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RELATIVE (p, q)-TH ORDER AND RELATIVE (p, q)-TH TYPE BASED ON SOME GROWTH PROPERTIES OF COMPOSITE P-ADIC ENTIRE FUNCTIONS

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ABSTRACT. Let us suppose that $\mathbb{K}$ be a complete ultrametric algebraically closed field and $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions on $\mathbb{K}$. The main aim of this paper is to study some growth properties of composite p-adic entire functions on the basis of their relative (p, q)-th order, relative (p, q)-th type and relative (p, q)-th weak type.

1. Introduction and Definitions

Suppose $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in]0, +\infty[$, the closed disk $\{x \in \mathbb{K} : |x - \alpha| \leq R\}$ and the open disk $\{x \in \mathbb{K} : |x - \alpha| < R\}$ are represented by $d(\alpha, R)$ and $d(\alpha, R^-)$ respectively. Also $C(\alpha, r)$ denotes the circle $\{x \in \mathbb{K} : |x - a| = r\}$. In addition $\mathcal{A}(\mathbb{K})$ signify the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [11, 14, 16]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [2] to [10], [12, 13]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, if f is not a constant, the $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ with r therefore there exists its inverse function $\widehat{|f|} : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s \to \infty} \widehat{|f|}(s) = \infty$.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm. Further we assume that throughout the present paper p, q, m, n and l always denote

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positive integers. Now taking this into account the (p,q)-th order and (p,q)-th lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ are define as follows:

Definition 1. [5] *Let $f \in \mathcal{A}(\mathbb{K})$ and $p,q \in \mathbb{N}$. Then the (p,q)-th order and (p,q)-th lower order of f are respectively define as:*

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}.$$

Definition 1 avoids the restriction $p \geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [15] in complex context.

When $q = 1$, we get the definitions of generalized order and generalized lower order of an entire function $f \in \mathcal{A}(\mathbb{K})$ which symbolize as $\rho^{(p)}(f)$ and $\lambda^{(p)}(f)$ respectively. If $p = 2$ and $q = 1$ then we write $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$ where $\rho(f)$ and $\lambda(f)$ are respectively known as order and lower order of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al. [2].

In this connection we just introduce the following definition:

Definition 2. *An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p,q) if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1,2,\cdots \end{cases}.$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$,

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1,2,\cdots \end{cases}.$$

An entire function $f \in \mathcal{A}(\mathbb{K})$ of index-pair (p,q) is said to be of regular (p,q)-th growth if its (p,q)-th order coincides with its (p,q)-th lower order, otherwise f is said to be of irregular (p,q)-th growth.

Next, to compare the growth of entire functions on $\mathbb{K}$ having the same (p,q)-th order, we give the definitions of (p,q)-th type and (p,q)-th lower type in the following manner :

Definition 3. [5] *Let $f \in \mathcal{A}(\mathbb{K})$. The (p,q)-th type and the (p,q)-th lower type of f having finite positive (p,q)-th order $\rho^{(p,q)}(f)$ $(0 < \rho^{(p,q)}(f) < \infty)$ are defined as:*

$$\begin{matrix} \sigma^{(p,q)}(f) \\ \overline{\sigma}^{(p,q)}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]}|f|(r)}{\left(\log^{[q-1]} r\right)^{\rho^{(p,q)}(f)}}.$$

Remark 1. *If $p = 2$ and $q = 1$ then we write $\sigma^{(p,q)}(f) = \sigma(f)$ where $\sigma(f)$ is known as type of $f \in \mathcal{A}(\mathbb{K})$ introduced by Boussaf et al.* [2].

Likewise, to compare the growth of entire functions on $\mathbb{K}$ having the same (p,q)-th lower order, one can also introduce the concepts of (p,q)-th weak type in the following manner:

Definition 4. [5] *Let $f \in \mathcal{A}(\mathbb{K})$. The (p,q)-th weak type of f having finite positive (p,q)-th lower order $\lambda^{(p,q)}(f)$ $(0 < \lambda^{(p,q)}(f) < \infty)$ is defined as:*

$$\tau^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p-1]}|f|(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}}.$$

Similarly one may define the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\overline{\tau}^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p-1]}|f|(r)}{\left(\log^{[q-1]} r\right)^{\lambda^{(p,q)}(f)}},\ 0 < \lambda^{(p,q)}(f) < \infty.$$

The notion of relative order was first introduced by Bernal [1]. In order to make some progress in the study of p-adic analysis, recently Biswas [4] introduce the definition of relative order and relative lower order of entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ in the following way:

$$\begin{matrix} \rho_g(f) \\ \lambda_g(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \widehat{|g|}(|f|(r))}{\log r}.$$

Further the function $f \in \mathcal{A}(\mathbb{K})$, for which relative order and relative lower order with respect to another function $g \in \mathcal{A}(\mathbb{K})$ are the same is called a function of regular relative growth with respect to g. Otherwise, f is said to be irregular relative growth.with respect to g.

