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# M-POLYNOMIAL AND DEGREE-BASED TOPOLOGICAL INDICES OF GRAPHS

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Abstract. For a graph $G$, the *M-polynomial* is defined as $M(G;x,y) = \sum_{i\leq j} m_{ij}(G)x^i y^j$, where $m_{ij}, (i,j \geq 1)$, is the number of edges $uv$ of $G$ such that $d_G(u) = i$ and $d_G(v) = j$. The topological indices play an important role in determining *physico-chemical properties* of chemical graphs, among them the degree-based topological indices can be easily driven from an algebraic expression corresponding to the chemical graphs called *M-polynomial.* In this note, we first compute $M-polynomial$ of some special graphs. Further, we derive some degree-based topological indices of these graphs from their respective $M-polynomial$.

## 1. Introduction

Throughout this paper, by a graph $G = (V, E)$ we mean a simple, undirected, finite graph of order $n$ and size $m$. Let $V(G)$ and $E(G)$ denote the vertex set and an edge set, respectively. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to it in $G$. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of $G$ and let $d_{v_i} = d_G(v_i)$. The *line graph* [13] $L(G)$ of a graph $G$ is a graph whose vertex set is one-to-one correspondence with the edge set of the graph $G$ and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in $G$. The *subdivision graph* [13] $S(G)$ of a graph $G$ is the graph obtained by inserting a new vertex onto each edge of $G$. The *product* [9, 13] $G \times H$ of graphs $G$ and $H$ has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(a, x)(b, y)$ is an edge of $G \times H$ if and only if [$a = b$ and $xy \in E(H)$] or [$x = y$ and $ab \in E(G)$]. The *corona* [9, 13] $G \circ H$ of graphs $G$ and $H$ is a graph obtained from $G$ and $H$ by taking one copy of $G$ and $|V(G)|$ copies of $H$ and then joining by an edge each vertex of the $i^{th}$ copy of $H$ is named $(H, i)$ with the $i^{th}$ vertex of $G$. The *join* [13] $G_1 + G_2$ of two graphs $G_1$ and $G_2$ is the graph obtained from $G_1$ and $G_2$ by joining every vertex of $G_1$ to all vertices of $G_2$. For undefined graph theoretic terminologies and notions, refer to [13, 15, 23].

It is always interesting to find some properties of graphs which are invariant. Topological indices and polynomials are foremost among them. Over the last decade

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there are numerous research papers devoted to topological indices and polynomials. Several topological indices have been defined in the literature. For details of topological indices one can refer to [7, 16]. For different topological indices and their applications one can refer to [1, 2, 3, 10, 11, 12]. The general form of degree-based topological index of a graph is given by

$$TI(G) = \sum_{e=uv \in E(G)} f(d_G(u), d_G(v))$$

where $f = f(x, y)$ is a function appropriately chosen for the computation. Table 1 gives the standard topological indices defined by $f(x, y)$.

There are many graph polynomials too [4, 25]. The Hosoya polynomial is the most well-known polynomial which plays a vital role in determining distance-based topological indices such as Wiener index [24], hyper Wiener index [4] of graphs. The $M-polynomial$ [5] is one among other algebraic polynomials which was introduced in 2015 and useful in determining many degree-based topological indices (listed in Table 1) [7, 16]. Recently, the study of $M-polynomial$ are reported in [18, 19, 20].

**Definition 1.** [5] *If $G$ is a graph, then $M-polynomial$ of $G$ is defined as*

$$M(G; x, y) = \sum_{i \leq j} m_{ij}(G) x^i y^j, \tag{1.1}$$

*where $m_{ij}, (i, j \geq 1)$, is the number* [6] *of edges $uv$ in $G$ such that $d_G(u) = i$ and $d_G(v) = j$.*

Table 1. Operations to Derive degree-based topological indices from $M-$polynomial [5].

| Notation | Topological Index | $f(x, y)$ | Derivation from $M(G; x, y)$ |
|---|---|---|---|
| $M_1(G)$ | First Zagreb | $x + y$ | $(D_x + D_y)(M(G; x, y))\|_{x=y=1}$ |
| $M_2(G)$ | Second Zagreb | $xy$ | $(D_x D_y)(M(G; x, y))\|_{x=y=1}$ |
| ${}^m M_2(G)$ | Second modified Zagreb | $\frac{1}{xy}$ | $(S_x S_y)(M(G; x, y))\|_{x=y=1}$ |
| $S_D(G)$ | Symmetric division index | $\frac{x^2+y^2}{xy}$ | $(D_x S_y + D_y S_x)(M(G; x, y))\|_{x=y=1}$ |
| $H(G)$ | Harmonic | $\frac{2}{x+y}$ | $2 S_x J(M(G; x, y))\|_{x=1}$ |
| $I_n(G)$ | Inverse sum index | $\frac{xy}{x+y}$ | $S_x J D_x D_y(M(G; x, y))\|_{x=1}$ |

where $D_x = x\frac{\partial f(x,y)}{\partial x}$, $D_y = y\frac{\partial f(x,y)}{\partial y}$, $S_x = \int_0^x \frac{f(t,y)}{t} dt$, $S_y = \int_0^y \frac{f(x,t)}{t} dt$ and $J(f(x, y)) = f(x, x)$ are the operators.

## 2. M-polynomial of some special Graphs

**Proposition 2.1.** *The $M$-polynomial of a path, a cycle, a complete graph, a complete bipartite graph, a wheel, a star and a double star are as follows:*

(i) *For a path $P_n$ of order $n$, we have*

$$M(P_n;x,y) = 2xy^2 + (n-3)x^2y^2.$$

(ii) *For a cycle $C_n$ of order $n$, we have*

$$M(C_n;x,y) = nx^2y^2.$$

(iii) *For a complete graph of order $n$, we have*

$$M(K_n;x,y) = \binom{n}{2}x^{n-1}y^{n-1}.$$

(iv) *For a complete bipartite graph $K_{a,b}$ of order $a+b$, we have*

$$M(K_{a,b};x,y) = abx^ay^b.$$

(v) *For a wheel $W_n$ of order $n+1$, we have*

$$M(W_n;x,y) = nx^3y^3 + nx^3y^n.$$

(vi) *For a star $S_n$ of order $n+1$, we have*

$$M(S_n;x,y) = nxy^n.$$

(vii) *For a double star $S_{a,b}$ of order $a+b+2$, we have*

$$M(S_{a,b};x,y) = axy^{a+1} + bxy^{b+1} + x^{a+1}y^{b+1}.$$

**Definition 2.** *The vertex splitting graph [22] $S'(G)$ of a graph $G$ is obtained from $G$ by adding for each vertex $v$ of $G$ a new vertex $v'$ so that $v'$ is adjacent to every vertex that is adjacent to $v$.*

**Theorem 2.2.** *If $G$ is a graph of order $n$ and size $m$ with the $M$-polynomial $M(G;x,y) = \sum_{i\leq j} m_{ij}(G)x^iy^j$, then*

$$M(S'(G);x,y) = \sum_{i\leq j} m_{ij}(S'(G))x^iy^j = \sum_{i\leq j} m_{ij}(G)x^{2i}y^{2j} + \sum_{a\leq b} m_{ab}(G)x^ay^b,$$

*where* $m_{ab}(G) = \begin{cases} m_{ij}(G) & \text{for } a=i, b=2j \text{ and } i\neq j, \\ 2m_{ij}(G) & \text{for } a=i, b=2j \text{ and } i=j. \end{cases}$

*Proof.* By definition of vertex splitting graph, we have the degree of the original vertices of $G$ in $S'(G)$ is twice the degree of that vertex in $G$ while the degree of the duplicates of those vertices are the same as the degree of corresponding vertices in $G$. Therefore, we have the following:

$$m_{2i2j}(S'(G)) = m_{ij}(G) \text{ and } m_{ab}(G) = \begin{cases} m_{ij}(G) & \text{for } a=i, b=2j \text{ and } i\neq j, \\ 2m_{ij}(G) & \text{for } a=i, b=2j \text{ and } i=j. \end{cases}$$

Thus, we get the desired result by substituting these values in Eq. (1.1). □

**Definition 3.** *The edge splitting graph [14] $L_S(G)$ of a graph $G$ is a graph with vertex set $E(G)\cup E_1(G)$ with two vertices adjacent if they correspond to adjacent edges of $G$ or one corresponds to an element ${e_i}'$ of $E_1(G)$ and the other to an element $e_j$ of $E(G)$ where $e_j \in N(e_i)$.*

**Theorem 2.3.** *If $G$ is a graph of order $n$ and size $m$ with the $M$-polynomial $M(G;x,y)=\sum_{i\leq j} m_{ij}(G)x^iy^j$, then*

$$M(L_S(G);x,y)=\sum_{i\leq j} m_{ij}(L_S(G))x^iy^j=\sum_{i\leq j} m_{ij}(L(G))x^{2i}y^{2j}+\sum_{a\leq b} m_{ab}(L(G))x^ay^b,$$

*where* $m_{ab}(L(G))=\begin{cases} m_{ij}(L(G)) & \text{for } a=i, b=2j \text{ and } i\neq j,\\ 2m_{ij}(L(G)) & \text{for } a=i, b=2j \text{ and } i=j.\end{cases}$

*Proof.* By definition of edge splitting graph, we have the degree of the original vertex of $L(G)$ in $L_S(G)$ is twice the degree of that edge vertex in $L(G)$ while the degree of the duplicates of those vertices are the same as the degree of corresponding vertices in $L(G)$. Therefore, we have the following:

$$m_{2i2j}(L_S(G))=m_{ij}(L(G)) \text{ and } m_{ab}(L(G))=\begin{cases} m_{ij}(L(G)) & \text{for } a=i, b=2j \text{ and } i\neq j,\\ 2m_{ij}(L(G)) & \text{for } a=i, b=2j \text{ and } i=j.\end{cases}$$

Thus, we get the desired result by substituting these values in Eq. (1.1). □

**Definition 4.** *The shadow graph [9] $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$, say $G'$, $G''$ and joining each vertex $v'$ in $G'$ to the neighbors of the corresponding vertex $v''$ in $G''$.*

**Theorem 2.4.** *If $G$ is a graph of order $n$ and $D_2(G)$ is the shadow graph of $G$, then*

$$M(D_2(G);x,y)=\sum_{i\leq j} 4m_{ij}(G)x^{2i}y^{2j}.$$

*Proof.* Let $D_2(G)$ be the shadow graph of a graph $G$ of order $n$ which has $2n$ vertices and $4m$ edges. Then we have by definition of shadow graph $d_{D_2(G)}(v')=2d_G(v)$ for each $v'\in V(D_2(G))$ corresponds to $v\in V(G)$. Thus,

$$\begin{aligned} |E_{\{2i,2j\}}| &= |uv\in E(D_2(G)) : d_u=2i \text{ and } d_v=2i|\\ &= 2|u'v'\in E(G') : d_{u'}=i \text{ and } d_{v'}=j|+2|u''v''\in E(G'') : d_{u''}=i \text{ and } d_{v''}=j|\\ &= 2m_{ij}(G)+2m_{ij}(G)\\ &= 4m_{ij}(G).\end{aligned}$$

