NEW CORONA AND NEW CLUSTER OF GRAPHS AND THEIR WIENER INDEX

S. GOYAL, P. GARG AND VISHNU NARAYAN MISHRA

ABSTRACT. The Wiener index of \( G \), denoted by \( W(G) \), is defined as

\[
W(G) = \sum_{u \neq v} d(u, v),
\]

where the sum is taken through all unordered pairs of vertices of \( G \). In this paper we introduce new operations namely, new corona and new cluster on graphs and study the Wiener indices of the resulting graphs.

1. Introduction

For a graph \( G = (V, E) \) if \( u, v \in V(G) \) then the distance \( d(u, v) \) between \( u \) and \( v \) is defined as the length of a shortest \( u-v \) path in \( G \). The Wiener index of \( G \), denoted by \( W(G) \), is defined as

\[
W(G) = \sum_{u \neq v} d(u, v),
\]

where the sum is taken through all unordered pairs of vertices of \( G \). Wiener index was introduced by Wiener [14]. It is related to boiling point, heat of evaporation, heat of formation, chromatographic retention times, surface tension, vapour pressure, partition coefficients, total electron energy of polymers, ultrasonic sound velocity, internal energy, etc., see [12]. For this reason Wiener index is widely studied by chemists. Mathematical properties and chemical applications of the Wiener index have been intensively studied over the past thirty years. For more information about the Wiener index in chemistry and mathematics see [10] and [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13], respectively. In this paper we introduce a new corona and new cluster of two graphs and study the Wiener indices of the resulting graphs.

The corona \( G_1 \circ G_2 \) is obtained by taking one copy of \( G_1 \) and \( |G_1| \) copies of \( G_2 \), and by joining each vertex of the ith copy of \( G_2 \) to the ith vertex of \( G_1 \), \( i = 1, 2, \ldots, |G_1| \). We are interested in giving new corona of graphs such that \( V(G_1) \cup E(G_1) \cup V(G_2) \) and \( V(G_1) \cup V'(G_1) \cup V(G_2) \) are the set of vertices. The cluster \( G_1\{G_2\} \), is obtained by taking one copy of \( G_1 \) and \( |G_1| \) copies of a rooted graph \( G_2 \), and by identifying the root of the ith copy of \( G_2 \) with the ith vertex of \( G_1 \), \( i = 1, 2, \ldots, |G_1| \). We are interested in giving new cluster of graphs such that \( V(G_1) \cup E(G_1) \cup (V(G_2) - v) \) and \( V(G_1) \cup V'(G_1) \cup (V(G_2) - v) \) are the set of vertices, where \( v \) is the root vertex of \( G_2 \). For this purpose we first recall some operations on graphs.

2010 Mathematics Subject Classification. 05C12, 05C76.
Key words and phrases. Wiener index, corona and cluster of two graphs, splitting graph, shadow graph.
Corresponding author.
The total graph $T(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if they are adjacent or incident in $G$. The middle graph or semi-total line graph $M(G)$ of $G$ is the graph whose vertex set is $V(G) \cup E(G)$. Two vertices in $M(G)$ are adjacent if and only if they are adjacent edges in $G$, or one is a vertex and another is an edge incident on it in $G$. The semi-total point graph $Q(G)$ of $G$ is the graph whose vertex set is $V(G) \cup E(G)$. Two vertices in $Q(G)$ are adjacent if and only if they are adjacent vertices in $G$, or one is a vertex and another is an edge incident on it in $G$. The subdivision graph $S(G)$ of $G$ is the graph whose vertex set is $V(G) \cup E(G)$. Two vertices in $S(G)$ are adjacent if and only if one is a vertex and another is an edge incident on it in $G$. The subdivision graph $S(G)$, the semi-total point graph $Q(G)$, the middle graph $M(G)$ and the total graph $T(G)$ of a graph $G$ are shown in Figure 1.