In the case of relative order, it therefore seems reasonable to define suitably the relative (p,q)-th order of entire function belonging to $\mathcal{A}(\mathbb{K})$. With this in view one may introduce the definition of relative (p,q)-th order $\rho_g^{(p,q)}(f)$ and relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$, in the light of index-pair which are as follows:

Definition 5. [5] *Let $f, g \in \mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p), respectively. Then the relative (p,q)-th order $\rho_g^{(p,q)}(f)$ and relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ of f with respect to g are defined as*

$$\begin{matrix} \rho_g^{(p,q)}(f) \\ \lambda_g^{(p,q)}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} \widehat{|g|}(|f|(r))}{\log^{[q]} r}.$$

In order to refine the above growth scale, now we introduce the definitions of an another growth indicator, called relative (p,q)-th type and relative (p,q)-th lower type respectively of entire function belonging to $\mathcal{A}(\mathbb{K})$ with respect to another entire function belonging to $\mathcal{A}(\mathbb{K})$ in the light of their index-pair which are as follows:

Definition 6. [5] *Let f, $g \in \mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p), respectively. The relative (p,q)-th type and relative (p,q)-th lower type of f with respect to g having finite positive relative (p,q)-th order $\rho_g^{(p,q)}(f)$ $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ are defined as:*

$$\begin{matrix} \sigma_g^{(p,q)}(f) \\ \overline{\sigma}_g^{(p,q)}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \widehat{|g|}(|f|(r))}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}}.$$

Analogously, to determine the relative growth of two entire functions belonging to $\mathcal{A}(\mathbb{K})$ and having same non zero finite relative (p,q)-th lower order with respect to another entire function belonging to $\mathcal{A}(\mathbb{K})$, one can introduce the definition of relative (p,q)-th weak type of an entire function $f \in \mathcal{A}(\mathbb{K})$ with respect to another entire function $g \in \mathcal{A}(\mathbb{K})$ of finite positive relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 7. [5] *Let $f,\ g \in \mathcal{A}(\mathbb{K})$. Also let the index-pair of f and g are (m,q) and (m,p), respectively. The relative (p,q)-th weak type and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of f with respect to g having finite positive relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ $\left(0 < \lambda_g^{(p,q)}(f) < \infty\right)$ are defined as:*

$$\begin{matrix} \overline{\tau}_g^{(p,q)}(f) \\ \tau_g^{(p,q)}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} \widehat{|g|}\left(|f|(r)\right)}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}}.$$

The main aim of this paper is to establish some results related to the growth rates of composite p-adic entire functions on the basis of relative (p,q)-th order, relative (p,q)-th type and relative (p,q)-th weak type.

2. **Lemma**

In this section we present the following lemma which can be found in [2] or [3] and will be needed in the sequel.

Lemma 1. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$. Then for all sufficiently large positive numbers of r the following equality holds*

$$|f \circ g|(r) = |f|(|g|(r)).$$

3. **Main Results**

In this section we present the main results of the paper.

Theorem 1. *Let $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\rho^{(m,n)}(g) > 0$. Then for every positive constant A,*

$$(i) \quad \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[n+1]} r\right)\right)}{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right)} = \infty \text{ if } q = m,$$

$$(ii) \quad \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[q+n+1-m]} r\right)\right)}{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right)} = \infty \text{ if } q > m$$

and

$$(iii) \quad \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[n-1]} r\right)\right)}{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right)} = \infty \text{ if } q \leq m-1 \text{ and } 0 < A < \rho^{(m,n)}(g).$$

Proof. From the definition of $\rho_h^{(p,q)}(f)$, we have for arbitrary positive $\varepsilon(>0)$ and for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) r^A. \tag{1}$$

Also we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[m]}|g|\left(\exp^{[q+n+1-m]}r\right)\geqslant\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}\exp^{[q+n+1-m]}r$$

$$i.e.,\ \log^{[m]}|g|\left(\exp^{[q+n+1-m]}r\right)\geqslant\left(\rho^{(m,n)}(g)-\varepsilon\right)\exp^{[q+1-m]}r$$

$$i.e.,\ \log^{[q-m]}\log^{[m]}|g|\left(\exp^{[q+n+1-m]}r\right)\geqslant\log^{[q-m]}\left[\left(\rho^{(m,n)}(g)-\varepsilon\right)\exp^{[q+1-m]}r\right]$$

$$i.e.,\ \log^{[q]}|g|\left(\exp^{[q+n+1-m]}r\right)\geqslant\exp r+O(1), \tag{2}$$

and

$$\log^{[m]}|g|\left(\exp^{[n-1]}r\right)\geqslant\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}\left(\exp^{[n-1]}r\right)$$

$$i.e.,\ \log^{[m-1]}|g|\left(\exp^{[n-1]}r\right)\geqslant r^{\left(\rho^{(m,n)}(g)-\varepsilon\right)}. \tag{3}$$

Since $\widehat{|h|}(r)$ is an increasing function of r, It follows from Lemma 1 for all sufficiently large positive numbers of r that

$$\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}\left(|g|(r)\right). \tag{4}$$