Thus, the $M-polynomial$ of $D_2(G)$ is

$$M(D_2(G);x,y)=\sum_{i\leq j} m_{ij}(D_2(G))x^iy^j=\sum_{i\leq j} 4m_{ij}(G)x^{2i}y^{2j}.$$

□

**Corollary 2.5.** *If $G$ is an $r$-regular graph of order $n$ and size $m$, then*

$$M(D_2(G);x,y)=4mx^{2r}y^{2r}$$

**Definition 5.** *For a graph $G=(V(G),E(G)$, the Mycielskian [21] $\mu(G)$ of $G$ is a graph with vertex set consisting the disjoint union $V(G)\cup V'(G)\cup\{u\}$, where $V'(G)=\{x' : x\in V(G)\}$, and the edge set $E(G)\cup\{x'y : xy\in E(G)\}\cup\{x'u : x'\in V'(G)\}$.*

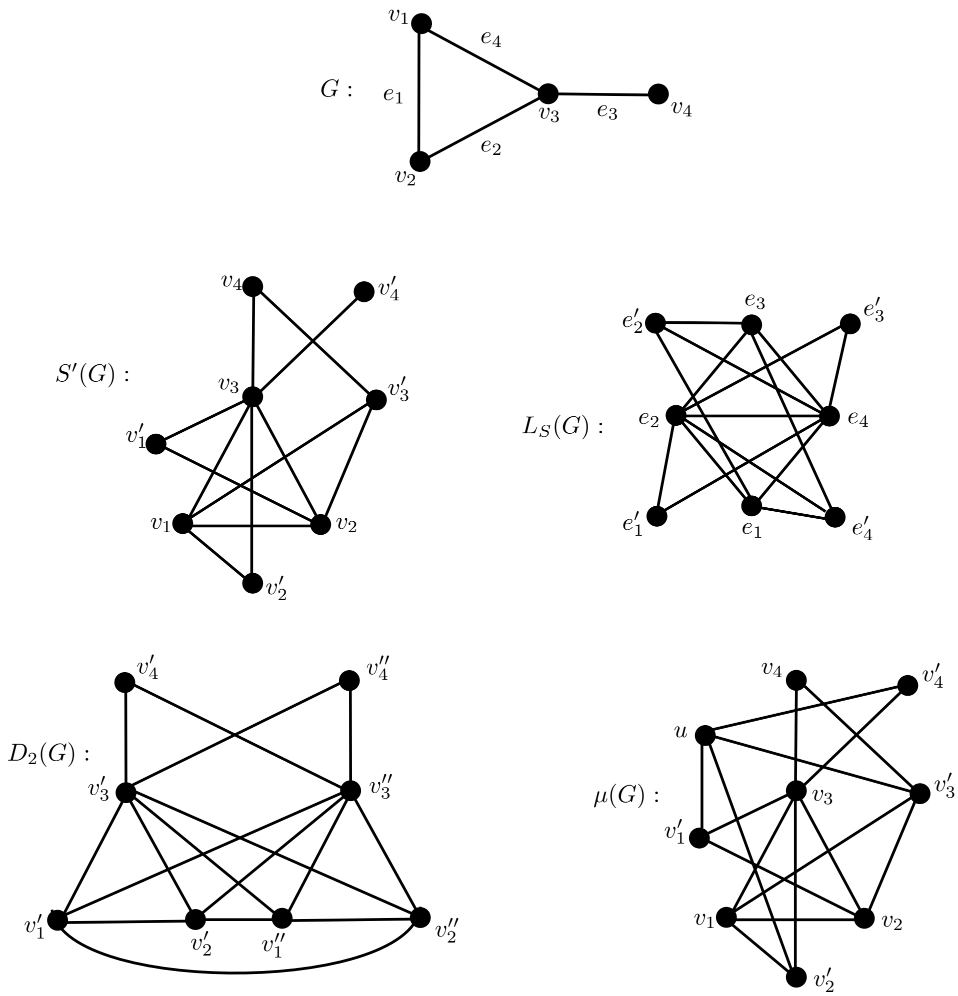


FIGURE 1. The graph $G$ with its vertex splitting graph $S'(G)$, line splitting graph $L_s(G)$, shadow graph $D_2(G)$ and Mycielskian $\mu(G)$.

**Theorem 2.6.** *If $G$ is a graph of order $n$ and size $m$ with the $M$-polynomial $M(G;x,y) = \sum\limits_{i\leq j} m_{ij}(G)x^iy^j$, then*

$$M(\mu(G);x,y) = \sum_{i\leq j} m_{ij}(G)x^{2i}y^{2j} + \sum_{a'\leq b'} m_{a'b'}(G)x^{a'}y^{b'},$$

*where $a' = min\{a,b\}, b' = max\{a,b\}$, and for $i' = min\{i,j\}, j' = max\{i,j\}$*

$$m_{a'b'}(G) = \begin{cases} m_{i'j'}(G) & if\ a = i+1, b = 2j \quad and\ \ i \neq j, \\ 2m_{i'j'}(G) & if\ a = i+1, b = 2j \quad and\ \ i = j, \\ |\{v : d_v = i\}| & if\ a = i+1, b = n \quad for\ \ i = 1,2,...,n-1. \end{cases}$$

*Proof.* By definition of mycielskian of a graph, we have the degree of the original vertices of $G$ in $\mu(G)$ is twice the degree of that vertex in $G$, the degree $d_{\mu(G)}(v_i') = d_G(v_i) + 1$ of the duplicates $v_i'$ of $v_i \in V(G)$ and the degree of the vertex $u \in \mu(G)$

is $n$. Therefore, we have the following:

$$m_{2i2j}(\mu(G)) = m_{ij}(G)$$

and

$$m_{a'b'}(G) = \begin{cases} m_{i'j'}(G) & if\ a = i+1, b = 2j \quad and\ \ i \neq j, \\ 2m_{i'j'}(G) & if\ a = i+1, b = 2j \quad and\ \ i = j, \\ |\{v : d_v = i\}| & if\ a = i+1, b = n \quad for\ \ i = 1, 2, ..., n-1. \end{cases}$$

Thus, we get the desired result by substituting these values in Eq. (1.1). □

**Corollary 2.7.** *If $M$-polynomial of Mycielskian of a graph $G$ is*

$$M(\mu(G); x, y) = \sum_{i\leq j} m_{ij}(G)x^{2i}y^{2j} + \sum_{a'\leq b'} m_{a'b'}(G)x^{a'}y^{b'},$$

*then*

$$\begin{aligned} M_1(\mu(G)) &= 2\sum_{i\leq j}(i+j)m_{ij}(G) + \sum_{a'\leq b'}(a'+b')m_{a'b'}(G), \\ M_2(\mu(G)) &= 4\sum_{i\leq j} ij m_{ij}(G) + \sum_{a'\leq b'} a'b' m_{a'b'}(G), \\ {}^{m}M_2(\mu(G)) &= \frac{1}{4}\sum_{i\leq j}\frac{m_{ij}(G)}{ij} + \sum_{a'\leq b'}\frac{m_{a'b'}(G)}{a'b'}, \\ S_D(\mu(G)) &= \sum_{i\leq j}\frac{(i^2+j^2)m_{ij}(G)}{ij} + \sum_{a'\leq b'}\frac{(a'^2+b'^2)m_{a'b'}(G)}{a'b'}, \\ H(\mu(G)) &= \sum_{i\leq j}\frac{ij m_{ij}(G)}{(i+j)} + 2\sum_{a'\leq b'}\frac{a'b' m_{a'b'}(G)}{(a'+b')}, \\ I_n(\mu(G)) &= \sum_{i\leq j} ij(i+j)m_{ij}(G) + \sum_{a'\leq b'} a'b'(a'+b')m_{a'b'}(G). \end{aligned}$$

*Proof.* We get the desired results by applying the appropriate operators to $M$-polynomial of $\mu(G)$. □

**Definition 6.** [8] *Let $P_3$ be the 3-vertex tree rooted at one its terminal vertices. See Fig. 2. For $k = 2, 3, ...$ construct the rooted tree $B_k$ by identifying the roots of $k$ copies of $P_3$. The vertex obtained by identifying the roots of $P_3$-trees is the root of $B_k$. The illustrative structure of the rooted tree $B_k$ is depicted in Fig. 2*

**Definition 7.** [8] *Let $d$ be an integer and $\beta_1, \beta_2, ..., \beta_d$ be rooted trees as specified in Definition 6, i.e., $\beta_1, \beta_2, ..., \beta_d \in \{B_2, B_3, ...\}$. A Kragujevac tree $T_k$ is a tree possessing a vertex of degree d, adjacent to the roots of $\beta_1, \beta_2, ..., \beta_d$. This vertex is said to be the central vertex of $T_k$. The subgraphs $\beta_1, \beta_2, ..., \beta_d$ are the branches of $T_k$. Note that, some (or all) branches of $T_k$ may be mutually isomorphic.*

**Theorem 2.8.** *If $T_k$ is a Kragujevac tree with $\beta_1, \beta_2, ..., \beta_d \in \{B_2, B_3, ...\}$ branches, then*

$$M(T_k; x, y) = \sum_{i\geq 2} ik_i xy^2 + \sum_{i\geq 2} ik_i x^2y^{i+1} + \sum_{i\geq 2} k_i x^d y^{i+1},$$

*where $k_i = |\{\beta_i : \beta_i$ is a branch of $T_k$ such that $\beta_i = B_i\}|$ for $i \geq 2$.*

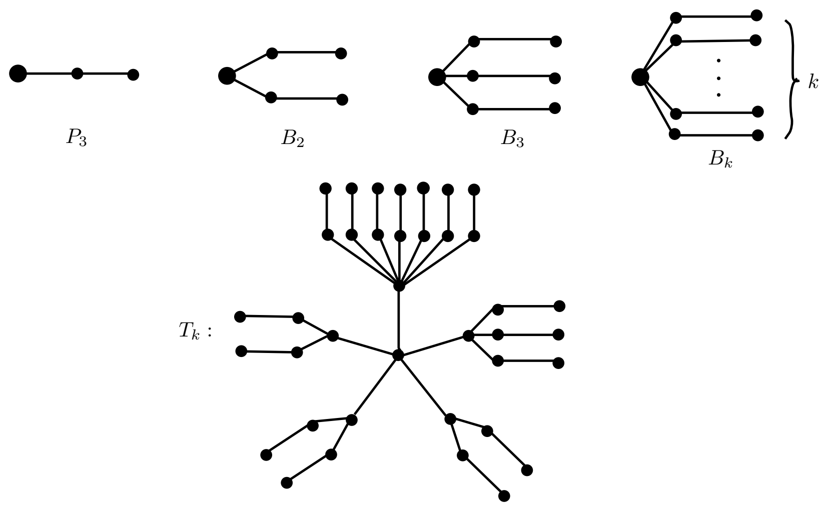


FIGURE 2. The rooted trees $B_k$'s and the Kragujevac tree $T_k$.

