]

**Definition 26.** The graph \( DW_n \) is a graph consisting of the two wheels \( W_n \) of the same order having the same central vertex.

**Theorem 2.31.** If \( DW_n \) is a graph with \((2n + 1)\) vertices and \(4n\) edges, then

\[
M(DW_n; x, y) = 2nx^3y^3 + 2nx^3y^{2n}.
\]

**Proof.** Let \( DW_n \) be a graph having \((2n + 1)\) vertices and \(4n\) edges. The edge partition of \( DW_n \) is given by,

\[
|E_{\{3,3\}}| = |uv \in E(DW_n) : d_u = 3 \text{ and } d_v = 3| = 2n,
\]

\[
|E_{\{3,2n\}}| = |uv \in E(DW_n) : d_u = 3 \text{ and } d_v = 2n|
\]

Thus, the \( M - \) polynomial of \( DW_n \) is

\[
M(DW_n; x, y) = \sum_{i \leq j} m_{ij}(DW_n)x^iy^j = 2nx^3y^3 + 2nx^3y^{2n}.
\]

\[
\Box
\]

**Definition 27.** The \( AC_n \) be a graph obtained from a cycle \( C_n : u_1u_2...u_nu_1 \) with the vertex set \( V(AC_n) = V(C_n) \cup \{v_i, w_i : 1 \leq i \leq n\} \) and the edge set \( E(AC_n) = E(C_n) \cup \{u_iv_i, v_iw_i : 1 \leq i \leq n\} \).

**Theorem 2.32.** If \( AC_n \) is a graph with \(3n\) vertices and \(3n\) edges, then

\[
M(AC_n; x, y) = nx^2y^2 + nx^2y^3 + nx^3y^3.
\]

**Proof.** Let \( AC_n \) is a graph having \(3n\) vertices and \(3n\) edges. The edge partition of \( AC_n \) is given by,

\[
|E_{\{1,2\}}| = |uv \in E(AC_n) : d_u = 1 \text{ and } d_v = 2| = n,
\]

\[
|E_{\{2,3\}}| = |uv \in E(AC_n) : d_u = 2 \text{ and } d_v = 3| = n,
\]

\[
|E_{\{3,3\}}| = |uv \in E(AC_n) : d_u = 3 \text{ and } d_v = 3|
\]

\[
= |E(AC_n)| - |E_{\{1,2\}}| - |E_{\{2,3\}}| = n.
\]

\[
\Box
\]
Thus, the $M$-polynomial of $AC_n$ is

$$M(AC_n; x, y) = \sum_{i \leq j} m_{ij}(AC_n) x^i y^j = nxy^2 + nx^2y^3 + nx^3y^3.$$ 

\[ \square \]

**Definition 28.** An umbrella $U_{m,n} = (P_m + K_1) \circ P_n$ is a graph of order $(m + n)$ and size $(2m + n - 2)$, where $P_m$ and $P_n$ are the two paths of lengths $(m - 1)$ and $(n - 1)$, respectively.

**Theorem 2.33.** If $U_{m,n}$ is an umbrella with $(m + n)$ vertices and $(2m + n - 2)$ edges, then

$$M(U_{m,n}; x, y) = xy^2 + (n - 3)x^2y^2 + 2x^2y^3 + 3x^3y^3 + (m - 3)x^3y^3 + (m - 2)x^3y^{m+1}.$$ 

**Proof.** Let an umbrella $U_{m,n}$ be a graph having $(m + n)$ vertices and $(2m + n - 2)$ edges. The edge partition of $U_{m,n}$ is given by,

$$|E_{(1,2)}| = |uv \in E(U_{m,n}) : d_u = 1 \text{ and } d_v = 2| = 1,$$

$$|E_{(2,2)}| = |uv \in E(U_{m,n}) : d_u = 2 \text{ and } d_v = 2| = n - 3,$$

$$|E_{(2,3)}| = |uv \in E(U_{m,n}) : d_u = 2 \text{ and } d_v = 3| = 2,$$

$$|E_{(2,m+1)}| = |uv \in E(U_{m,n}) : d_u = 2 \text{ and } d_v = m + 1| = 3,$$

$$|E_{(3,3)}| = |uv \in E(U_{m,n}) : d_u = 3 \text{ and } d_v = 3| = m - 3,$$

$$|E_{(3,m+1)}| = |uv \in E(U_{m,n}) : d_u = 3 \text{ and } d_v = m + 1| = |E(U_{m,n})| - |E_{(1,2)}| - |E_{(2,2)}| - |E_{(2,3)}| - |E_{(2,m+1)}| - |E_{(3,3)}| = m - 2.$$