Case I. Let $q=m$. Then it follows from (4) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n+1]}r\right)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\left(\rho^{(m,n)}(g)-\varepsilon\right)\exp r. \tag{5}$$

Case II. Let $q>m$. Then we get from (2) and (4) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[q+n+1-m]}r\right)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\exp r+O(1). \tag{6}$$

Case III. Again let $q\leq m-1$. Then we have from (3) and (4) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}|g|\left(\exp^{[n-1]}r\right)$$

$$i.e.,\ \log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[m-1]}|g|\left(\exp^{[n-1]}r\right)$$

$$i.e.,\ \log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)r^{\left(\rho^{(m,n)}(g)-\varepsilon\right)}. \tag{7}$$

Now combining (1) and (5) of Case I it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n+1]}r\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^A\right)\right)}\geq\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\left(\rho^{(m,n)}(g)-\varepsilon\right)\exp r}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^A}.$$

Since $\frac{\exp r}{r^A}\to\infty$ as $r\to\infty$, then from above it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n+1]}r\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^A\right)\right)}=\infty,$$

from which the first part of the theorem follows.

Again combining (1) and (6) of Case II we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[q+n+1-m]}r\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)} \geq \frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\exp r+O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^{A}}$$

$$i.e.\ \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[q+n+1-m]}r\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)} = \infty.$$

This establishes the second part of the theorem.

Once more, it follows from (1) and (7) of Case III for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)} \geq \frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)r^{\left(\rho^{(m,n)}(g)-\varepsilon\right)}}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^{A}}.$$

As $A<\rho^{(m,n)}(g)$, therefore we obtain from above that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)}=\infty.$$

This proves the third part of the theorem.

Thus the theorem follows . □

In view of Theorem 1 the following theorem can be carried out:

Theorem 2. *Let* $f,\ g,\ h,k\in\mathcal{A}(\mathbb{K})$. *Also let* $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$ *and* $\lambda^{(m,n)}(g)>0$. *Then for every positive constant* A,

$$(i)\ \lim_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n+1]}r\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)}=\infty\ \text{if}\ q=m$$

and

$$(ii)\ \lim_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[q+n+1-m]}r\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)}=\infty\ \text{if}\ q>m.$$

Theorem 3. *Let* $f,\ g,\ h\in\mathcal{A}(\mathbb{K})$. *Also let* $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, $\rho^{(m,n)}(g)>0$ *and* $\rho_k^{(l,n)}(g)<\infty$ *Then for every positive constant* A,

$$(i)\quad \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n+1]}r\right)\right)}{\log^{[l]}\widehat{|k|}\left(|g|\left(\exp^{[n]}r^{A}\right)\right))} = \infty\ \text{if}\ q=m,$$

$$(ii)\ \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[q+n+1-m]}r\right)\right)}{\log^{[l]}\widehat{|k|}\left(|g|\left(\exp^{[n]}r^{A}\right)\right)} = \infty\ \text{if}\ q>m$$

and

$$(iii)\ \limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)}{\log^{[l]}\widehat{|k|}\left(|g|\left(\exp^{[n]}r^{A}\right)\right)} = \infty\ \text{if}\ q\leq m-1\ \text{and}$$

$$0 < A<\rho^{(m,n)}(g).$$

Proof. Suppose
$$0 < A < A_0 < \rho^{(m,n)}(g). \tag{8}$$

Case I. Let $q = m$. Then in view of the first part of Theorem 1, we have for a sequence of positive numbers of r tending to infinity and $K > 1$
$$\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[n+1]} r\right)\right) > K\left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}. \tag{9}$$

Case II. Let $q > m$. Then we get from the second part of Theorem 1 for a sequence of positive numbers of r tending to infinity and $K > 1$
$$\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[q+n+1-m]} r\right)\right) > K\left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}. \tag{10}$$

Case III. Let $q \leq m-1$. Then we obtain from the third part of Theorem 1 for a sequence of positive numbers of r tending to infinity and $K > 1$
$$\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[n-1]} r\right)\right) > K\left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}. \tag{11}$$

Now from the definition of $\rho_h^{(p,n)}(g)$ we get for all sufficiently large positive numbers of r that
$$\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^A\right)\right) \leq \left(\rho_k^{(l,n)}(g) + \varepsilon\right) r^A. \tag{12}$$

Now combining (9) of Case I and (12) it follows for a sequence of positive numbers of r tending to infinity that
$$\frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[n+1]} r\right)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^A\right)\right)} > \frac{K\left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}}{\left(\rho_k^{(l,n)}(g) + \varepsilon\right) r^A}. \tag{13}$$

Since $A_0 > A$, from (13) it follows that
$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[n+1]} r\right)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^A\right)\right)} = \infty,$$
from which the first part of the theorem follows.

Similarly for $A_0 > A$, we have from (10) of Case II and (12) for a sequence of positive numbers of r tending to infinity that
$$\frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[q+n+1-m]} r\right)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^A\right)\right)} > \frac{K\left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}}{\left(\rho_k^{(l,n)}(g) + \varepsilon\right) r^A}$$
$$i.e. \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[q+n+1-m]} r\right)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^A\right)\right)} = \infty.$$

This establishes the second part of the theorem .