![Figure 1. The graphs $G$, $S(G)$, $Q(G)$, $M(G)$, $T(G)$](image)

The splitting graph $\Lambda(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup V'(G)$, where $V'(G)$ is the copy of $V(G)$ (i.e., $V'(G) = v' : v \in V(G)$) and the edge set $E(G) \cup \{xy' : xy \in E(G)\}$. The vertex $v'$ is called the twin of the vertex $v$, (and $v$ the twin of $v'$). The closed splitting graph $\overline{\Lambda}(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup V'(G)$, where $V'(G)$ is the copy of $V(G)$ (i.e., $V'(G) = v' : v \in V(G)$) and the edge set $E(G) \cup \{xx' : x \in V(G)\} \cup \{xy' : xy \in E(G)\}$. The shadow graph of $G$, denoted by $D_2(G)$, has as the vertex set $V(G) \cup V'(G)$, and the edge set $E(G) \cup \{x'y' : x'y \in E(G)\} \cup \{xy' : xy \in E(G)\}$. The closed shadow graph of $G$, denoted by $D_2[G]$, has as the vertex set $V(G) \cup V'(G)$, and the edge set $E(G) \cup \{x'y' : x'y \in E(G)\} \cup \{xy' : xy \in E(G)\} \cup \{xx' : x \in V(G)\}$. The vertex $v'$ is called the twin of the vertex $v$, (and $v$ the twin of $v'$). We illustrate these definitions in Figure 2.

Let $F$ be one of the symbols $S(G), M(G), Q(G), T(G), \Lambda(G), \overline{\Lambda}(G), D_2(G), D_2[G]$. The $F$-corona $F(G_1)oG_2$ is obtained by taking one copy of $F(G_1)$ and $|G_1|$ copies of $G_2$, and by joining each vertex of the $i$th copy of $G_2$ to the $i$th vertex of $G_1$, $i = 1, 2,\ldots, |G_1|$. We illustrate these definitions in Figure 3.

Let $F$ be one of the symbols $S(G), M(G), Q(G), T(G), \Lambda(G), \overline{\Lambda}(G), D_2(G), D_2[G]$. The $F$-cluster $F(G_1)[G_2]$, is obtained by taking one copy of $F(G_1)$ and $|G_1|$ copies of a rooted graph $G_2$, and by identifying the root of the $i$th copy of $G_2$ with the $i$th vertex of $G_1$, $i = 1, 2,\ldots, |G_1|$. We illustrate these definitions in Figure 4.

2. Wiener index of cluster of two graphs

**Lemma 1.** \[\] Let $G_1$ and $G_2$ be two connected graphs. Then,

$$W(G_1 \circ G_2) = (|G_2| + 1)^2 W(G_1) + |G_1| \left| |G_2|^2 - |E(G_2)| \right| + \left| (|G_1|^2 - |G_1|) |G_2| - (|G_2| + 1) \right|.$$
Figure 2. The graphs $G$, $P = \Lambda(G)$, $Q = \overline{\Lambda}(G)$, $R = D_2(G)$ and $S = D_2[G]$

Lemma 2. [11] Let $G_1$ and $G_2$ be two connected graphs. Then,

$$W(G_1\{G_2\}) = (|G_2|^2)W(G_1) + |G_1|W(G_2) + (|G_1|^2 - |G_1||G_2|d(r|G_2),$$

where $d(r|G_2)$ is the sum of distances of all the vertices of $G_2$ from the root vertex of $G_2$.

