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VALUE DISTRIBUTION OF A ALGEBROID FUNCTION AND ITS LINEAR COMBINATIONS OF DERIVATIVES ON ANNULI

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Abstract. In this paper, we establish analogous of Milloux inequality and Hayman's alternative for algebroid functions on annuli. As an application of our results, we deduce some interesting analogous results for algebroid function on annuli.

1. Introduction

The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. The uniqueness problem of algebroid functions was firstly considered by Valiron, afterwards some scholars have got several uniqueness theorems of algebroid functions in the complex plane $\mathbb{C}$, (see [1]-[3], [8]-[11], [14], [16], [18]-[35]). In 2005, Khrystiyanyn and Kondratyuk [6]-[7] gave an extension of the Nevanlinna value distribution theory for meromorphic functions in annuli. In their extension the main characteristics of meromorphic function are one-parameter and posses the same properties as in the classical case of a simply connected domain. After [6]-[7], Fernandez [5] study the value distribution of meromorphic functions in the punctured plane.In 2015,Yang Tan [12], Yang Tan and Yue Wang [13] proved some interesting results on the multiple values and uniqueness of algebroid functions on annuli. Thus it is interesting to consider the uniqueness problem of algebroid functions in multiply connected domains. By Doubly connected mapping theorem [17] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leq r < R \leq +\infty$. We consider only two cases : $r = 0$, $R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus $\mathbb{A} = \mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

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2. **Definitions and main results**

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and algebroid functions (see [4] and [15]).

Let $A_v(z), A_{v-1}(z), ..., A_0(z)$ be a group of analytic functions which have no common zeros and define on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$,

$$\psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + ... + A_1(z)W + A_0(z) = 0. \qquad (1)$$

Then irreducible equation (1) defines a v-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$.

Let $W(z)$ be a v-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, we use the notations

$$m(r, W) = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r, w_j) = \frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta,$$

$$N_1(r, W) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{n_1(t, W)}{t}dt, \qquad N_2(r, W) = \frac{1}{\nu}\int_1^r \frac{n_2(t, W)}{t}dt,$$

$$\overline{N}_1\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1\left(t, \frac{1}{W-a}\right)}{t}dt, \qquad \overline{N}_2\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_1^r \frac{\overline{n}_2\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$m_0(r, W) = m(r, W) + m\left(\frac{1}{r}, W\right) - 2m(1, W), \quad N_0(r, W) = N_1(r, W) + N_2(r, W),$$

$$\overline{N}_0\left(r, \frac{1}{W-a}\right) = \overline{N}_1\left(r, \frac{1}{W-a}\right) + \overline{N}_2\left(r, \frac{1}{W-a}\right),$$

$$N_x(r, W) = N_{x_1}(r, W) + N_{x_2}(r, W).$$

where $w_j(z)(j = 1, 2, ..., \nu)$ is one valued branch of $W(z)$, $n_1(t, W)$ is the counting functions of poles of the function $W(z)$ in $\{z : t < |z| \leq 1\}$ and $n_2(t, W)$ is the counting functions of poles of the function $W(z)$ in $\{z : 1 < |z| \leq t\}$ (both counting multiplicity).$\overline{n}_1\left(t, \frac{1}{W-a}\right)$ is the counting functions of poles of the function $\frac{1}{W-a}$ in $\{z : t < |z| \leq 1\}$ and $\overline{n}_2\left(t, \frac{1}{W-a}\right)$ is the counting functions of poles of the function $\frac{1}{W-a}$ in $\{z : 1 < |z| \leq t\}$ (both ignoring multiplicity).

Let $W(z)$ be a v-valued algebroid function which determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, when $a \in \mathbb{C}$, $n_0\left(r, \frac{1}{W-a}\right) = n_0\left(r, \frac{1}{\psi(z,a)}\right)$, $N_0\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}N_0\left(r, \frac{1}{\psi(z,a)}\right)$. In particular, when $a = 0$, $N_0\left(r, \frac{1}{W}\right) = \frac{1}{\nu}N_0\left(r, \frac{1}{A_0}\right)$. When $a = \infty$, $N_0(r, W) = \frac{1}{\nu}N_0\left(r, \frac{1}{A_v}\right)$; where $n_0\left(r, \frac{1}{W-a}\right)$ and $n_0\left(r, \frac{1}{\psi(z,a)}\right)$ are the counting function of zeros of $W(z) - a$ and $\psi(z, a)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, respectively.

Definition 1. [12] Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, the function

$$T_0(r, W) = m_0(r, W) + N_0(r, W), \quad 1 \leq r < R_0$$

is called Nevanlinna characteristic of $W(z)$.

Lemma 1. [12] (The first fundamental theorem on annuli) Let $W(z)$ be ν-valued algebroid function which is determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, $a \in \mathbb{C}$

$$m_0(r,a) + N_0(r,a) = T_0\,(r,W) + O(1).$$

Lemma 2. [13] (The second fundamental theorem on annuli). Let $W(z)$ be ν-valued algebroid function which is determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, a_k $(k = 1, 2, .., p)$ are p distinct complex numbers (finite or infinite), then we have

$$(p - 2v)T_0\,(r,W) \leq \sum_{k=1}^{p} N_0\left(r, \frac{1}{W - a_k}\right) - N_1(r,W) + S_0(r,W) \qquad (2)$$

$N_1(r,W)$ is the density index of all multiple values including finite or infinite, every τ multiple value counts $\tau - 1$, and

$$S_0(r,W) = m_0\left(r, \frac{W'}{W}\right) + \sum_{j=1}^{p} m_0\left(r, \frac{W'}{W - a_k}\right) + O(1).$$

The remainder of the second fundamental theorem is the following formula

$$S_0(r,W) = O\left(\log T_0(r,W)\right) + O(\log r),$$

outside a set of finite linear measure, if $r \to R_0 = +\infty$, while

$$S_0(r,W) = O\left(\log T_0(r,W)\right) + O\left(\log \frac{1}{R_0 - r}\right),$$

outside a set E of r such that $\int_E \frac{dr}{R_0 - r} < +\infty$, when $r \to R_0 < +\infty$.