Again it follows from (11) of Case III and (12) for a sequence of positive numbers of r tending to infinity that
$$\frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|\left(\exp^{[n-1]} r\right)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^A\right)\right)} > \frac{K\left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}}{\left(\rho_k^{(l,n)}(g) + \varepsilon\right) r^A}.$$

Therefore in view of (8) and above we get that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^{A}\right)\right)} = \infty,$$

This proves the third part of the theorem.

Thus the theorem is establish. □

Theorem 4. *Let f, g, $h, k \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$, $\lambda^{(m,n)}(g) > 0$ and and $\rho_k^{(l,n)}(g) < \infty$. Then for every positive constant A,*

$$(i)\ \lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[n+1]} r\right)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^{A}\right)\right)} = \infty \text{ if } q = m$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[q+n+1-m]} r\right)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^{A}\right)\right)} = \infty \text{ if } q > m.$$

The proof of Theorem 4 is omitted as it can be carried out with the help of Theorem 2 and Theorem 3.

Theorem 5. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $\lambda^{(m,n)}(g) < \infty$. Then*

$$(i)\ \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^{A}\right)\right)}{\log^{[p]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[n]} r\right)\right)} = \infty \text{ if } q \ge m \text{ and } A > 1,$$

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^{A}\right)\right)}{\log^{[p]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)} = \infty \text{ if } q \ge m \text{ or } q = m-1 \text{ with } m \ne 1 \text{ and } \lambda^{(m,n)}(g) < A$$

and

$$(iii)\ \limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^{A}\right)\right)}{\log^{[p+m-q-1]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)} = \infty \text{ if } m > q+1 \text{ and } A > \lambda^{(m,n)}(g).$$

Proof. From the definition of $\lambda_h^{(p,q)}(f)$, we get for arbitrary positive $\varepsilon\,(>0)$ and for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^{A}\right)\right) \ge \left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A}. \tag{14}$$

Also we have for a sequence of positive numbers of r tending to infinity that

$$\log^{[m]} |g|\left(\exp^{[n-1]} r\right) \le \left(\lambda^{(m,n)}(g) + \varepsilon\right) \log r$$

$$i.e.,\ \log^{[m]} |g|\left(\exp^{[n-1]} r\right) \le \log r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)} \tag{15}$$

$$i.e.,\ \log^{[m-1]} |g|\left(\exp^{[n-1]} r\right) \le r^{\left(\lambda^{(m,n)}(g)+\varepsilon\right)}. \tag{16}$$

Since $\widehat{|h|}(r)$ is an increasing function of r, It follows from Lemma 1 for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}\left(|f\circ g|(r)\right) \le \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]}\left(|g|(r)\right).. \tag{17}$$

Case I. Let $q \geq m$. Then it follows from (17) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n]} r \right) \right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[m]} \left(|g| \left(\exp^{[n]} r \right) \right)$$

$$i.e.,\ \log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n]} r \right) \right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon \right) \left(\lambda^{(m,n)}(g) + \varepsilon \right) r. \quad (18)$$

Case II. Let $q \geq m$ or $q = m - 1$ with $m \neq 1$. Then also we get from (16) and (17) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[m-1]} |g| \left(\exp^{[n-1]} r \right)$$

$$i.e.,\ \log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon \right) r^{\left(\lambda^{(m,n)}(g) + \varepsilon \right)}. \quad (19)$$

Case III. Let $m > q + 1$. Then we obtain from (15) and (17) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon \right) \log^{[q-m]} \log^{[m]} |g| \left(\exp^{[n-1]} r \right)$$

$$i.e.,\ \log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon \right) \exp^{[m-q]} \log r^{\left(\lambda^{(m,n)}(g) + \varepsilon \right)}$$

$$i.e.,\ \log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n-1]} r \right) \right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon \right) \exp^{[m-q-1]} r^{\left(\lambda^{(m,n)}(g) + \varepsilon \right)}$$

$$i.e.,\ \log^{[p+m-q-1]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n-1]} r \right) \right) \leq r^{\left(\lambda^{(m,n)}(g) + \varepsilon \right)} + O(1). \quad (20)$$

Now if $q \geq m$ and $A > 1$, we have from (14) and (18) of Case I for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h|} \left(|f| \left(\exp^{[q]} r^A \right) \right)}{\log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n]} r \right) \right)} \geq \frac{\left(\lambda_h^{(p,q)}(f) - \varepsilon \right) r^A}{\left(\rho_h^{(p,q)}(f) + \varepsilon \right) \left(\lambda^{(m,n)}(g) + \varepsilon \right) r},$$

from which the first part of the theorem follows.