*Proof.* By definition of Kragujevac tree $T_k$, we have $\sum_{i\geq 2} ik_i$ vertices of degree 1, $\sum_{i\geq 2} ik_i$ vertices of degree 2 and $k_i$ vertices of degree $i+1$. Therefore, the edge partition of $T_k$ is given as follows:

$$\begin{aligned} |E_{\{1,2\}}| &= |uv \in E(T_k) : d_u = 1 \ \ and \ \ d_v = 2| = \sum_{i\geq 2} ik_i, \\ |E_{\{2,i+1\}}| &= |uv \in E(T_k) : d_u = 2 \ \ and \ \ d_v = i+1| = ik_i, \\ |E_{\{d,i+1\}}| &= |uv \in E(T_k) : d_u = d \ \ and \ \ d_v = i+1| = k_i. \end{aligned}$$

Thus, the $M-polynomial$ of $T_k$ is

$$M(T_k; x, y) = \sum_{i\leq j} m_{ij}(T_k) x^i y^j = \sum_{i\geq 2} ik_i xy^2 + \sum_{i\geq 2} ik_i x^2 y^{i+1} + \sum_{i\geq 2} k_i x^d y^{i+1}.$$

□

**Corollary 2.9.** *If M-polynomial of Kragujevac tree $T_k$ is*

$$M(T_k; x, y) = \sum_{i\geq 2} ik_i xy^2 + \sum_{i\geq 2} ik_i x^2 y^{i+1} + \sum_{i\geq 2} k_i x^d y^{i+1},$$

*then*

$$\begin{aligned} M_1(T_k) &= \sum_{i\geq 2}(i^2+7i+d+1)k_i, \\ M_2(T_k) &= \sum_{i\geq 2}(2i^2+(4+d)i+d)k_i, \\ {}^mM_2(T_k) &= \sum_{i\geq 2}\frac{(i^2+5i+2d)}{2(i+1)}k_i, \\ S_D(T_k) &= \sum_{i\geq 2}\frac{(7i^2+13i+2d+2)}{2(i+1)}k_i, \\ H(T_k) &= \sum_{i\geq 2}\frac{(2i^3+2(d+7)i^2+6(2d+3)i+18)}{3(i+3)(d+i+1)}k_i, \\ I_n(T_k) &= \sum_{i\geq 2}\frac{(8i^3+(11d+20)i^2+12(2d+1)i+9d)}{3(i+3)(d+i+1)}k_i. \end{aligned}$$

*Proof.* We get the desired results by applying the appropriate operators on $M$-polynomial of $T_k$. □

The definitions of the special graphs used in this paper can be found in [9]. In this section, we obtain $M-$ polynomials of some special graphs. We also derive some topological indices (mentioned in section 1) of these graphs from the respective $M-$ polynomials.

**Definition 8.** *The book graph $B_m = S_m \times P_2$ is a graph with $2(m+1)$ vertices and $(3m+1)$ edges, where $S_m$ is a star of order $(m+1)$ and $P_2$ is a path of length one.*

**Theorem 2.10.** *If $B_m$ is a book graph of order $2(m+1)$ and size $(3m+1)$, then*

$$M(B_m;x,y) = mx^2y^2 + 2mx^2y^{m+1} + x^{m+1}y^{m+1}.$$

*Proof.* The book graph $B_m$ has $2(m+1)$ vertices and $(3m+1)$ edges. The edge set of $B_m$ can be partitioned as,

$$\begin{aligned} |E_{\{2,2\}}| &= |uv \in E(B_m) : d_u = 2 \ \ and \ \ d_v = 2| = m, \\ |E_{\{2,m+1\}}| &= |uv \in E(B_m) : d_u = 2 \ \ and \ \ d_v = m+1| = 2m, \\ |E_{\{m+1,m+1\}}| &= |uv \in E(B_m) : d_u = m+1 \ \ and \ \ d_v = m+1| \\ &= |E(B_m) - |E_{\{2,2\}}| - |E_{\{2,m+1\}}| = 1. \end{aligned}$$

Thus, the $M-polynomial$ of $B_m$ is
$M(B_m;x,y) = \sum_{i\leq j} m_{ij}(B_m)x^iy^j = mx^2y^2 + 2mx^2y^{m+1} + x^{m+1}y^{m+1}$.
□

**Corollary 2.11.** *If $M$-polynomial of the book graph $B_m$ is $M(B_m; x, y) = mx^2y^2 + 2mx^2y^{m+1} + x^{m+1}y^{m+1}$, then*

$$\begin{aligned} M_1(B_m) &= 2(m^2 + 6m + 1), \\ M_2(B_m) &= 5m^2 + 10m + 1, \\ {}^mM_2(B_m) &= \frac{m^3 + 6m^2 + 5m + 4}{4(m^2 + 2m + 1)}, \\ S_D(B_m) &= \frac{m^3 + 4m^2 + 9m + 2}{m + 1}, \\ H(B_m) &= \frac{m^3 + 12m^2 + 13m + 6}{2(m^2 + 4m + 3)}, \\ I_n(B_m) &= \frac{11m^2 + 18m + 3}{2(m + 3)}. \end{aligned}$$

*Proof.* We have, the $M$-polynomial of the book graph $B_m$ as

$$M(B_m; x, y) = mx^2y^2 + 2mx^2y^{m+1} + x^{m+1}y^{m+1}.$$

Therefore,

$$\begin{aligned} D_x &= x\frac{\partial f(x, y)}{\partial x} = 2mx^2y^2 + 4mx^2y^{m+1} + (m + 1)x^{m+1}y^{m+1}, \\ D_y &= y\frac{\partial f(x, y)}{\partial y} = 2mx^2y^2 + 2m(m + 1)x^2y^{m+1} + (m + 1)x^{m+1}y^{m+1}, \\ S_x &= \int_0^x \frac{f(t, y)}{t} dt = \frac{m}{2}x^2y^2 + mx^2y^{m+1} + \frac{1}{(m + 1)}x^{m+1}y^{m+1}, \\ S_y &= \int_0^y \frac{f(x, t)}{t} dt = \frac{m}{2}x^2y^2 + \frac{2m}{(m + 1)}x^2y^{m+1} + \frac{1}{(m + 1)}x^{m+1}y^{m+1}, \\ J(f(x, y)) &= f(x, x) = mx^4 + 2mx^{m+3} + x^{2(m+1)}. \end{aligned}$$

Thus, we get

$$\begin{aligned} M_1(B_m) &= (D_x + D_y)(M(B_m; x, y))|_{x=y=1} = 2(m^2 + 6m + 1), \\ M_2(B_m) &= (D_xD_y)(M(B_m; x, y))|_{x=y=1} = 5m^2 + 10m + 1, \\ {}^mM_2(B_m) &= (S_xS_y)(M(B_m; x, y))|_{x=y=1} = \frac{m^3 + 6m^2 + 5m + 4}{4(m^2 + 2m + 1)}, \\ S_D(B_m) &= (D_xS_y + D_yS_x)(M(B_m; x, y))|_{x=y=1} = \frac{m^3 + 4m^2 + 9m + 2}{m + 1}, \\ H(B_m) &= 2S_xJ(M(B_m; x, y))|_{x=1} = \frac{m^3 + 12m^2 + 13m + 6}{2(m^2 + 4m + 3)}, \\ I_n(B_m) &= S_xJD_xD_y(M(B_m; x, y))|_{x=1} = \frac{11m^2 + 18m + 3}{2(m + 3)}. \end{aligned}$$

□

**Definition 9.** *The Ladder $L_n = P_n \times P_2$ is a graph of order $2n$ and size $(3n - 2)$, where $P_n$ and $P_2$ are two paths of length $(n - 1)$ and $1$, respectively.*

**Theorem 2.12.** *If $L_n$ is a ladder, then*

$$M(L_n; x, y) = 2x^2y^2 + 4x^2y^3 + (3n - 8)x^3y^3.$$

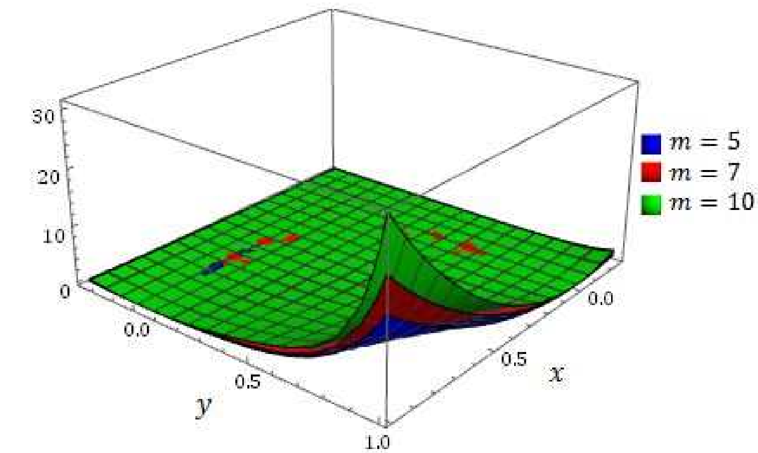


FIGURE 3. Plot of $M$-polynomial of the book graph $B_{10}$

*Proof.* The ladder $L_n$ has $2n$ vertices and $(3n-2)$ edges. The edge set of $L_n$ can be partitioned as,

$$\begin{aligned}
|E_{\{2,2\}}| &= |uv \in E(L_n) : d_u = 2 \ \ and \ \ d_v = 2| = 2, \\
|E_{\{2,3\}}| &= |uv \in E(L_n) : d_u = 2 \ \ and \ \ d_v = 3| = 4, \\
|E_{\{3,3\}}| &= |uv \in E(L_n) : d_u = 3 \ \ and \ \ d_v = 3| \\
&= |E(L_n) - |E_{\{2,2\}}| - |E_{\{2,3\}}| = 3n-8.
\end{aligned}$$

Thus, the $M - polynomial$ of $L_n$ is
$$M(L_n; x, y) = \sum_{i \le j} m_{ij}(L_n) x^i y^j = 2x^2y^2 + 4x^2y^3 + (3n-8)x^3y^3.$$ □

**Corollary 2.13.** *If the $M$-polynomial of the ladder $L_n$ is $M(L_n; x, y) = 2x^2y^2 + 4x^2y^3 + (3n-8)x^3y^3$, then*

$$\begin{aligned}
M_1(L_n) &= 2(9n-10), \\
M_2(L_n) &= 27n-40, \\
{}^mM_2(L_n) &= \frac{6n+5}{18}, \\
S_D(L_n) &= \frac{2(9n-5)}{3}, \\
H(L_n) &= \frac{15n-1}{15}, \\
I_n(L_n) &= \frac{45n-52}{10}.
\end{aligned}$$