Theorem 1. Let $G_1$ and $G_2$ be two connected graphs and $F = S(G), Q(G), M(G)$, or $T(G)$. Then,

$$W(F(G_1)\{G_2\}) = |G_1|W(G_2) + |G_2|^2 \sum_{u,v \in V_1} d(u,v|F(G_1)) + (|G_1|^2 - |G_1||G_2|d(r|G_2)$$

$$+ W(F(G_1)) - \sum_{u,v \in V_1} d(u,v|F(G_1)) + (|G_2| - 1) \sum_{u \in V_1, v \in E_1} d(u,v|F(G_1))$$

Further if $F(G_1) = Q(G_1)$ or $T(G_1)$, then

$$W(F(G_1)\{G_2\}) = |G_1|W(G_2) + |G_2|^2W(G_1) + (|G_1|^2 - |G_1||G_2|d(r|G_2)$$

$$+ W(F(G_1)) - W(G_1) + (|G_2| - 1) \sum_{u \in V_1, v \in E_1} d(u,v|F(G_1))$$

$$+ |G_1||E_1|d(r|G_2).$$

Proof. If two vertices $u$ and $v$ belong to the same copy of $G_2$, then

$$d(u,v|F(G_1)\{G_2\}) = d(u,v|G_2).$$

The respective contribution to $W(F(G_1)\{G_2\})$ is clearly,

$$W_1 = |G_1|W(G_2).$$
Figure 3. The graphs $G$, $H$, $S(G)\circ H$, $Q(G)\circ H$, $M(G)\circ H$, $T(G)\circ H$, $P = \Lambda(G)\circ H$, $Q = \overline{\Lambda}(G)\circ H$, $D_2(G)\circ H$ and $D_2[G]\circ H$.

If, however, the vertices $u$ and $v$ of $F(G_1)[G_2]$ belong to different copies of $G_2$, then

$$d(u, v|F(G_1)[G_2]) = d(u, r|G_2) + d(i, j|F(G_1)) + d(v, r|G_2),$$
where \( i \) and \( j \) denote the vertices of \( F(G_1) \) to which the copies of \( G_2 \) are attached. For each fixed pair \( i, j \) there are \( |G_2|^2 \) such pairs \( u, v \) and their contribution to \( W(F(G_1\{G_2\}) \) amounts \( 2|G_2|d(r|G_2| + |G_2|^2d(i, j|F(G_1)) \). Summing these contribution over all the \( |G_1|C_2 \) distinct pairs \( i, j \), we arrive at

\[
W_2 = 2|G_2|^2 |G_1|C_2 d(r|G_2| + |G_2|^2 \sum_{u,v \in V_1} d(u, v|F(G_1)).
\]

Now, if the two vertices \( u \) and \( v \) of \( F(G_1\{G_2 \) belong to a copy of \( G_2 \) and \( F(G_1) \) (not an identifying vertex of \( F(G_1) \)), respectively, then

\[
d(u, v|F(G_1\{G_2\}) = d(u, r|G_2| + d(i, v|F(G_1)),
\]

where \( i \) denote the vertex of \( F(G_1) \) to which the copy of \( G_2 \) is attached. For each fixed \( i \), there are \( |G_2| \) such pairs \( u, v \) and their contribution to \( W(F(G_1\{G_2\}) \) is,

\[
W_3 = |E_1| |G_1| d(r|G_2| + |G_2| \sum_{u \in V_1, v \in E_1} d(u, v|F(G_1)).
\]
If the two vertices $u$ and $v$ of $F(G_1)\{G_2\}$ belongs to $F(G_1)$ (corresponding to the edges of $G_1$), then
\[ d(u, v|F(G_1)\{G_2\}) = d(u, v|F(G_1)). \]
So, their contribution is clearly,
\[ W_4 = \sum_{u, v \in E_1} d(u, v|F(G_1)). \]
Hence,
\[
W(F(G_1)\{G_2\}) = W_1 + W_2 + W_3 + W_4
\]
\[
= |G_1|W(G_2) + 2|G_2|d(r|G_2) + |G_2|^2 \sum_{u, v \in V_1} d(u, v|F(G_1)) + |G_1|d(r|G_2) + \sum_{u, v \in E_1} d(u, v|F(G_1))
\]
\[
= |G_1|W(G_2) + |G_2|^2 \sum_{u, v \in V_1} d(u, v|F(G_1)) + (|G_1|^2 - |G_1|)G_2|d(r|G_2)
\]
\[
+ W(F(G_1)) - \sum_{u, v \in V_1} d(u, v|F(G_1)) + (|G_2| - 1) \sum_{u, v \in E_1} d(u, v|F(G_1))
\]
\[
+ |G_1||E_1|d(r|G_2).
\]