Again combining (14) and (19) of Case II we get for a sequence of positive numbers of r tending to infinity when $q \geq m$ or $q = m\,(\neq 1) - 1$

$$\frac{\log^{[p]} \widehat{|h|} \left(|f| \left(\exp^{[q]} r^A \right) \right)}{\log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n-1]} r \right) \right)} \geq \frac{\left(\lambda_h^{(p,q)}(f) - \varepsilon \right) r^A}{\left(\rho_h^{(p,q)}(f) + \varepsilon \right) r^{\left(\lambda^{(m,n)}(g) + \varepsilon \right)}}. \quad (21)$$

As $A > \lambda^{(m,n)}(g)$ we can choose $\varepsilon\,(> 0)$ in such a way that

$$\lambda^{(m,n)}(g) + \varepsilon < A. \quad (22)$$

Thus from (21) and (22) we get that

$$\limsup_{r \to \infty} \frac{\log^{[p]} \widehat{|h|} \left(|f| \left(\exp^{[q]} r^A \right) \right)}{\log^{[p]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n-1]} r \right) \right)} = \infty.$$

This establishes the second part of the theorem .

When $m > q + 1$, it follows from (14) and (20) of Case III for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]} \widehat{|h|} \left(|f| \left(\exp^{[q]} r^A \right) \right)}{\log^{[p+m-q-1]} \widehat{|h|} \left(|f \circ g| \left(\exp^{[n-1]} r \right) \right)} \geq \frac{\left(\lambda_h^{(p,q)}(f) - \varepsilon \right) r^A}{r^{\left(\lambda^{(m,n)}(g) + \varepsilon \right)} + O(1)}. \quad (23)$$

Now from (22) and (23) we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)}{\log^{[p+m-q-1]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)}=\infty.$$

This proves the third part of the theorem.
Thus the theorem follows . □

In the line of Theorem 5 we may state the following theorem without proof.

Theorem 6. *Let f, g, $h,k\in\mathcal{A}(\mathbb{K})$. Also let $\rho_h^{(p,q)}(f)<\infty$, $\lambda^{(m,n)}(g)<\infty$ and and $\lambda_k^{(l,n)}(g)<\infty$. Then for every positive constant A,*

$$(i)\quad \limsup_{r\to\infty}\frac{\log^{[l]}\widehat{|k|}\left(|g|\left(\exp^{[n]}r^{A}\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n]}r\right)\right)}=\infty \text{ if } q\geq m \text{ and } A>1,$$

$$(ii)\quad \limsup_{r\to\infty}\frac{\log^{[l]}\widehat{|k|}\left(|g|\left(\exp^{[n]}r^{A}\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)}=\infty \text{ if } q\geq m$$

$$\text{or } q=m-1 \text{ with } m\neq 1 \text{ and } \lambda^{(m,n)}(g)<A$$

and

$$(iii)\quad \limsup_{r\to\infty}\frac{\log^{[l]}\widehat{|k|}\left(|g|\left(\exp^{[n]}r^{A}\right)\right)}{\log^{[p+m-q-1]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)}=\infty \text{ if } m>q+1 \text{ and } A>\lambda^{(m,n)}(g).$$

Theorem 7. *Let f, g, $h\in\mathcal{A}(\mathbb{K})$. Also let $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$ and $\rho^{(m,n)}(g)<\infty$. Then*

$$(i)\quad \lim_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n]}r\right)\right)}=\infty \text{ if } q\geq m \text{ and } A>1,$$

$$(ii)\quad \lim_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)}=\infty \text{ if } q\geq m$$

$$\text{or } q=m-1 \text{ with } m\neq 1 \text{ and } \rho^{(m,n)}(g)<A$$

and

$$(iii)\quad \lim_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f|\left(\exp^{[q]}r^{A}\right)\right)}{\log^{[p+m-q-1]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)}=\infty \text{ if } m>q+1 \text{ and } A>\rho^{(m,n)}(g).$$

Theorem 8. *Let f, g, $h,k\in\mathcal{A}(\mathbb{K})$. Also let $\rho_h^{(p,q)}(f)<\infty$, $\rho^{(m,n)}(g)<\infty$ and $\lambda_k^{(l,n)}(g)<\infty$. Then for every positive constant A,*

$$(i)\quad \lim_{r\to\infty}\frac{\log^{[l]}\widehat{|k|}\left(|g|\left(\exp^{[n]}r^{A}\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n]}r\right)\right)}=\infty \text{ if } q\geq m \text{ and } A>1,$$

$$(ii)\quad \lim_{r\to\infty}\frac{\log^{[l]}\widehat{|k|}\left(|g|\left(\exp^{[n]}r^{A}\right)\right)}{\log^{[p]}\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]}r\right)\right)}=\infty \text{ if } q\geq m$$

$$or\ q = m-1\ with\ m \neq 1\ and\ \rho^{(m,n)}(g) < A$$

and

$$(iii)\ \lim_{r\to\infty} \frac{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]} r^A\right)\right)}{\log^{[p+m-q-1]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)} = \infty\ if\ m > q+1\ and\ A > \rho^{(m,n)}(g).$$

We omit the proof of Theorem 7 and Theorem 8 as those can be carried out in the line of Theorem 5 and Theorem 6 respectively.