*Proof.* We have, the $M$-polynomial of the ladder $L_n$ as

$$M(L_n; x, y) = 2x^2y^2 + 4x^2y^3 + (3n-8)x^3y^3.$$

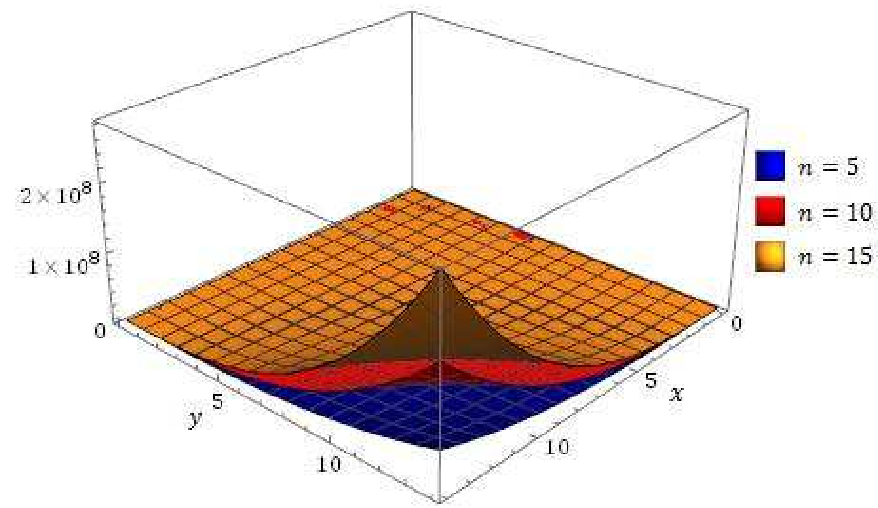


FIGURE 4. Plot of $M$-polynomial of the ladder $L_{10}$

Therefore,

$$\begin{aligned} D_x &= x\frac{\partial f(x,y)}{\partial x} = 4x^2y^2 + 8x^2y^3 + 3(3n-8)x^3y^3, \\ D_y &= y\frac{\partial f(x,y)}{\partial y} = 4x^2y^2 + 12x^2y^3 + 3(3n-8)x^3y^3, \\ S_x &= \int_0^x \frac{f(t,y)}{t}dt = x^2y^2 + 2x^2y^3 + \frac{(3n-8)}{3}x^3y^3, \\ S_y &= \int_0^y \frac{f(x,t)}{t}dt = x^2y^2 + \frac{4}{3}x^2y^3 + \frac{(3n-8)}{3}x^3y^3, \\ J(f(x,y)) &= f(x,x) = 2x^4 + 4x^5 + (3n-8)x^6. \end{aligned}$$

Thus, we get

$$\begin{aligned} M_1(L_n) &= (D_x + D_y)(M(L_n;x,y))|_{x=y=1} = 2(9n-10), \\ M_2(L_n) &= (D_xD_y)(M(L_n;x,y))|_{x=y=1} = 27n-40, \\ {}^mM_2(L_n) &= (S_xS_y)(M(L_n;x,y))|_{x=y=1} = \frac{6n+5}{18}, \\ S_D(L_n) &= (D_xS_y + D_yS_x)(M(L_n;x,y))|_{x=y=1} = \frac{2(9n-5)}{3}, \\ H(L_n) &= 2S_xJ(M(L_n;x,y))|_{x=1} = \frac{15n-1}{15}, \\ I_n(L_n) &= S_xJD_xD_y(M(L_n;x,y))|_{x=1} = \frac{45n-52}{10}. \end{aligned}$$

□

The surfaces in Figures 3 and 4 are plotted by using Mathematica. These surfaces are obtained by $M$-polynomial of the respective graph which shows different behaviours for different parameters $m, n, x$ and $y$.

**Definition 10.** *A planar grid $P_m \times P_n$, is a graph obtained by the product of two paths $P_m$ and $P_n$ of lengths $(m-1)$ and $(n-1)$, respectively.*

**Theorem 2.14.** *If $P_m \times P_n$ is a planar grid, then*
$M(P_m \times P_n; x, y) = 8x^2y^3 + 2(m+n-6)x^3y^3 + 2(m+n-4)x^3y^4 + (2mn-5m-5n+12)x^4y^4$.

*Proof.* The planar grid $P_m \times P_n$ has $mn$ vertices and $(2mn - m - n)$ edges. The edge set of $P_m \times P_n$ can be partitioned as,

$$\begin{aligned} |E_{\{2,3\}}| &= |uv \in E(P_m \times P_n) : d_u = 2 \ \ and \ \ d_v = 3| = 8, \\ |E_{\{3,3\}}| &= |uv \in E(P_m \times P_n) : d_u = 3 \ \ and \ \ d_v = 3| = 2(m+n-6), \\ |E_{\{3,4\}}| &= |uv \in E(P_m \times P_n) : d_u = 3 \ \ and \ \ d_v = 4| = 2(m+n-4), \\ |E_{\{4,4\}}| &= |uv \in E(P_m \times P_n) : d_u = 4 \ \ and \ \ d_v = 4| \\ &= |E(P_m \times P_n) - |E_{\{2,3\}}| - |E_{\{3,3\}}| - |E_{\{3,4\}}| = 2mn - 5m - 5n + 12. \end{aligned}$$

Thus, the $M-polynomial$ of $P_m \times P_n$ is

$$\begin{aligned} M(P_m \times P_n; x, y) &= \sum_{i \le j} m_{ij}(P_m \times P_n)x^iy^j \\ &= 8x^2y^3 + 2(m+n-6)x^3y^3 + 2(m+n-4)x^3y^4 + (2mn-5m-5n+12)x^4y^4. \end{aligned}$$

□

**Definition 11.** *The prism $\Pi_n = C_n \times P_2$ is a 3-regular graph of order $2n$ and size $3n$, where $C_n$ is cycle of order $n$ and $P_2$ is a path of length one.*

**Theorem 2.15.** *If $\Pi_n$ is a prism, then*

$$M(\Pi_n; x, y) = 3nx^3y^3.$$

*Proof.* Let prism $\Pi_n$ be a 3-regular graph having $2n$ vertices and $3n$ edges. The edge partition of $\Pi_n$ is given by,

$$|E_{\{3,3\}}| = |uv \in E(\Pi_n) : d_u = 3 \ \ and \ \ d_v = 3| = 3n.$$

Thus, the $M-polynomial$ of the prism $\Pi_n$ is

$$M(\Pi_n; x, y) = \sum_{i \le j} m_{ij}(\Pi_n)x^iy^j = 3nx^3y^3.$$ □

**Definition 12.** *The book graph with triangular pages $B_m^t = P_2 + mK_1$ is a graph with $(n+2)$ vertices and $(2n+1)$ edges, where $P_2$ is a path of length one and $mK_1$ are the $m$ isolated vertices.*

**Theorem 2.16.** *If $B_m^t$ is a book graph with triangular pages having $(n+2)$ vertices and $(2n+1)$ edges, then*

$$M(B_m^t; x, y) = 2mx^2y^{m+1} + x^{m+1}y^{m+1}.$$

*Proof.* Let $B_m^t$ is a book graph with triangular pages having $(n+2)$ vertices and $(2n+1)$ edges. The edge partition of $B_m^t$ is given by,

$$\begin{aligned} |E_{\{2,m+1\}}| &= |uv \in E(B_m^t) : d_u = 2 \ \ and \ \ d_v = m+1| = 2m, \\ |E_{\{m+1,m+1\}}| &= |uv \in E(B_m^t) : d_u = m+1 \ \ and \ \ d_v = m+1| \\ &= |E(B_m^t) - |E_{\{2,m+1\}}| = 1. \end{aligned}$$

Thus, $M(B_m^t; x, y) = \sum_{i \le j} m_{ij}(B_m^t)x^iy^j = 2mx^2y^{m+1} + x^{m+1}y^{m+1}$. □

**Definition 13.** *The corona $P_n \circ K_1$ of a path $P_n$ of length $(n-1)$ with an isolated vertex $K_1$ is called a comb graph and the corona $P_n \circ 2K_1$ of a path $P_n$ of length $(n-1)$ with two isolated vertices $2K_1$ is called a double comb graph.*

**Theorem 2.17.** *If $P_n \circ K_1$ is a comb graph, then*

$$M(P_n \circ K_1; x, y) = 2xy^2 + (n-2)xy^3 + 2x^2y^3 + (n-3)x^3y^3.$$

*Proof.* The comb graph $P_n \circ K_1$ has $2n$ vertices and $(2n-1)$ edges. The edge set of $P_n \circ K_1$ can be partitioned as,

$$\begin{aligned} |E_{\{1,2\}}| &= |uv \in E(P_n \circ K_1) : d_u = 1 \ \ and \ \ d_v = 2| = 2, \\ |E_{\{1,3\}}| &= |uv \in E(P_n \circ K_1) : d_u = 1 \ \ and \ \ d_v = 3| = (n-2), \\ |E_{\{2,3\}}| &= |uv \in E(P_n \circ K_1) : d_u = 2 \ \ and \ \ d_v = 3| = 2, \\ |E_{\{3,3\}}| &= |uv \in E(P_n \circ K_1) : d_u = 3 \ \ and \ \ d_v = 3| \\ &= |E(P_n \circ K_1) - |E_{\{1,2\}}| - |E_{\{1,3\}}| - |E_{\{2,3\}}| = n-3. \end{aligned}$$

Thus, the $M-polynomial$ of $P_n \circ K_1$ is

$$\begin{aligned} M(P_n \circ K_1; x, y) &= \sum_{i \leq j} m_{ij}(P_n \circ K_1) x^i y^j \\ &= 2xy^2 + (n-2)xy^3 + 2x^2y^3 + (n-3)x^3y^3. \end{aligned}$$

□

**Theorem 2.18.** *If $P_n \circ 2K_1$ is a double comb graph, then*

$$M(P_n \circ 2K_1; x, y) = 4xy^3 + 2(n-2)xy^4 + 2x^3y^4 + (n-3)x^4y^4.$$

*Proof.* The double comb graph $P_n \circ 2K_1$ has $3n$ vertices and $(3n-1)$ edges. The edge set of $P_n \circ 2K_1$ can be partitioned as,

$$\begin{aligned} |E_{\{1,3\}}| &= |uv \in E(P_n \circ 2K_1) : d_u = 1 \ \ and \ \ d_v = 3| = 4, \\ |E_{\{1,4\}}| &= |uv \in E(P_n \circ 2K_1) : d_u = 1 \ \ and \ \ d_v = 4| = 2(n-2), \\ |E_{\{3,4\}}| &= |uv \in E(P_n \circ 2K_1) : d_u = 3 \ \ and \ \ d_v = 4| = 2, \\ |E_{\{4,4\}}| &= |uv \in E(P_n \circ 2K_1) : d_u = 4 \ \ and \ \ d_v = 4| \\ &= |E(P_n \circ 2K_1) - |E_{\{1,3\}}| - |E_{\{1,4\}}| - |E_{\{3,4\}}| = n-3. \end{aligned}$$

Thus, the $M-polynomial$ of $P_n \circ 2K_1$ is

$$\begin{aligned} M(P_n \circ 2K_1; x, y) &= \sum_{i \leq j} m_{ij}(P_n \circ 2K_1) x^i y^j \\ &= 4xy^3 + 2(n-2)xy^4 + 2x^3y^4 + (n-3)x^4y^4. \end{aligned}$$

□

**Definition 14.** *A jelly fish $J(m, n)$ is a graph obtained from a cycle $C_4 : uxvyu$ by joining $x$ and $y$ with an edge and appending $m$ pendant edges to $u$ and $n$ pendant edges to $v$.*

**Theorem 2.19.** *If $J(m, n)$ is a jelly fish graph, then*

$$M(J(m,n); x, y) = mxy^{m+2} + nxy^{n+2} + 2x^3y^{m+2} + 2x^3y^{n+2} + x^3y^3.$$