Thus, the $M$-polynomial of $U_{m,n}$ is

$$M(U_{m,n}; x, y) = \sum_{i \leq j} m_{ij}(U_{m,n}) x^i y^j = xy^2 + (n - 3)x^2y^2 + 2x^2y^3 + 3x^3y^3 + (m - 3)x^3y^3 + (m - 2)x^3y^{m+1}.$$ 

\[ \square \]

**Definition 29.** A Dumbbell $Db_n$ is a graph obtained from two cycles of length $n$ by joining a vertex from one cycle to a vertex of another cycle.

**Theorem 2.34.** If $Db_n$ is a dumbbell with $2n$ vertices and $(2n + 1)$ edges, then

$$M(Db_n; x, y) = 2(n - 2)x^2y^2 + 4x^2y^3 + x^3y^3.$$ 

**Proof.** Let a dumbbell $Db_n$ be a graph having $2n$ vertices and $(2n + 1)$ edges. The edge partition of $Db_n$ is given by,

$$|E_{(2,2)}| = |uv \in E(Db_n) : d_u = 2 \text{ and } d_v = 2| = 2(n - 2),$$

$$|E_{(2,3)}| = |uv \in E(Db_n) : d_u = 2 \text{ and } d_v = 3| = 2,$$

$$|E_{(3,3)}| = |uv \in E(Db_n) : d_u = 3 \text{ and } d_v = 3| = 2 - |E_{(2,2)}| - |E_{(2,3)}| = 1.$$

Thus, the $M$-polynomial of $Db_n$ is

$$M(Db_n; x, y) = \sum_{i \leq j} m_{ij}(Db_n) x^i y^j = 2(n - 2)x^2y^2 + 4x^2y^3 + x^3y^3.$$ 

\[ \square \]

**Definition 30.** The slanting ladder $SL_n$ is a graph obtained from two paths $u_1v_2...u_n$ and $v_1v_2...v_n$ by joining each $u_i$ with $v_{i+1}$, $(1 \leq i \leq n - 1)$.
Theorem 2.35. If \( SL_n \) is a slanting ladder with \( 2n \) vertices and \( 3(n - 1) \) edges, then
\[
M(SL_n; x, y) = 2xy^3 + 4x^2y^3 + 3(n - 3)x^3y^3.
\]

Proof. Let a slanting ladder \( SL_n \) be a graph having \( 2n \) vertices and \( 3(n - 1) \) edges. The edge partition of \( SL_n \) is given by,
\[
\begin{align*}
|E_{1,3}| &= |uv \in E(SL_n) : d_u = 1 \text{ and } d_v = 3| = 2, \\
|E_{2,3}| &= |uv \in E(SL_n) : d_u = 2 \text{ and } d_v = 3| = 4, \\
|E_{3,3}| &= |uv \in E(SL_n) : d_u = 3 \text{ and } d_v = 3| = |E(SL_n)| - |E_{1,3}| - |E_{2,3}| = 3(n - 3).
\end{align*}
\]
Thus, the \( M \)-polynomial of \( SL_n \) is
\[
M(SL_n; x, y) = \sum_{i \leq j} m_{ij}(SL_n)x^iy^j = 2xy^3 + 4x^2y^3 + 3(n - 3)x^3y^3.
\]
\( \square \)

Definition 31. The triangular ladder \( TL_n \) with vertex set \( V(TL_n) = \{u_i, v_i : 1 \leq i \leq n \} \) and the edge set \( E(TL_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\} \).