As an application of Theorem 1 and Theorem 5, we may state the following theorem:

Theorem 9. *Let $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\lambda^{(m,n)}(g) < A < \rho^{(m,n)}(g)$. Then for $q = m-1$ and $m \neq 1$,*

$$\liminf_{r\to\infty} \frac{\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)}{\widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right)} \leq 1 \leq \limsup_{r\to\infty} \frac{\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)}{\widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right)}.$$

Proof. In view of Theorem 1 we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right) > \log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right)$$

$$i.e.,\ \widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right) > \widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right)$$

$$i.e.,\ \limsup_{r\to\infty} \frac{\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)}{\widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right)} > 1. \tag{24}$$

Again from Theorem 5 we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right) \geq \log^{[p]} \widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)$$

$$i.e.,\ \widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right) \geq \widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)$$

$$i.e.,\ \liminf_{r\to\infty} \frac{\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)}{\widehat{|h|}\left(|f|\left(\exp^{[q]} r^A\right)\right)} < 1. \tag{25}$$

Thus the theorem follows from (24) and (25) . □

In view of Theorem 3 and Theorem 6, the following theorem can be carried out:

Theorem 10. *Let $f,\ g,\ h, k \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $0 < \lambda_k^{(p,n)}(g) \leq \rho_k^{(p,n)}(g) < \infty$ and $\lambda^{(m,n)}(g) < A < \rho^{(m,n)}(g)$. Then for $q = m-1$ and $m \neq 1$,*

$$\liminf_{r\to\infty} \frac{\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)}{\widehat{|k|}\left(|g|\left(\exp^{[n]} r^A\right)\right)} \leq 1 \leq \limsup_{r\to\infty} \frac{\widehat{|h|}\left(|f\circ g|\left(\exp^{[n-1]} r\right)\right)}{\widehat{|k|}\left(|g|\left(\exp^{[n]} r^A\right)\right)}.$$

The proof is omitted.

Theorem 11. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\overline{\sigma}_g(m,n) > 0$ where $q = m-1$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \frac{\overline{\sigma}^{(m,n)}(g)\cdot \lambda_h^{(p,q)}(f)}{\rho_h^{(p,q)}(f)}.$$

Proof. Since $q = m-1$, so we have from (4) for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}\left(|f\circ g|(r)\right) \geq \left(\lambda_h^{(p,q)}(f) - \varepsilon\right)\log^{[m-1]}\left(|g|(r)\right)$$

$$i.e.,\ \log^{[p]} \widehat{|h|}\left(|f\circ g|(r)\right) \geq \left(\lambda_h^{(p,q)}(f) - \varepsilon\right)\left(\overline{\sigma}^{(m,n)}(g) - \varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}. \tag{26}$$

Now from the definition of $\lambda_h^{(p,q)}(f)$, we get for all sufficiently large positive numbers of r that

$$\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}. \tag{27}$$

Therefore from (26) and (27), it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \frac{\left(\lambda_h^{(p,q)}(f) - \varepsilon\right)\left(\overline{\sigma}^{(m,n)}(g) - \varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}}{\left(\rho_h^{(p,q)}(f) + \varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}}$$

$$i.e.,\ \liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \frac{\overline{\sigma}^{(m,n)}(g)\cdot \lambda_h^{(p,q)}(f)}{\rho_h^{(p,q)}(f)}.$$

Thus the theorem is established. □

Remark 2. *In Theorem 11, if we will replace "$\overline{\sigma}_g(m,n)$" by "$\sigma_g(m,n)$", then Theorem 11 remains valid with "superior limit" replaced by "inferior limit".*

Remark 3. *We remark that in Theorem 11, if we will replace the condition "$0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$" by "$0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$", then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \overline{\sigma}_g(m,n).$$

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 11.

Theorem 12. *Let f, g, $h, k \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f)$, $\rho_k^{(l,n)}(g) < \infty$ and $\overline{\sigma}^{(m,n)}(g) > 0$ where $q = m-1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|(r)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \frac{\overline{\sigma}^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}.$$

Remark 4. *In Theorem 12, if we will replace "$\overline{\sigma}^{(m,n)}(g)$" by "$\sigma^{(m,n)}(g)$", then Theorem 12 remains valid with "inferior limit" replaced by "superior limit".*

Remark 5. *We remark that in Theorem 12, if we will replace the condition "$\rho_k^{(l,n)}(g) < \infty$" by "$\lambda_k^{(l,n)}(g) < \infty$", then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|(r)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho^{(m,n)}(g)}\right)\right)} \geq \frac{\overline{\sigma}^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}. \quad (28)$$

Remark 6. *In Remark 5, if we will replace the conditions "$0 < \lambda_h^{(p,q)}(f)$ and $\lambda_h^{(p,n)}(g) < \infty$" by "$0 < \rho_h^{(p,q)}(f)$ and $\rho_h^{(p,n)}(g) < \infty$" respectively, then is need to go the same replacement in right part of* (28).