*Proof.* The jelly fish graph $J(m,n)$ has $(4+m+n)$ vertices and $(5+m+n)$ edges. The edge set of $J(m,n)$ can be partitioned as,

$$\begin{aligned}
|E_{\{1,m+2\}}| &= |uv \in E(J(m,n)) : d_u = 1 \ \ and \ \ d_v = m+2| = m, \\
|E_{\{1,n+2\}}| &= |uv \in E(J(m,n)) : d_u = 1 \ \ and \ \ d_v = n+2| = n, \\
|E_{\{3,m+2\}}| &= |uv \in E(J(m,n)) : d_u = 3 \ \ and \ \ d_v = m+2| = 2, \\
|E_{\{3,n+2\}}| &= |uv \in E(J(m,n)) : d_u = 3 \ \ and \ \ d_v = n+2| = 2, \\
|E_{\{3,3\}}| &= |uv \in E(J(m,n)) : d_u = 3 \ \ and \ \ d_v = 3| \\
&= |E(J(m,n)) - |E_{\{1,m+2\}}| - |E_{\{1,n+2\}}| - |E_{\{3,m+2\}}| - |E_{\{3,n+2\}}| = 1.
\end{aligned}$$

Thus, the $M-polynomial$ of $J(m,n)$ is

$$\begin{aligned}
M(J(m,n);x,y) &= \sum_{i\leq j} m_{ij}(J(m,n))x^i y^j \\
&= mxy^{m+2} + nxy^{n+2} + 2x^3y^{m+2} + 2x^3y^{n+2} + x^3y^3.
\end{aligned}$$

□

**Definition 15.** *A butterfly graph $By_{m,n}$ is obtained from two even cycles of the same order $n$ for $n \geq 3$, sharing a common vertex with $m$ pendant edges attached at the common vertex.*

**Theorem 2.20.** *If $By_{m,n}$ is a butterfly graph, then*

$$M(By_{m,n};x,y) = mxy^{m+4} + 4x^2y^{m+4} + (2n-4)x^2y^2.$$

*Proof.* The butterfly graph $By_{m,n}$ has $(2n+m-1)$ vertices and $(2n+m)$ edges. The edge set of $By_{m,n}$ can be partitioned as,

$$\begin{aligned}
|E_{\{1,m+4\}}| &= |uv \in E(By_{m,n}) : d_u = 1 \ \ and \ \ d_v = m+4| = m, \\
|E_{\{2,m+4\}}| &= |uv \in E(By_{m,n}) : d_u = 2 \ \ and \ \ d_v = m+4| = 4, \\
|E_{\{2,2\}}| &= |uv \in E(By_{m,n}) : d_u = 2 \ \ and \ \ d_v = 2| \\
&= |E(By_{m,n}) - |E_{\{1,m+4\}}| - |E_{\{2,m+4\}}| = 2n-4.
\end{aligned}$$

Thus, the $M-polynomial$ of $By_{m,n}$ is

$$\begin{aligned}
M(By_{m,n};x,y) &= \sum_{i\leq j} m_{ij}(By_{m,n})x^i y^j \\
&= mxy^{m+4} + 4x^2y^{m+4} + (2n-4)x^2y^2.
\end{aligned}$$

□

**Definition 16.** *The triangular snake* [17] *$T_n$ is a graph obtained from the path $P_n$ of length $(n-1)$, by replacing each edge of the path by a triangle $C_3$.*

**Theorem 2.21.** *If $T_n$ is a triangular snake, then*

$$M(T_n;x,y) = 2x^2y^2 + 2(n-1)x^2y^4 + (n-3)x^4y^4.$$

*Proof.* Let triangular snake $T_n$ be a graph having $(2n-1)$ vertices and $3(n-1)$ edges. The edge partition of $T_n$ is given by,

$$\begin{aligned}
|E_{\{2,2\}}| &= |uv \in E(T_n) : d_u = 2 \ \ and \ \ d_v = 2| = 2, \\
|E_{\{2,4\}}| &= |uv \in E(T_n) : d_u = 2 \ \ and \ \ d_v = 4| = 2(n-1), \\
|E_{\{4,4\}}| &= |uv \in E(T_n) : d_u = 4 \ \ and \ \ d_v = 4| \\
&= |E(T_n)| - |E_{\{2,2\}}| - |E_{\{2,4\}}| = n-3.
\end{aligned}$$

Thus, the $M-polynomial$ of $T_n$ is

$$M(T_n;x,y)=\sum_{i\le j}m_{ij}(T_n)x^iy^j=2x^2y^2+2(n-1)x^2y^4+(n-3)x^4y^4.$$

□

**Definition 17.** *The double triangular snake $DT_n$ is a graph consisting of two triangular snakes that have a common path. i.e., a double triangular snake is obtained from the path $P_n: u_1u_2...u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i,(1\le i\le n-1)$ and to a new vertex $w_i,(1\le i\le n-1)$.*

**Theorem 2.22.** *If $DT_n$ is a double triangular snake, then*

$$M(DT_n;x,y)=4x^2y^3+4(n-2)x^2y^6+2x^3y^6+(n-3)x^6y^6.$$

*Proof.* Let double triangular snake $DT_n$ be a graph having $(3n-2)$ vertices and $5(n-1)$ edges. The edge partition of $DT_n$ is given by,

$$\begin{aligned}
|E_{\{2,3\}}| &= |uv\in E(DT_n): d_u=2 \ \ and \ \ d_v=3|=4,\\
|E_{\{2,6\}}| &= |uv\in E(DT_n): d_u=2 \ \ and \ \ d_v=6|=4(n-2),\\
|E_{\{3,6\}}| &= |uv\in E(DT_n): d_u=3 \ \ and \ \ d_v=6|=2,\\
|E_{\{6,6\}}| &= |uv\in E(DT_n): d_u=6 \ \ and \ \ d_v=6|\\
&= |E(DT_n)|-|E_{\{2,3\}}|-|E_{\{2,6\}}|-|E_{\{3,6\}}|=n-3.
\end{aligned}$$

Thus, the $M-polynomial$ of $DT_n$ is

$$M(DT_n;x,y)=\sum_{i\le j}m_{ij}(DT_n)x^iy^j=4x^2y^3+4(n-2)x^2y^6+2x^3y^6+(n-3)x^6y^6.$$

□

**Definition 18.** *An irregular triangular snake $IT_n$ is a graph obtained from the path $P_n: u_1u_2...u_n$ with vertex set $V(IT_n)=V(P_n)\cup\{v_i:1\le i\le n-2\}$ and the edge set $E(IT_n)=E(P_n)\cup\{u_iv_i,v_iu_{i+2}:1\le i\le n-2\}$.*

**Theorem 2.23.** *If $IT_n$ is an irregular triangular snake, then*

$$M(IT_n;x,y)=2x^2y^2+4x^2y^3+2x^3y^4+2(n-4)x^2y^4+(n-5)x^4y^4.$$

*Proof.* Let an irregular triangular snake $IT_n$ be a graph having $2(n-1)$ vertices and $(3n-5)$ edges. The edge partition of $IT_n$ is given by,

$$\begin{aligned}
|E_{\{2,2\}}| &= |uv\in E(IT_n): d_u=2 \ \ and \ \ d_v=2|=2,\\
|E_{\{2,3\}}| &= |uv\in E(IT_n): d_u=2 \ \ and \ \ d_v=3|=4,\\
|E_{\{2,4\}}| &= |uv\in E(IT_n): d_u=2 \ \ and \ \ d_v=4|=2(n-4),\\
|E_{\{3,4\}}| &= |uv\in E(IT_n): d_u=3 \ \ and \ \ d_v=4|=2,\\
|E_{\{4,4\}}| &= |uv\in E(IT_n): d_u=4 \ \ and \ \ d_v=4|\\
&= |E(IT_n)|-|E_{\{2,2\}}|-|E_{\{2,3\}}|-|E_{\{2,4\}}|-|E_{\{3,4\}}|=n-5.
\end{aligned}$$

Thus, the $M-polynomial$ of $IT_n$ is

$$M(IT_n;x,y)=\sum_{i\le j}m_{ij}(IT_n)x^iy^j=2x^2y^2+4x^2y^3+2x^3y^4+2(n-4)x^2y^4+(n-5)x^4y^4.$$

□

**Definition 19.** *The alternate triangular snake $A(T_n)$ is obtained from a path $v_1v_2...v_n$ by joining $v_i$ and $v_{i+1}$ (alternatively) to new vertex $v_i$, that is, every alternate edge of a path is replaced by $C_3$.*

**Theorem 2.24.** *If $A(T_n)$ is an alternate triangular snake, then*

$$M(A(T_n);x,y)=\begin{cases} 2x^2y^2+nx^2y^3+(n-3)x^3y^3 & \text{if } n \text{ is even,}\\ xy^3+x^2y^2+(n-1)x^2y^3+(n-3)x^3y^3 & \text{if } n \text{ is odd .}\end{cases}$$

*Proof.* Let an alternate triangular snake $A(T_n)$ be a graph having $(n+\lfloor\frac{n}{2}\rfloor)$ vertices and $(n-1+\lfloor\frac{n}{2}\rfloor)$ edges. The edge partition of $A(T_n)$ is given as follows:
If $n$ is even, then there will be no pendant edge in $A(T_n)$. Therefore, we have

$$\begin{aligned}|E_{\{2,2\}}| &= |uv\in E(A(T_n)): d_u=2 \;\; and \;\; d_v=2|=2,\\ |E_{\{2,3\}}| &= |uv\in E(A(T_n)): d_u=2 \;\; and \;\; d_v=3|=n,\\ |E_{\{3,3\}}| &= |uv\in E(A(T_n)): d_u=3 \;\; and \;\; d_v=3|\\ &= |E(A(T_n))|-|E_{\{2,2\}}|-|E_{\{2,3\}}|=n-3.\end{aligned}$$

If $n$ is odd, then there will be a pendant edge in $A(T_n)$. Therefore, we have

$$\begin{aligned}|E_{\{1,3\}}| &= |uv\in E(A(T_n)): d_u=1 \;\; and \;\; d_v=3|=1,\\ |E_{\{2,2\}}| &= |uv\in E(A(T_n)): d_u=2 \;\; and \;\; d_v=2|=1,\\ |E_{\{2,3\}}| &= |uv\in E(A(T_n)): d_u=2 \;\; and \;\; d_v=3|=n-1,\\ |E_{\{3,3\}}| &= |uv\in E(A(T_n)): d_u=3 \;\; and \;\; d_v=3|\\ &= |E(A(T_n))|-|E_{\{1,3\}}|-|E_{\{2,2\}}|-|E_{\{2,3\}}|=n-3.\end{aligned}$$

Thus, the $M-polynomial$ of $A(T_n)$ is

$$M(A(T_n);x,y)=\sum_{i\le j} m_{ij}(A(T_n))x^iy^j=\begin{cases} 2x^2y^2+nx^2y^3+(n-3)x^3y^3 & \text{if } n \text{ is even,}\\ xy^3+x^2y^2+(n-1)x^2y^3+(n-3)x^3y^3 & \text{if } n \text{ is odd.}\end{cases}$$

□

**Definition 20.** *A double alternate triangular snake $DA(T_n)$ consists of two alternate triangular snakes that have a common path.*

**Theorem 2.25.** *Let $DA(T_n)$ be a double alternate triangular snake. Then*

$$M(DA(T_n);x,y)=\begin{cases} 4x^2y^3+(4\left\lfloor\frac{n}{2}\right\rfloor-4)x^2y^4+2x^3y^4+(n-3)x^4y^4 & \text{if } n \text{ is even,}\\ xy^4+2x^2y^3+(4\left\lfloor\frac{n}{2}\right\rfloor-2)x^2y^4+x^3y^4+(n-3)x^4y^4 & \text{if } n \text{ is odd.}\end{cases}$$