Theorem 2.36. If \( TL_n \) is a triangular ladder with \( 2n \) vertices and \( (4n - 3) \) edges, then
\[
M(TL_n; x, y) = 2x^2y^3 + 2x^2y^4 + 4x^3y^4 + (4n - 11)x^4y^4.
\]

Proof. Let a triangular ladder \( TL_n \) be a graph having \( 2n \) vertices and \( (4n - 3) \) edges. The edge partition of \( TL_n \) is given by,
\[
\begin{align*}
|E_{2,3}| &= |uv \in E(TL_n) : d_u = 2 \text{ and } d_v = 3| = 2, \\
|E_{2,4}| &= |uv \in E(TL_n) : d_u = 2 \text{ and } d_v = 4| = 2, \\
|E_{3,4}| &= |uv \in E(TL_n) : d_u = 3 \text{ and } d_v = 4| = 4, \\
|E_{4,4}| &= |uv \in E(TL_n) : d_u = 4 \text{ and } d_v = 4| = |E(TL_n)| - |E_{2,3}| - |E_{2,4}| - |E_{3,4}| = (4n - 11).
\end{align*}
\]
Thus, the \( M \)-polynomial of \( TL_n \) is
\[
M(TL_n; x, y) = \sum_{i \leq j} m_{ij}(TL_n)x^iy^j = 2x^2y^3 + 2x^2y^4 + 4x^3y^4 + (4n - 11)x^4y^4.
\]
\( \square \)

Definition 32. The \( n \)-cone graph \( C_m + K_n \) is a graph where \( C_m \) is a cycle of order \( n \) and \( K_n \) is a complete graph of order \( n \).

Theorem 2.37. If \( C_m + K_n \) is a \( n \)-cone with \( (m + n) \) vertices and \( m(n + 1) \) edges, then
\[
M(C_m + K_n; x, y) = mnx^my^{n+2} + mx^{n+2}y^{n+2}.
\]

Proof. Let a \( n \)-cone graph \( C_m + K_n \) be a graph having \( (m + n) \) vertices and \( m(n + 1) \) edges. The edge partition of \( C_m + K_n \) is given by,
\[
\begin{align*}
|E_{m,n+2}| &= |uv \in E(C_m + K_n) : d_u = m \text{ and } d_v = n + 2| = mn, \\
|E_{n+2,m+2}| &= |uv \in E(C_m + K_n) : d_u = n + 2 \text{ and } d_v = n + 2| = |E(C_m + K_n)| - |E_{m,n+2}| = m.
\end{align*}
\]
Thus, the $M$ – polynomial of $C_m + \overline{K_n}$ is
\[ M(C_m + \overline{K_n}; x, y) = \sum_{i \leq j} m_{ij}(C_m + \overline{K_n})x^iy^j = mnx^my^{n+2} + mnx^{n+2}y^{n+2}. \]

□

**Definition 33.** The graph $C_n^{(m,t)}$ is obtained by identifying one vertex of $C_n$ with one end vertex of $m$ paths each of length $t$. In particular, $C_n^{(1,t)}$ is a tadpole.

**Theorem 2.38.** If $C_n^{(m,t)}$ is a graph with $(n + t)$ vertices and $(mt + n)$ edges, then
\[ M(C_n^{(m,t)}; x, y) = nxy^2 + (m + n - 2)x^2y^2 + (m + 2)x^2y^{n+2}. \]

**Proof.** Let $C_n^{(m,t)}$ be a graph having $(n + t)$ vertices and $(mt + n)$ edges. The edge partition of $C_n^{(m,t)}$ is given by,
\[
|E_{(1,2)}| = |uw \in E(C_n^{(m,t)}): d_u = 1 \text{ and } d_v = 2| = m,
\]
\[
|E_{(2,2)}| = |uw \in E(C_n^{(m,t)}): d_u = 2 \text{ and } d_v = 2| = m + n - 2,
\]
\[
|E_{(2,m+2)}| = |uw \in E(C_n^{(m,t)}): d_u = 2 \text{ and } d_v = m + 2| = |E(C_n^{(m,t)})| - |E_{(1,2)}| - |E_{(2,2)}| = m + 2.
\]

Thus, the $M$ – polynomial of $C_n^{(m,t)}$ is
\[ M(C_n^{(m,t)}; x, y) = \sum_{i \leq j} m_{ij}(C_n^{(m,t)})x^iy^j = nx^2y^2 + (m + n - 2)x^2y^2 + (m + 2)x^2y^{n+2}. \]

□

**Definition 34.** The graph $\theta(C_m)^n$ is obtained from $n$ copies of $C_m$ that shares an edge in common, where $C_m$ is a cycle of length $m$. i.e., an $n$ page book graph with $m$-polygonal pages.