Using the concept of (m,n)-th weak type of a p-adic entire function g, we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 11 and Theorem 12 respectively.

Theorem 13. *Let f, g, $h \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\tau^{(m,n)}(g) > 0$ where $q = m-1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|(r)\right)}{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\lambda^{(m,n)}(g)}\right)\right)} \geq \frac{\tau^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\rho_h^{(p,q)}(f)}.$$

Remark 7. *In Theorem 13, if we will replace "$\tau^{(m,n)}(g)$" by "$\overline{\tau}^{(m,n)}(g)$", then Theorem 13 remains valid with "superior limit" replaced by "inferior limit".*

Remark 8. *We remark that in Theorem 13, if we will replace the condition "$0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$" by "$0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$", then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|(r)\right)}{\log^{[p]} \widehat{|h|}\left(|f|\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\lambda^{(m,n)}(g)}\right)\right)} \geq \tau^{(m,n)}(g).$$

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 11 and Theorem 13 respectively.

Theorem 14. *Let f, g, $h, k \in \mathcal{A}(\mathbb{K})$. Also let $0 < \lambda_h^{(p,q)}(f)$, $\rho_k^{(l,n)}(g) < \infty$ and $\tau^{(m,n)}(g) > 0$ where $q = m-1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}\left(|f \circ g|(r)\right)}{\log^{[l]} \widehat{|k|}\left(|g|\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\lambda^{(m,n)}(g)}\right)\right)} \geq \frac{\tau^{(m,n)}(g) \cdot \lambda_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}.$$

Remark 9. *In Theorem 14, if we will replace "$\tau^{(m,n)}(g)$" by "$\overline{\tau}^{(m,n)}(g)$", then Theorem 14 remains valid with "superior limit" replaced by "inferior limit".*

Remark 10. *We remark that in Theorem 14, if we will replace the condition "$0<\lambda_h^{(p,q)}(f)$" by "$0<\rho_h^{(p,q)}(f)$", then*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[l]}\widehat{|k|}\left(|g|\left(\exp^{[n]}\left(\log^{[n-1]}r\right)^{\lambda^{(m,n)}(g)}\right)\right)}\geq\frac{\tau^{(m,n)}(g)\cdot\rho_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}. \quad (29)$$

Remark 11. *In Remark 10, if we will replace the conditions "$0<\rho_h^{(p,q)}(f)$ and $0<\rho_h^{(p,n)}(g)$" by "$0<\lambda_h^{(p,q)}(f)$ and $0<\lambda_h^{(p,n)}(g)$", then is need to go the same replacement in right part of* (29).

Theorem 15. *Let $f,\ g,\ h\in\mathcal{A}(\mathbb{K})$. Also let $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, $\rho_h^{(p,q)}(f)=\rho^{(m,n)}(g)$, $\overline{\sigma}^{(m,n)}(g)<\infty$ and $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ where $q=n=m-1$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)}\geq\frac{\lambda_h^{(p,q)}(f)\cdot\overline{\sigma}^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}. \quad (30)$$

Proof. In view of the condition $\rho_h^{(p,q)}(f)=\rho^{(m,n)}(g)$ and $q=m-1$, we get from (4) for all sufficiently large positive numbers of r that

$$\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\left(\overline{\sigma}^{(m,n)}(g)-\varepsilon\right)\left(\log^{[n-1]}r\right)^{\rho_h^{(p,q)}(f)}. \quad (31)$$

Further in view of definition of $\overline{\sigma}_h^{(p,q)}(f)$, we have for a sequence of positive numbers of r tending to infinity that

$$\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)\leq\left(\overline{\sigma}_h^{(p,q)}(f)+\varepsilon\right)\left(\log^{[n-1]}r\right)^{\rho_h^{(p,q)}(f)}. \quad (32)$$

Now from (31) and (32),it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)}\geq\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\left(\overline{\sigma}^{(m,n)}(g)-\varepsilon\right)\left(\log^{[n-1]}r\right)^{\rho_h^{(p,q)}(f)}}{\left(\overline{\sigma}_h^{(p,q)}(f)+\varepsilon\right)\left(\log^{[n-1]}r\right)^{\rho_h^{(p,q)}(f)}}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)}\geq\frac{\lambda_h^{(p,q)}(f)\cdot\overline{\sigma}^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

□

Remark 12. *In Theorem 15, if we will replace the conditions "$\overline{\sigma}^{(m,n)}(g)<\infty$" and "$0<\overline{\sigma}_h^{(p,q)}(f)<\infty$" by "$\sigma^{(m,n)}(g)<\infty$" and "$0<\sigma_h^{(p,q)}(f)<\infty$", then is need to go the same replacement in right part of* (30). *Also if we replace the conditions $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$ and $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ of Theorem 15 by $0<\rho_h^{(p,q)}(f)<\infty$ and $0<\sigma_h^{(p,q)}(f)<\infty$ respectively, then*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)}\geq\frac{\rho_h^{(p,q)}(f)\cdot\overline{\sigma}^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ *of Theorem 15 by* $0 < \sigma_h^{(p,q)}(f) < \infty$*, then Theorem 15 remains valid with "inferior limit" replaced by "superior limit".*

Now we state the following three theorems without their proofs as those can easily be carried out in the line of Theorem 15.