*Proof.* Let a double alternate triangular snake $DA(T_n)$ be a graph having $(n+2\lfloor\frac{n}{2}\rfloor)$ vertices and $(n-1+4\lfloor\frac{n}{2}\rfloor)$ edges. The edge partition of $DA(T_n)$ is given as follows:
If $n$ is even, then there will be no pendant edge in $DA(T_n)$. Therefore, we have

$$\begin{aligned}|E_{\{2,3\}}| &= |uv\in E(DA(T_n)): d_u=2 \;\; and \;\; d_v=3|=4,\\ |E_{\{2,4\}}| &= |uv\in E(DA(T_n)): d_u=2 \;\; and \;\; d_v=4|=4\left\lfloor\frac{n}{2}\right\rfloor-4,\\ |E_{\{3,4\}}| &= |uv\in E(DA(T_n)): d_u=3 \;\; and \;\; d_v=4|=2,\\ |E_{\{4,4\}}| &= |uv\in E(DA(T_n)): d_u=4 \;\; and \;\; d_v=4|\\ &= |E(DA(T_n))|-|E_{\{2,3\}}|-|E_{\{2,4\}}|-|E_{\{3,4\}}|=n-3.\end{aligned}$$

If $n$ is odd, then there will be a pendant edge in $DA(T_n)$. Therefore, we have

$$\begin{aligned}
|E_{\{1,4\}}| &= |uv \in E(DA(T_n)) : d_u = 1 \ \ and \ \ d_v = 4| = 1, \\
|E_{\{2,3\}}| &= |uv \in E(DA(T_n)) : d_u = 2 \ \ and \ \ d_v = 3| = 2, \\
|E_{\{2,4\}}| &= |uv \in E(DA(T_n)) : d_u = 2 \ \ and \ \ d_v = 4| = 4\left\lfloor \frac{n}{2} \right\rfloor - 2, \\
|E_{\{3,4\}}| &= |uv \in E(DA(T_n)) : d_u = 3 \ \ and \ \ d_v = 4| = 1, \\
|E_{\{4,4\}}| &= |uv \in E(DA(T_n)) : d_u = 4 \ \ and \ \ d_v = 4| \\
&= |E(DA(T_n))| - |E_{\{1,4\}}| - |E_{\{2,3\}}| - |E_{\{2,4\}}| - |E_{\{3,4\}}| = n - 3.
\end{aligned}$$

Thus, the $M-polynomial$ of $DA(T_n)$ is

$$\begin{aligned}
M(DA(T_n); x, y) &= \sum_{i \le j} m_{ij}(DA(T_n))x^i y^j \\
&= \begin{cases} 4x^2y^3 + (4\left\lfloor \frac{n}{2} \right\rfloor - 4)x^2y^4 + 2x^3y^4 + (n-3)x^4y^4 & \text{if } n \text{ is even,} \\ xy^4 + 2x^2y^3 + (4\left\lfloor \frac{n}{2} \right\rfloor - 2)x^2y^4 + x^3y^4 + (n-3)x^4y^4 & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

□

**Definition 21.** *The quadrilateral snake $Q_n$ is obtained from the path $P_n$ by replacing each edge of the path by a quadrilateral $C_4$.*

**Theorem 2.26.** *If $Q_n$ is a quadrilateral snake, then*

$$M(Q_n; x, y) = 4x^2y^2 + 4(n-2)x^2y^4.$$

*Proof.* Let quadrilateral snake $Q_n$ be a graph having $(3n-2)$ vertices and $4(n-1)$ edges. The edge partition of $Q_n$ is given by,

$$\begin{aligned}
|E_{\{2,2\}}| &= |uv \in E(Q_n) : d_u = 2 \ \ and \ \ d_v = 2| = 4, \\
|E_{\{2,4\}}| &= |uv \in E(Q_n) : d_u = 2 \ \ and \ \ d_v = 4| \\
&= |E(Q_n)| - |E_{\{2,2\}}| = 4(n-2).
\end{aligned}$$

Thus, the $M-polynomial$ of $Q_n$ is

$$M(Q_n; x, y) = \sum_{i \le j} m_{ij}(Q_n)x^i y^j = 4x^2y^2 + 4(n-2)x^2y^4.$$

□

**Definition 22.** *A double quadrilateral snake $DQ_n$ is a graph consisting two quadrilateral snakes that have a common path.*

**Theorem 2.27.** *If $DQ_n$ is a double quadrilateral snake, then*

$$M(DQ_n; x, y) = 2(n-1)x^2y^2 + 4x^2y^3 + 4(n-2)x^2y^6 + 2x^3y^6 + (n-3)x^6y^6.$$

*Proof.* Let a double quadrilateral snake $DQ_n$ be a graph having $(5n-4)$ vertices and $7(n-1)$ edges. The edge partition of $DQ_n$ is given by,

$$\begin{aligned}
|E_{\{2,2\}}| &= |uv \in E(DQ_n) : d_u = 2 \ \ and \ \ d_v = 2| = 2(n-1),\\
|E_{\{2,3\}}| &= |uv \in E(DQ_n) : d_u = 2 \ \ and \ \ d_v = 3| = 4,\\
|E_{\{2,6\}}| &= |uv \in E(DQ_n) : d_u = 2 \ \ and \ \ d_v = 6| = 4(n-2),\\
|E_{\{3,6\}}| &= |uv \in E(DQ_n) : d_u = 3 \ \ and \ \ d_v = 6| = 2,\\
|E_{\{6,6\}}| &= |uv \in E(DQ_n) : d_u = 6 \ \ and \ \ d_v = 6|\\
&= |E(DQ_n)| - |E_{\{2,2\}}| - |E_{\{2,3\}}| - |E_{\{2,6\}}| - |E_{\{3,6\}}| = n-3.
\end{aligned}$$

Thus, the $M-polynomial$ of $DQ_n$ is

$$M(DQ_n;x,y) = \sum_{i\le j} m_{ij}(DQ_n)x^iy^j = 2(n-1)x^2y^2+4x^2y^3+4(n-2)x^2y^6+2x^3y^6+(n-3)x^6y^6.$$

□

**Definition 23.** *The alternate quadrilateral snake $A(Q_n)$ is obtained from a path $v_1v_2...v_n$ by joining $v_i, v_{i+1}$ (alternatively) to new vertices $v_i, w_i$ respectively and then joining $v_i$ and $w_i$. i.e., every alternate edge of a path is replaced by a cycle $C_4$.*

**Theorem 2.28.** *If $A(Q_n)$ is an alternate quadrilateral snake, then*

$$M(A(Q_n);x,y) = \begin{cases} \left(\frac{n}{2}+2\right)x^2y^2 + nx^2y^3 + (n-3)x^3y^3 & \textit{if } n \textit{ is even},\\ xy^3 + \left(\lfloor\frac{n}{2}\rfloor+1\right)x^2y^2 + 2\lfloor\frac{n}{2}\rfloor x^2y^3 + (n-3)x^3y^3 & \textit{if } n \textit{ is odd}.\end{cases}$$

*Proof.* Let an alternate quadrilateral snake $A(Q_n)$ be a graph having $(n+2\lfloor\frac{n}{2}\rfloor)$ vertices and $(3\lfloor\frac{n}{2}\rfloor+n-1)$ edges. The edge partition of $A(Q_n)$ is given as follows:
If $n$ is even, then there will be no pendant edge in $A(T_n)$. Therefore, we have

$$\begin{aligned}
|E_{\{2,2\}}| &= |uv \in E(A(Q_n)) : d_u = 2 \ \ and \ \ d_v = 2| = \frac{n}{2}+2,\\
|E_{\{2,3\}}| &= |uv \in E(A(Q_n)) : d_u = 2 \ \ and \ \ d_v = 3| = n,\\
|E_{\{3,3\}}| &= |uv \in E(A(Q_n)) : d_u = 3 \ \ and \ \ d_v = 3|\\
&= |E(A(Q_n))| - |E_{\{2,2\}}| - |E_{\{2,3\}}| = n-3.
\end{aligned}$$

If $n$ is odd, then there will be a pendant edge in $A(T_n)$. Therefore, we have

$$\begin{aligned}
|E_{\{1,3\}}| &= |uv \in E(A(Q_n)) : d_u = 1 \ \ and \ \ d_v = 3| = 1,\\
|E_{\{2,2\}}| &= |uv \in E(A(Q_n)) : d_u = 2 \ \ and \ \ d_v = 2| = \left\lfloor\frac{n}{2}\right\rfloor+1,\\
|E_{\{2,3\}}| &= |uv \in E(A(Q_n)) : d_u = 2 \ \ and \ \ d_v = 3| = 2\left\lfloor\frac{n}{2}\right\rfloor,\\
|E_{\{3,3\}}| &= |uv \in E(A(Q_n)) : d_u = 3 \ \ and \ \ d_v = 3|\\
&= |E(A(Q_n))| - |E_{\{1,3\}}| - |E_{\{2,2\}}| - |E_{\{2,3\}}| = n-3.
\end{aligned}$$

Thus, the $M-polynomial$ of $A(Q_n)$ is

$$\begin{aligned}
M(A(Q_n);x,y) &= \sum_{i\le j} m_{ij}(A(Q_n))x^iy^j\\
&= \begin{cases} \left(\frac{n}{2}+2\right)x^2y^2 + nx^2y^3 + (n-3)x^3y^3 & \textit{if } n \textit{ is even},\\ xy^3 + \left(\lfloor\frac{n}{2}\rfloor+1\right)x^2y^2 + 2\lfloor\frac{n}{2}\rfloor x^2y^3 + (n-3)x^3y^3 & \textit{if } n \textit{ is odd}.\end{cases}
\end{aligned}$$

□

**Definition 24.** *An irregular quadrilateral snake $IQ_n$ is a graph obtained from the path $P_n : u_1u_2...u_n$ with vertex set $V(IQ_n) = V(P_n) \cup \{v_i, w_i : 1 \leq i \leq n-2\}$ and the edge set $E(IQ_n) = E(P_n) \cup \{u_iv_i, w_iu_{i+2} : 1 \leq i \leq n-2\}$.*

**Theorem 2.29.** *If $IQ_n$ is an irregular quadrilateral snake, then*

$$M(IQ_n; x, y) = nx^2y^2 + 4x^2y^3 + 2(n-4)x^2y^4 + 2x^3y^4 + (n-5)x^4y^4.$$

*Proof.* Let an irregular quadrilateral snake $IQ_n$ be a graph having $(3n-4)$ vertices and $(4n-7)$ edges. The edge partition of $IQ_n$ is given by,

$$\begin{aligned}
|E_{\{2,2\}}| &= |uv \in E(IQ_n) : d_u = 2 \ \ and \ \ d_v = 2| = n, \\
|E_{\{2,3\}}| &= |uv \in E(IQ_n) : d_u = 2 \ \ and \ \ d_v = 3| = 4, \\
|E_{\{2,4\}}| &= |uv \in E(IQ_n) : d_u = 2 \ \ and \ \ d_v = 4| = 2(n-4), \\
|E_{\{3,4\}}| &= |uv \in E(IQ_n) : d_u = 3 \ \ and \ \ d_v = 4| = 2, \\
|E_{\{4,4\}}| &= |uv \in E(IQ_n) : d_u = 4 \ \ and \ \ d_v = 4| \\
&= |E(IQ_n)| - |E_{\{2,2\}}| - |E_{\{2,3\}}| - |E_{\{2,4\}}| - |E_{\{3,4\}}| = n-5.
\end{aligned}$$