Further if $F(G_1) = Q(G_1)$ or $T(G_1)$, then
\[
\sum_{u, v \in V_1} d(u, v|F(G_1)) = \sum_{u, v \in V_1} d(u, v|G_1) = W(G_1),
\]
Therefore,
\[
W(F(G_1)\{G_2\}) = |G_1|W(G_2) + |G_2|^2W(G_1) + (|G_1|^2 - |G_1|)G_2|d(r|G_2)
\]
\[
+ W(F(G_1)) - W(G_1) + (|G_2| - 1) \sum_{u, v \in V_1} d(u, v|F(G_1))
\]
\[
+ |G_1||E_1|d(r|G_2).
\]

**Theorem 2.** Let $G_1$ and $G_2$ be two connected graphs and $F = \Lambda(G), \Xi(G), D_2(G),$ or $D_2[G]$. Then,
\[
W(F(G_1)\{G_2\}) = |G_1|W(G_2) + |G_2|^2W(G_1) + (|G_1|^2 - |G_1|)G_2|d(r|G_2)
\]
\[
+ W(F(G_1)) - W(G_1) + (|G_2| - 1) \sum_{u, v \in V_1} d(u, v|F(G_1))
\]
\[
+ |G_1|^2d(r|G_2).
\]

Further if $F(G_1) = \Lambda(G_1)$ or $D_2(G_1)$, then
\[
W(F(G_1)\{G_2\}) = |G_1|W(G_2) + W(G_1)((|G_2|^2 + 2|G_1| - 3) + (\sum_{u \neq v} d(u, v|G_1)) + \sum_{u \neq v} d(u, v|G_2))
\]
\[
+ W(F(G_1)) + 2G_1| G_2| - 1).
\]
And if $F(G_1) = \Xi(G_1)$ or $D_2[G_1]$, then
\[
W(F(G_1)\{G_2\}) = |G_1|W(G_2) + W(G_1)((|G_2|^2 + 2|G_1| - 3) + (\sum_{u \neq v} d(u, v|G_1)) + \sum_{u \neq v} d(u, v|G_2))
\]
\[
+ W(F(G_1)) + G_1| G_2| - 1).
\]

**Proof.** If two vertices $u$ and $v$ belong to the same copy of $G_2$, then
\[ d(u, v|F(G_1)\{G_2\}) = d(u, v|G_2). \]
The respective contribution to $W(F(G_1\{G_2\}))$ is clearly,

$$W_1 = |G_1|W(G_2).$$

If, however, the vertices $u$ and $v$ of $F(G_1\{G_2\})$ belong to different copies of $G_2$, then

$$d(u, v|F(G_1\{G_2\})) = d(u, r|G_2) + d(i, j|F(G_1)) + d(v, r|G_2),$$

where $i$ and $j$ denote the vertices of $F(G_1)$ to which the copies of $G_2$ are attached. For each fixed pair $i$, $j$ there are $|G_2|^2$ such pairs $u$, $v$ and their contribution to $W(F(G_1\{G_2\}))$ amounts $2|G_2|d(r|G_2) + |G_2|^2d(i, j|F(G_1))$. Summing these contribution over all the $|G_1|C_2$ distinct pairs $i$, $j$, we arrive at

$$W_2 = 2|G_2|^2d(r|G_2) + |G_2|^2\sum_{u, v \in V_1} d(u, v|F(G_1)).$$

Now, if the two vertices $u$ and $v$ of $F(G_1\{G_2\})$ belong to a copy of $G_2$ and $F(G_1)$ (not an identifying vertex of $F(G_1)$), respectively, then

$$d(u, v|F(G_1\{G_2\})) = d(u, r|G_2) + d(i, v|F(G_1)),$$

where $i$ denote the vertex of $F(G_1)$ to which the copy of $G_2$ is attached. For each fixed $i$, there are $|G_2|$ such pairs $u$, $v$ and their contribution to $W(F(G_1\{G_2\}))$ is,