Theorem 16. *Let* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$*. Also let* $0 < \lambda_h^{(p,q)}(f) < \infty$, $\lambda_h^{(p,q)}(f) = \lambda^{(m,n)}(g)$, $\tau^{(m,n)}(g) < \infty$ *and* $0 < \tau_h^{(p,q)}(f) < \infty$ *where* $q = n = m - 1$*. Then*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)} \geq \frac{\lambda_h^{(p,q)}(f)\cdot\tau^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}. \tag{33}$$

Remark 13. *In Theorem 16, if we will replace the conditions "*$\tau^{(m,n)}(g) < \infty$*" and "*$0 < \tau_h^{(p,q)}(f) < \infty$*" by "*$\overline{\tau}^{(m,n)}(g) < \infty$*" and "*$0 < \overline{\tau}_h^{(p,q)}(f) < \infty$*", then is need to go the same replacement in right part of* (33)*. Also if we replace the conditions* $0 < \lambda_h^{(p,q)}(f) < \infty$ *and* $0 < \tau_h^{(p,q)}(f) < \infty$ *of Theorem 16 by* $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ *and* $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ *respectively, then*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)} \geq \frac{\rho_h^{(p,q)}(f)\cdot\tau^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \tau_h^{(p,q)}(f) < \infty$ *of Theorem 16 by* $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$*, then Theorem 16 remains valid with "inferior limit" replaced by "superior limit".*

Theorem 17. *Let* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$*. Also let* $0 < \lambda_h^{(p,q)}(f) < \infty$, $\lambda_h^{(p,q)}(f) = \rho^{(m,n)}(g)$, $\overline{\sigma}^{(m,n)}(g) < \infty$ *and* $0 < \tau_h^{(p,q)}(f) < \infty$ *where* $q = n = m - 1$*. Then*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)} \geq \frac{\lambda_h^{(p,q)}(f)\cdot\overline{\sigma}^{(m,n)}(g)}{\tau_h^{(p,q)}(f)}. \tag{34}$$

Remark 14. *In Theorem 17, if we will replace the conditions "*$\overline{\sigma}^{(m,n)}(g) < \infty$*" and "*$0 < \tau_h^{(p,q)}(f) < \infty$*" by "*$\sigma^{(m,n)}(g) < \infty$*" and "*$0 < \overline{\tau}_h^{(p,q)}(f) < \infty$*", then is need to go the same replacement in right part of* (34)*. Also if we replace the conditions* $0 < \lambda_h^{(p,q)}(f) < \infty$ *and* $0 < \tau_h^{(p,q)}(f) < \infty$ *of Theorem 17 by* $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ *and* $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$ *respectively, then*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)} \geq \frac{\rho_h^{(p,q)}(f)\cdot\overline{\sigma}^{(m,n)}(g)}{\overline{\tau}_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \tau_h^{(p,q)}(f) < \infty$ *of Theorem 17 by* $0 < \overline{\tau}_h^{(p,q)}(f) < \infty$*, then Theorem 17 remains valid with "inferior limit" replaced by "superior limit".*

Theorem 18. *Let* $f,\ g,\ h \in \mathcal{A}(\mathbb{K})$*. Also let* $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $\rho_h^{(p,q)}(f) = \lambda^{(m,n)}(g)$, $\tau^{(m,n)}(g) < \infty$ *and* $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ *where* $q = n = m-1$*. Then*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\widehat{|h|}\left(|f\circ g|(r)\right)}{\log^{[p-1]}\widehat{|h|}\left(|f|(r)\right)} \geq \frac{\lambda_h^{(p,q)}(f)\cdot\tau^{(m,n)}(g)}{\overline{\sigma}_h^{(p,q)}(f)}. \tag{35}$$

Remark 15. *In Theorem 18, if we will replace the conditions "$\tau^{(m,n)}(g) < \infty$" and "$0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$" by "$\overline{\tau}^{(m,n)}(g) < \infty$" and "$0 < \sigma_h^{(p,q)}(f) < \infty$", then is need to go the same replacement in right part of* (35)*. Also if we replace the conditions $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$ and $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ of Theorem 18 by $0 < \rho_h^{(p,q)}(f) < \infty$ and $0 < \sigma_h^{(p,q)}(f) < \infty$ respectively, then*

$$\limsup_{r\to\infty} \frac{\log^{[p]} \widehat{|h|}(|f \circ g|(r))}{\log^{[p-1]} \widehat{|h|}(|f|(r))} \ge \frac{\rho_h^{(p,q)}(f) \cdot \tau^{(m,n)}(g)}{\sigma_h^{(p,q)}(f)}.$$

Further if we replace the condition $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ of Theorem 18 by $0 < \sigma_h^{(p,q)}(f) < \infty$, then Theorem 18 remains valid with "inferior limit" replaced by "superior limit".

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