**Theorem 2.39.** If $\theta(C_m)^n$ is an $n$ page book graph with $m$-polygonal pages, then
\[ M(\theta(C_m)^n; x, y) = n(m - 3)x^2y^2 + 2nx^2y^{n+1} + x^{n+1}y^{n+1}. \]

**Proof.** Let $\theta(C_m)^n$ be a graph having $n(m - 2) + 2$ vertices and $n(m - 1) + 1$ edges. The edge partition of $\theta(C_m)^n$ is given by,
\[
|E_{(2,2)}| = |uw \in E(\theta(C_m)^n): d_u = 2 \text{ and } d_v = 2| = n(m - 3),
\]
\[
|E_{(2,n+1)}| = |uw \in E(\theta(C_m)^n): d_u = 2 \text{ and } d_v = n + 1| = 2n,
\]
\[
|E_{(n+1,n+1)}| = |uw \in E(\theta(C_m)^n): d_u = n + 1 \text{ and } d_v = n + 1| = |E(\theta(C_m)^n)| - |E_{(2,2)}| - |E_{(2,n+1)}| = 1.
\]

Thus, the $M$ – polynomial of $\theta(C_m)^n$ is
\[ M(\theta(C_m)^n; x, y) = \sum_{i \leq j} m_{ij}(\theta(C_m)^n)x^iy^j = n(m - 3)x^2y^2 + 2nx^2y^{n+1} + x^{n+1}y^{n+1}. \]

□

**Definition 35.** The kayak paddle graph $KP(k, m, l)$ is a graph obtained by joining two cycles $C_k$ and $C_m$ by a path of length $l$.

**Theorem 2.40.** If $KP(k, m, l)$ is a kayak paddle graph, then
\[ M(KP(k, m, l); x, y) = (k + m + l - 6)x^2y^2 + 6x^2y^3. \]
Proof. Let $KP(k, m, l)$ be a graph having $(k + m + l - 1)$ vertices and $(k + m + l)$ edges. The edge partition of $KP(k, m, l)$ is given by,

\[
\begin{align*}
|E_{\{2,2\}}| &= |uv \in E(KP(k, m, l)) : d_u = 2 \text{ and } d_v = 2| = k + m + l - 6, \\
|E_{\{2,3\}}| &= |uv \in E(KP(k, m, l)) : d_u = 2 \text{ and } d_v = 3| \\
&= |E(KP(k, m, l))| - |E_{\{2,2\}}| = 6.
\end{align*}
\]

Thus, the $M$-polynomial of $KP(k, m, l)$ is

\[M(KP(k, m, l); x, y) = \sum_{i \leq j} m_{ij}(KP(k, m, l))x^iy^j = (k + m + l - 6)x^2y^2 + 6x^2y^3.\]

\[\Box\]

**Definition 36.** The graph $C_n^{(t)}$ is obtained from the one-point union of $t$ cycles of length $n$.

**Theorem 2.41.** If $C_n^{(t)}$ is a graph with $t(n - 1) + 1$ vertices and $nt$ edges, then

\[M(C_n^{(t)}; x, y) = t(n - 2)x^2y^2 + 2tx^2y^{2t}.\]

**Proof.** Let $C_n^{(t)}$ be a graph having $t(n - 1) + 1$ vertices and $nt$ edges. The edge partition of $C_n^{(t)}$ is given by,

\[
\begin{align*}
|E_{\{2,2\}}| &= |uv \in E(C_n^{(t)}) : d_u = 2 \text{ and } d_v = 2| = t(n - 2), \\
|E_{\{2,2t\}}| &= |uv \in E(C_n^{(t)}) : d_u = 2 \text{ and } d_v = 2t| \\
&= |E(C_n^{(t)})| - |E_{\{2,2\}}| = 2t.
\end{align*}
\]

Thus, the $M$-polynomial of $C_n^{(t)}$ is

\[M(C_n^{(t)}; x, y) = \sum_{i \leq j} m_{ij}(C_n^{(t)})x^iy^j = t(n - 2)x^2y^2 + 2tx^2y^{2t}.\]

\[\Box\]

Note that, the topological indices (that are mentioned in Table 1) of all these special graphs can be obtained by using respective $M$-polynomial and column 4 of Table 1. The process of obtaining these topological indices is given in two Corollaries 2.11 and 2.13 as an illustration.

3. Conclusion

In this paper, we have obtained $M$-polynomial of some special graphs and some topological indices of these graphs. The advantage of $M$-polynomial is that, from that one expression we can obtain several degree-based topological indices. It is very challenging to bring all the degree-based topological indices under $M$-polynomial.

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