$$W_3 = |G_1|^2d(r|G_2) + |G_2|\sum_{u, v \in V_1'} d(u, v|F(G_1)).$$

If the two vertices $u$ and $v$ of $F(G_1\{G_2\})$ belongs to $F(G_1)$ (corresponding to the twin of the vertices of $G_1$), then

$$d(u, v|F(G_1\{G_2\})) = d(u, v|F(G_1)).$$

So, their contribution is clearly,

$$W_4 = \sum_{u, v \in V_1'} d(u, v|F(G_1)).$$

Hence,

$$W(F(G_1\{G_2\})) = W_1 + W_2 + W_3 + W_4$$

$$= |G_1|W(G_2) + 2|G_2|^2|G_1|C_2d(r|G_2) + |G_2|^2\sum_{u, v \in V_1} d(u, v|F(G_1))$$

$$+ |G_1|^2d(r|G_2) + |G_2|\sum_{u, v \in V_1'} d(u, v|F(G_1)) + \sum_{u, v \in V_1'} d(u, v|F(G_1))$$

$$= |G_1|W(G_2) + |G_2|^2\sum_{u, v \in V_1} d(u, v|F(G_1)) + (|G_1|^2 - |G_1|)|G_2|d(r|G_2)$$

$$+ W(F(G_1)) - \sum_{u, v \in V_1} d(u, v|F(G_1)) + (|G_2| - 1) \sum_{u, v \in V_1'} d(u, v|F(G_1))$$

$$+ |G_1|^2d(r|G_2).$$

Also since $F(G) = \Lambda(G), \overline{\lambda}(G), D_2(G)$ or $D_2[G]$,

$$\sum_{u, v \in V_1} d(u, v|F(G_1)) = d(u, v|G_1) = W(G_1).$$

Therefore,

$$W(F(G_1\{G_2\})) = |G_1|W(G_2) + |G_2|^2W(G_1) + (|G_1|^2 - |G_1|)|G_2|d(r|G_2)$$

$$+ W(F(G_1)) - W(G_1) + (|G_2| - 1) \sum_{u, v \in V_1, v \in V_1'} d(u, v|F(G_1))$$

$$+ |G_1|^2d(r|G_2).$$
Further if \( F(G) = \Lambda(G) \) or \( D_2(G) \), then
\[
\sum_{u,v \in V_i} d(u,v|F(G_1)) = 2W(G_1) + 2|G_1|.
\]
Therefore,
\[
W(F(G_1)\{G_2\}) = |G_1|W(G_2) + W(G_1)(|G_2|^2 + 2|G_1| - 3) + ((|G_1|^2 - |G_1|)|G_2| + |G_1|^2)d(r|G_2)
+ W(F(G_1)) + 2|G_1|(|G_2| - 1).
\]
Now if \( F(G) = \overline{\Lambda}(G) \) or \( D_2[G] \), then
\[
\sum_{u,v \in V_i} d(u,v|F(G_1)) = 2W(G_1) + |G_1|.
\]
Therefore,
\[
W(F(G_1)\{G_2\}) = |G_1|W(G_2) + W(G_1)(|G_2|^2 + 2|G_1| - 3) + ((|G_1|^2 - |G_1|)|G_2| + |G_1|^2)d(r|G_2)
+ W(F(G_1)) + |G_1|(|G_2| - 1).
\]