Thus, the $M-polynomial$ of $IQ_n$ is

$$M(IQ_n; x, y) = \sum_{i \leq j} m_{ij}(IQ_n)x^iy^j = nx^2y^2+4x^2y^3+2(n-4)x^2y^4+2x^3y^4+(n-5)x^4y^4.$$

□

**Definition 25.** *A double alternate quadrilateral snake $DA(Q_n)$ consists of two alternate quadrilateral snakes that have a common path.*

**Theorem 2.30.** *If $DA(Q_n)$ is a double alternate quadrilateral snake, then*

$$M(DA(Q_n); x, y) = \begin{cases} nx^2y^2 + 4x^2y^3 + 2(n-2)x^2y^4 + 2x^3y^4 + (n-3)x^4y^4 & \textit{if n is even}, \\ xy^4 + 2\lfloor \frac{n}{2} \rfloor x^2y^2 + 2x^2y^3 + 2(n-2)x^2y^4 + x^3y^4 + (n-3)x^4y^4 & \textit{if n is odd}. \end{cases}$$

*Proof.* Let a double alternate quadrilateral snake $DA(Q_n)$ be a graph having $(n + 4\lfloor \frac{n}{2} \rfloor)$ vertices and $(6\lfloor \frac{n}{2} \rfloor + n - 1)$ edges. The edge partition of $DA(Q_n)$ is given as follows:
If $n$ is even, then there will be no pendant edge in $DA(T_n)$. Therefore, we have

$$\begin{aligned}
|E_{\{2,2\}}| &= |uv \in E(DA(Q_n)) : d_u = 2 \ \ and \ \ d_v = 2| = n, \\
|E_{\{2,3\}}| &= |uv \in E(DA(Q_n)) : d_u = 2 \ \ and \ \ d_v = 3| = 4, \\
|E_{\{2,4\}}| &= |uv \in E(DA(Q_n)) : d_u = 2 \ \ and \ \ d_v = 4| = 2(n-2), \\
|E_{\{3,4\}}| &= |uv \in E(DA(Q_n)) : d_u = 3 \ \ and \ \ d_v = 4| = 2, \\
|E_{\{4,4\}}| &= |uv \in E(DA(Q_n)) : d_u = 4 \ \ and \ \ d_v = 4| \\
&= |E(DA(Q_n))| - |E_{\{2,2\}}| - |E_{\{2,3\}}| - |E_{\{2,4\}}| - |E_{\{3,4\}}| = n-3.
\end{aligned}$$

If $n$ is odd, then there will be a pendant edge in $DA(T_n)$. Therefore, we have

$$\begin{aligned}
|E_{\{1,4\}}| &= |uv \in E(DA(Q_n)) : d_u = 1 \;\; and \;\; d_v = 4| = 1, \\
|E_{\{2,2\}}| &= |uv \in E(DA(Q_n)) : d_u = 2 \;\; and \;\; d_v = 2| = 2\left\lfloor \frac{n}{2} \right\rfloor, \\
|E_{\{2,3\}}| &= |uv \in E(DA(Q_n)) : d_u = 2 \;\; and \;\; d_v = 3| = 2, \\
|E_{\{2,4\}}| &= |uv \in E(DA(Q_n)) : d_u = 2 \;\; and \;\; d_v = 4| = 2(n-2), \\
|E_{\{3,4\}}| &= |uv \in E(DA(Q_n)) : d_u = 3 \;\; and \;\; d_v = 4| = 1, \\
|E_{\{4,4\}}| &= |uv \in E(DA(Q_n)) : d_u = 4 \;\; and \;\; d_v = 4| \\
&= |E(DA(Q_n))| - |E_{\{1,4\}}| - |E_{\{2,2\}}| - |E_{\{2,3\}}| - |E_{\{2,4\}}| - |E_{\{3,4\}}| = n-3.
\end{aligned}$$

Thus, the $M-polynomial$ of $DA(Q_n)$ is

$$\begin{aligned}
M(DA(Q_n); x, y) &= \sum_{i \leq j} m_{ij}(DA(Q_n)) x^i y^j \\
&= \begin{cases} nx^2y^2 + 4x^2y^3 + 2(n-2)x^2y^4 + 2x^3y^4 + (n-3)x^4y^4 & \text{if } n \text{ is even}, \\ xy^4 + 2\lfloor \frac{n}{2} \rfloor x^2y^2 + 2x^2y^3 + 2(n-2)x^2y^4 + x^3y^4 + (n-3)x^4y^4 & \text{if } n \text{ is odd}. \end{cases}
\end{aligned}$$

□

**Definition 26.** *The graph $DW_n$ is a graph consisting of the two wheels $W_n$ of the same order having the same central vertex.*

**Theorem 2.31.** *If $DW_n$ is a graph with $(2n+1)$ vertices and $4n$ edges, then*

$$M(DW_n; x, y) = 2nx^3y^3 + 2nx^3y^{2n}.$$

*Proof.* Let $DW_n$ be a graph having $(2n+1)$ vertices and $4n$ edges. The edge partition of $DW_n$ is given by,

$$\begin{aligned}
|E_{\{3,3\}}| &= |uv \in E(DW_n) : d_u = 3 \;\; and \;\; d_v = 3| = 2n, \\
|E_{\{3,2n\}}| &= |uv \in E(DW_n) : d_u = 3 \;\; and \;\; d_v = 2n| \\
&= |E(DW_n)| - |E_{\{3,3\}}| = 2n.
\end{aligned}$$

Thus, the $M-polynomial$ of $DW_n$ is

$$M(DW_n; x, y) = \sum_{i \leq j} m_{ij}(DW_n) x^i y^j = 2nx^3y^3 + 2nx^3y^{2n}.$$

□

**Definition 27.** *The $AC_n$ be a graph obtained from a cycle $C_n : u_1u_2...u_nu_1$ with the vertex set $V(AC_n) = V(C_n) \cup \{v_i, w_i : 1 \leq i \leq n\}$ and the edge set $E(AC_n) = E(C_n) \cup \{u_iv_i, v_iw_i : 1 \leq i \leq n\}$.*

**Theorem 2.32.** *If $AC_n$ is a graph with $3n$ vertices and $3n$ edges, then*

$$M(AC_n; x, y) = nxy^2 + nx^2y^3 + nx^3y^3.$$

*Proof.* Let $AC_n$ is a graph having $3n$ vertices and $3n$ edges. The edge partition of $AC_n$ is given by,

$$\begin{aligned}
|E_{\{1,2\}}| &= |uv \in E(AC_n) : d_u = 1 \;\; and \;\; d_v = 2| = n, \\
|E_{\{2,3\}}| &= |uv \in E(AC_n) : d_u = 2 \;\; and \;\; d_v = 3| = n, \\
|E_{\{3,3\}}| &= |uv \in E(AC_n) : d_u = 3 \;\; and \;\; d_v = 3| \\
&= |E(AC_n)| - |E_{\{1,2\}}| - |E_{\{2,3\}}| = n.
\end{aligned}$$

Thus, the $M - polynomial$ of $AC_n$ is

$$M(AC_n; x, y) = \sum_{i \leq j} m_{ij}(AC_n) x^i y^j = nxy^2 + nx^2y^3 + nx^3y^3.$$

□

**Definition 28.** *An umbrella* $U_{m,n} = (P_m + K_1) \circ P_n$ *is a graph of order* $(m+n)$ *and size* $(2m + n - 2)$*, where* $P_m$ *and* $P_n$ *are the two paths of lengths* $(m-1)$ *and* $(n-1)$*, respectively.*

**Theorem 2.33.** *If* $U_{m,n}$ *is an umbrella with* $(m+n)$ *vertices and* $(2m+n-2)$ *edges, then*

$$M(U_{m,n}; x, y) = xy^2 + (n-3)x^2y^2 + 2x^2y^3 + 3x^2y^{m+1} + (m-3)x^3y^3 + (m-2)x^3y^{m+1}.$$

*Proof.* Let an umbrella $U_{m,n}$ be a graph having $(m+n)$ vertices and $(2m+n-2)$ edges. The edge partition of $U_{m,n}$ is given by,

$$\begin{aligned}
|E_{\{1,2\}}| &= |uv \in E(U_{m,n}) : d_u = 1 \ \ and \ \ d_v = 2| = 1, \\
|E_{\{2,2\}}| &= |uv \in E(U_{m,n}) : d_u = 2 \ \ and \ \ d_v = 2| = n - 3, \\
|E_{\{2,3\}}| &= |uv \in E(U_{m,n}) : d_u = 2 \ \ and \ \ d_v = 3| = 2, \\
|E_{\{2,m+1\}}| &= |uv \in E(U_{m,n}) : d_u = 2 \ \ and \ \ d_v = m + 1| = 3, \\
|E_{\{3,3\}}| &= |uv \in E(U_{m,n}) : d_u = 3 \ \ and \ \ d_v = 3| = m - 3, \\
|E_{\{3,m+1\}}| &= |uv \in E(U_{m,n}) : d_u = 3 \ \ and \ \ d_v = m + 1| \\
&= |E(U_{m,n})| - |E_{\{1,2\}}| - |E_{\{2,2\}}| - |E_{\{2,3\}}| - |E_{\{2,m+1\}}| - |E_{\{3,3\}}| = m - 2.
\end{aligned}$$

Thus, the $M - polynomial$ of $U_{m,n}$ is

$$M(U_{m,n}; x, y) = \sum_{i \leq j} m_{ij}(U_{m,n}) x^i y^j = xy^2 + (n-3)x^2y^2 + 2x^2y^3 + 3x^2y^{m+1} + (m-3)x^3y^3 + (m-2)x^3y^{m+1}.$$

□

**Definition 29.** *A Dumbbell* $Db_n$ *is a graph obtained from two cycles of length* $n$ *by joining a vertex from one cycle to a vertex of another cycle.*

**Theorem 2.34.** *If* $Db_n$ *is a dumbbell with* $2n$ *vertices and* $(2n+1)$ *edges, then*

$$M(Db_n; x, y) = 2(n-2)x^2y^2 + 4x^2y^3 + x^3y^3.$$

*Proof.* Let a dumbbell $Db_n$ be a graph having $2n$ vertices and $(2n+1)$ edges. The edge partition of $Db_n$ is given by,

$$\begin{aligned}
|E_{\{2,2\}}| &= |uv \in E(Db_n) : d_u = 2 \ \ and \ \ d_v = 2| = 2(n-2), \\
|E_{\{2,3\}}| &= |uv \in E(Db_n) : d_u = 2 \ \ and \ \ d_v = 3| = 4, \\
|E_{\{3,3\}}| &= |uv \in E(Db_n) : d_u = 3 \ \ and \ \ d_v = 3| \\
&= |E(Db_n)| - |E_{\{2,2\}}| - |E_{\{2,3\}}| = 1.
\end{aligned}$$

Thus, the $M - polynomial$ of $Db_n$ is

$$M(Db_n; x, y) = \sum_{i \leq j} m_{ij}(Db_n) x^i y^j = 2(n-2)x^2y^2 + 4x^2y^3 + x^3y^3.$$

□

**Definition 30.** *The slanting ladder* $SL_n$ *is a graph obtained from two paths* $u_1u_2...u_n$ *and* $v_1v_2...v_n$ *by joining each* $u_i$ *with* $v_{i+1}, (1 \leq i \leq n-1)$.