3. Wiener index of corona of two graphs

**Theorem 3.** Let \( G_1 \) and \( G_2 \) be two connected graphs and \( F = S(G), Q(G), M(G), \) or \( T(G) \). Then,
\[
W(F(G_1) \circ G_2) = |G_1|W(G_2) + W(G_1)(|G_2| + 1)^2 \sum_{u,v \in V_i} d(u,v|F(G_1)) + (|G_1|^2 - |G_1|)|G_2|
+ 1)|G_2| + W(F(G_1)) - \sum_{u,v \in V_i} d(u,v|F(G_1)) + |G_2| \sum_{u \in V_1, e \in E_1} d(u,e|F(G_1))
+ |G_1||E_1||G_2|.
\]
Further if \( F(G_1) = Q(G_1) \) or \( T(G_1) \), then
\[
W(F(G_1) \circ G_2) = |G_1|W(G_2) + W(G_1)(|G_2| + 1)^2W(G_1) + (|G_1|^2 - |G_1|)(|G_2| + 1)|G_2|
+ W(F(G_1)) - W(G_1) + |G_2| \sum_{u \in V_1, e \in E_1} d(u,e|F(G_1)) + |G_1||E_1||G_2|.
\]

**Proof.** It is a special case of Theorem[1] We have \( F(G_1) \circ G_2 \equiv F(G_1)\{G_2 + K_1\} \), where \( K_1 \) is a one vertex graph and where the root of \( G_2 + K_1 \) is chosen to be the vertex belonging to \( V(K_1) \). □

**Theorem 4.** Let \( G_1 \) and \( G_2 \) be two connected graphs and \( F = \Lambda(G), \overline{\Lambda}(G), D_2(G), \) or \( D_2[G] \). Then,
\[
W(F(G_1) \circ G_2) = |G_1|W(G_2) + W(G_1)(|G_2|^2 + 2|G_1| - 3) + ((|G_1|^2 - |G_1|)|G_2| + |G_1|^2)d(r|G_2)
+ W(F(G_1)) + 2|G_1|(|G_2| - 1).
\]
Further if \( F(G_1) = \Lambda(G_1) \) or \( D_2(G_1) \), then
\[
W(F(G_1) \circ G_2) = |G_1|W(G_2) + W(G_1)(|G_2|^2 + 2|G_1| - 3) + ((|G_1|^2 - |G_1|)|G_2| + |G_1|^2)d(r|G_2)
+ W(F(G_1)) + 2|G_1|(|G_2| - 1).
\]
And if \( F(G_1) = \overline{\Lambda}(G_1) \) or \( D_2[G_1] \), then
\[
W(F(G_1) \circ G_2) = |G_1|W(G_2) + W(G_1)(|G_2|^2 + 2|G_1| - 3) + ((|G_1|^2 - |G_1|)|G_2| + |G_1|^2)d(r|G_2)
+ W(F(G_1)) + |G_1|(|G_2| - 1).
\]

**Proof.** It is a special case of Theorem[2] We have \( F(G_1) \circ G_2 \equiv F(G_1)\{G_2 + K_1\} \), where \( K_1 \) is a one vertex graph and where the root of \( G_2 + K_1 \) is chosen to be the vertex belonging to \( V(K_1) \). □
4. Conclusion

In this paper, we have introduced the new type of corona and cluster operations of two graphs. Further, we have studied the results related to the Wiener index of the said graph operations.

REFERENCES


S. Goyal
DEPARTMENT OF MATHEMATICS & STATISTICS, BANASTHALI UNIVERSITY, BANASTHALI, RAJASTHAN, INDIA
E-mail address: shanugoyalneval@gmail.com

P. Garg
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF RAJASTHAN, JAIPUR, RAJASTHAN, INDIA
E-mail address: garg.pravin@gmail.com

Vishnu Narayan Mishra
DEPARTMENT OF MATHEMATICS, INDIRA GANDHI NATIONAL TRIBAL UNIVERSITY, LALPUR, AMARKANTAK 484887, MADHYA PRADESH, INDIA
L. 1627 AWADH PURI COLONY BENGALI, PHASE -III, OPPOSITE - INDUSTRIAL TRAINING INSTITUTE (I.T.I.), AYODHYA MAIN ROAD, AYODHYA 224 001, UTTAR PRADESH, INDIA
E-mail address: vishnuunarayamishra@gmail.com, vnm@igntu.ac.in