**Theorem 2.35.** *If $SL_n$ is a slanting ladder with $2n$ vertices and $3(n-1)$ edges, then*
$$M(SL_n;x,y) = 2xy^3 + 4x^2y^3 + 3(n-3)x^3y^3.$$

*Proof.* Let a slanting ladder $SL_n$ be a graph having $2n$ vertices and $3(n-1)$ edges. The edge partition of $SL_n$ is given by,
$$\begin{aligned}
|E_{\{1,3\}}| &= |uv \in E(SL_n) : d_u = 1 \ \ and \ \ d_v = 3| = 2,\\
|E_{\{2,3\}}| &= |uv \in E(SL_n) : d_u = 2 \ \ and \ \ d_v = 3| = 4,\\
|E_{\{3,3\}}| &= |uv \in E(SL_n) : d_u = 3 \ \ and \ \ d_v = 3|\\
&= |E(SL_n)| - |E_{\{1,3\}}| - |E_{\{2,3\}}| = 3(n-3).
\end{aligned}$$
Thus, the $M-polynomial$ of $SL_n$ is
$$M(SL_n;x,y) = \sum_{i\le j} m_{ij}(SL_n)x^iy^j = 2xy^3 + 4x^2y^3 + 3(n-3)x^3y^3.$$
□

**Definition 31.** *The triangular ladder $TL_n$ with vertex set $V(TL_n) = \{u_i, v_i : 1 \le i \le n\}$ and the edge set $E(TL_n) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_{i+1} : 1 \le i \le n\} \cup \{u_iv_i : 1 \le i \le n\}$.*

**Theorem 2.36.** *If $TL_n$ is a triangular ladder with $2n$ vertices and $(4n-3)$ edges, then*
$$M(TL_n;x,y) = 2x^2y^3 + 2x^2y^4 + 4x^3y^4 + (4n-11)x^4y^4.$$

*Proof.* Let a triangular ladder $TL_n$ be a graph having $2n$ vertices and $(4n-3)$ edges. The edge partition of $TL_n$ is given by,
$$\begin{aligned}
|E_{\{2,3\}}| &= |uv \in E(TL_n) : d_u = 2 \ \ and \ \ d_v = 3| = 2,\\
|E_{\{2,4\}}| &= |uv \in E(TL_n) : d_u = 2 \ \ and \ \ d_v = 4| = 2,\\
|E_{\{3,4\}}| &= |uv \in E(TL_n) : d_u = 3 \ \ and \ \ d_v = 4| = 4,\\
|E_{\{4,4\}}| &= |uv \in E(TL_n) : d_u = 4 \ \ and \ \ d_v = 4|\\
&= |E(TL_n)| - |E_{\{2,3\}}| - |E_{\{2,4\}}| - |E_{\{3,4\}}| = (4n-11).
\end{aligned}$$
Thus, the $M-polynomial$ of $TL_n$ is
$$M(TL_n;x,y) = \sum_{i\le j} m_{ij}(TL_n)x^iy^j = 2x^2y^3 + 2x^2y^4 + 4x^3y^4 + (4n-11)x^4y^4.$$
□

**Definition 32.** *The n-cone graph $C_m + \overline{K_n}$ is a graph where $C_m$ is a cycle of order $m$ and $K_n$ is a complete graph of order $n$.*

**Theorem 2.37.** *If $C_m + \overline{K_n}$ is a n-cone with $(m+n)$ vertices and $m(n+1)$ edges, then*
$$M(C_m + \overline{K_n};x,y) = mnx^my^{n+2} + mx^{n+2}y^{n+2}.$$

*Proof.* Let a $n$-cone graph $C_m + \overline{K_n}$ be a graph having $(m+n)$ vertices and $m(n+1)$ edges. The edge partition of $C_m + \overline{K_n}$ is given by,
$$\begin{aligned}
|E_{\{m,n+2\}}| &= |uv \in E(C_m + \overline{K_n}) : d_u = m \ \ and \ \ d_v = n+2| = mn,\\
|E_{\{n+2,n+2\}}| &= |uv \in E(C_m + \overline{K_n}) : d_u = n+2 \ \ and \ \ d_v = n+2|\\
&= |E(C_m + \overline{K_n})| - |E_{\{m,n+2\}}| = m.
\end{aligned}$$

Thus, the $M-polynomial$ of $C_m+\overline{K_n}$ is

$$M(C_m+\overline{K_n};x,y)=\sum_{i\le j}m_{ij}(C_m+\overline{K_n})x^iy^j=mnx^my^{n+2}+mx^{n+2}y^{n+2}.$$

□

**Definition 33.** *The graph $C_n^{+(m,t)}$ is obtained by identifying one vertex of $C_n$ with one end vertex of $m$ paths each of length $t$. In particular, $C_n^{+(1,t)}$ is a tadpole.*

**Theorem 2.38.** *If $C_n^{+(m,t)}$ is a graph with $(n+t)$ vertices and $(mt+n)$ edges, then*

$$M(C_n^{+(m,t)};x,y)=mxy^2+(m+n-2)x^2y^2+(m+2)x^2y^{m+2}.$$

*Proof.* Let $C_n^{+(m,t)}$ be a graph having $(n+t)$ vertices and $(mt+n)$ edges. The edge partition of $C_n^{+(m,t)}$ is given by,

$$\begin{aligned}|E_{\{1,2\}}| &= |uv\in E(C_n^{+(m,t)}):d_u=1\ \ and\ \ d_v=2|=m,\\ |E_{\{2,2\}}| &= |uv\in E(C_n^{+(m,t)}):d_u=2\ \ and\ \ d_v=2|=m+n-2,\\ |E_{\{2,m+2\}}| &= |uv\in E(C_n^{+(m,t)}):d_u=2\ \ and\ \ d_v=m+2|\\ &= |E(C_n^{+(m,t)})|-|E_{\{1,2\}}|-|E_{\{2,2\}}|=m+2.\end{aligned}$$

Thus, the $M-polynomial$ of $C_n^{+(m,t)}$ is

$$M(C_n^{+(m,t)};x,y)=\sum_{i\le j}m_{ij}(C_n^{+(m,t)})x^iy^j=mxy^2+(m+n-2)x^2y^2+(m+2)x^2y^{m+2}.$$

□

**Definition 34.** *The graph $\theta(C_m)^n$ is obtained from $n$ copies of $C_m$ that shares an edge in common, where $C_m$ is a cycle of length $m$. i.e., an $n$ page book graph with $m$-polygonal pages.*

**Theorem 2.39.** *If $\theta(C_m)^n$ is an $n$ page book graph with $m$-polygonal pages, then*

$$M(\theta(C_m)^n;x,y)=n(m-3)x^2y^2+2nx^2y^{n+1}+x^{n+1}y^{n+1}.$$

*Proof.* Let $\theta(C_m)^n$ be a graph having $n(m-2)+2$ vertices and $n(m-1)+1$ edges. The edge partition of $\theta(C_m)^n$ is given by,

$$\begin{aligned}|E_{\{2,2\}}| &= |uv\in E(\theta(C_m)^n):d_u=2\ \ and\ \ d_v=2|=n(m-3),\\ |E_{\{2,n+1\}}| &= |uv\in E(\theta(C_m)^n):d_u=2\ \ and\ \ d_v=n+1|=2n,\\ |E_{\{n+1,n+1\}}| &= |uv\in E(\theta(C_m)^n):d_u=n+1\ \ and\ \ d_v=n+1|\\ &= |E(\theta(C_m)^n)|-|E_{\{2,2\}}|-|E_{\{2,n+1\}}|=1.\end{aligned}$$

Thus, the $M-polynomial$ of $\theta(C_m)^n$ is

$$M(\theta(C_m)^n;x,y)=\sum_{i\le j}m_{ij}(\theta(C_m)^n)x^iy^j=n(m-3)x^2y^2+2nx^2y^{n+1}+x^{n+1}y^{n+1}.$$

□

**Definition 35.** *The kayak paddle graph $KP(k,m,l)$ is a graph obtained by joining two cycles $C_k$ and $C_m$ by a path of length $l$.*

**Theorem 2.40.** *If $KP(k,m,l)$ is a kayak paddle graph, then*

$$M(KP(k,m,l);x,y)=(k+m+l-6)x^2y^2+6x^2y^3.$$

*Proof.* Let $KP(k,m,l)$ be a graph having $(k+m+l-1)$ vertices and $(k+m+l)$ edges. The edge partition of $KP(k,m,l)$ is given by,

$$\begin{aligned} |E_{\{2,2\}}| &= |uv \in E(KP(k,m,l)) : d_u = 2 \ \ and \ \ d_v = 2| = k+m+l-6, \\ |E_{\{2,3\}}| &= |uv \in E(KP(k,m,l)) : d_u = 2 \ \ and \ \ d_v = 3| \\ &= |E(KP(k,m,l))| - |E_{\{2,2\}}| = 6. \end{aligned}$$

Thus, the $M-polynomial$ of $KP(k,m,l)$ is

$$M(KP(k,m,l);x,y) = \sum_{i \le j} m_{ij}(KP(k,m,l))x^i y^j = (k+m+l-6)x^2y^2 + 6x^2y^3.$$

□

**Definition 36.** *The graph $C_n^{(t)}$ is obtained from the one-point union of $t$ cycles of length $n$.*

**Theorem 2.41.** *If $C_n^{(t)}$ is a graph with $t(n-1)+1$ vertices and $nt$ edges, then*

$$M(C_n^{(t)};x,y) = t(n-2)x^2y^2 + 2tx^2y^{2t}.$$

*Proof.* Let $C_n^{(t)}$ be a graph having $t(n-1)+1$ vertices and $nt$ edges. The edge partition of $C_n^{(t)}$ is given by,

$$\begin{aligned} |E_{\{2,2\}}| &= |uv \in E(C_n^{(t)}) : d_u = 2 \ \ and \ \ d_v = 2| = t(n-2), \\ |E_{\{2,2t\}}| &= |uv \in E(C_n^{(t)}) : d_u = 2 \ \ and \ \ d_v = 2t| \\ &= |E(C_n^{(t)})| - |E_{\{2,2\}}| = 2t. \end{aligned}$$

Thus, the $M-polynomial$ of $C_n^{(t)}$ is

$$M(C_n^{(t)};x,y) = \sum_{i \le j} m_{ij}(C_n^{(t)})x^i y^j = t(n-2)x^2y^2 + 2tx^2y^{2t}.$$

□

Note that, the topological indices (that are mentioned in Table 1) of all these special graphs can be obtained by using respective $M-$polynomial and column 4 of Table 1. The process of obtaining these topological indices is given in two Corollaries 2.11 and 2.13 as an illustration.

## 3. Conclusion

In this paper, we have obtained $M$-polynomial of some special graphs and some topological indices of these graphs. The advantage of $M$-polynomial is that, from that one expression we can obtain several degree-based topological indices. It is very challenging to bring all the degree-based topological indices under $M-$polynomial.

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