Lemma 3. [12] Let $W(z)$ be ν-valued algebroid function which is determined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, if the following conditions are satisfied

$$\liminf_{r \to \infty} \frac{T_0\,(r,W)}{\log r} < \infty, \quad R_0 = +\infty,$$
$$\liminf_{r \to R_0^-} \frac{T_0\,(r,W)}{\log \frac{1}{(R_0 - r)}} < \infty, \quad R_0 < +\infty,$$

then $W(z)$ is an algebraic function.

Lemma 4. [13] Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively, then

$$N_x(r,W) \leq 2(\nu - 1)T_0(r,W) + O(1).$$

Lemma 5. Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively, then

$$T_0(r,W') < 2\nu T_0(r,W) + S_0(r,W).$$

Proof. By Lemma 2 and Lemma 4, we have

$$\begin{aligned} T_0\left(r, W'\right) &= m_0(r, W') + N_0(r, W') \\ &\leq m_0\left(r, W\right) + N_0(r, W) + \overline{N}(r, W) \\ &\quad +2(\nu-1)T_0(r, W) + S_0(r, W) \\ &\leq (2\nu-1)T_0\left(r, W\right) + \overline{N}_0(r, W) + S_0(r, W) \\ &\leq 2\nu T_0\left(r, W\right) + S_0(r, W). \end{aligned}$$

□

In the value distribution theory, it is very important to introduce and study the derivative of a given function. It is natural to ask whether can we establish the analogous of Milloux inequality and Hayman's alternative for algebroid function on annuli.

In this paper, we prove the following theorems and establish an interesting and remarkable result of the Milloux inequality and Heyman's alternative for algebroid function on annuli.

Theorem 1. Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively. Let

$$\Theta(z) = \sum_{l=0}^{k} a_l\, f^{(l)}(z) \tag{3}$$

for any positive integer k.Where $a_0, a_1, a_2, a_3, ..., a_k$ are small functions of $W(z)$. Then

$$m_0\left(R, \frac{\Theta}{W}\right) = S_0(r, W) \tag{4}$$

and

$$\begin{aligned} T_0\left(r, \Theta\right) &\leq [2(\nu-1)(2k-1)+1]T_0\left(r, W\right) + k\overline{N}_0(r, W) + S_0(r, W), \\ &\leq [2\nu(2k-1) - 3(k-1)]T_0\left(r, W\right) + S_0(r, W). \end{aligned} \tag{5}$$

Theorem 2. Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively and $\Theta(z)$ be the function defined by (2.6). If $\Theta(z)$ is not a constant, then

$$\begin{aligned} T_0(r, W) &< \overline{N}_0(r, W) + N_0\left(r, \frac{1}{W}\right) + \sum_{j=1}^{2\nu} \overline{N}_0\left(r, \frac{1}{\Theta - a}\right) \\ &\quad +2N_x(r, W) - N_0^{(0)}\left(r, \frac{1}{\Theta'}\right) + S_0(r, W) \end{aligned} \tag{6}$$

where $(a_j \neq 0, \infty)$ and $N_0^{(0)}\left(r, \frac{1}{\Theta'}\right)$ counts only zeros of Θ' but not the repeated roots of $\Theta = a_j (j = 1, 2, ..., 2\nu)$ in $\mathbb{A}$.

Theorem 3. Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively. $\Theta = W^{(k)}$ and $N_0^{(0)}\left(r, \frac{1}{\Theta'}\right)$

be defined as in Theorem 2.2. Then

$$k\,N_0^1(r,W) \le \overline{N}_0^{(2}(r,W) + \overline{N}_0\left(r,\frac{1}{\Theta-a}\right) + N_0^{(0)}\left(r,\frac{1}{\Theta'}\right) + N_x(r,W) + S_0(r,W) \tag{7}$$

where $N_0^1(r,W)$ counts the simple poles of W and $\overline{N}_0^{(2}(r,W)$ counts the multiple poles of W, not including multiplicity in $\mathbb{A}$.

Theorem 4. Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ respectively. Then

$$T_0(r,W) \le \left(2+\frac{1}{k}\right) N_0\left(r,\frac{1}{W}\right) + \left(2+\frac{2}{k}\right)\overline{N}_0\left(r,\frac{1}{\Theta-a}\right) + 2N_x(r,W) + S_0(r,W). \tag{8}$$

By replacing $\Theta = W^{(k)}$ in the Theorem 2, we get the following result

Corollary 1. Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ respectively and k is any positive integer. Then

$$T_0\,(r,W) \le \overline{N}_0\,(r,W) + N_0\left(r,\frac{1}{W}\right) + \overline{N}_0\left(r,\frac{1}{W^{(k)}-a}\right) - N_0^{(0)}\left(r,\frac{1}{W^{(k+1)}}\right) + N_x(r,W) + S_0(r,W).$$

By Theorem 2, we get the following Corollary

Corollary 2. Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ respectively, with only a finite number of zeros and poles. Then every function Θ as defined in (2.6) assumes every finite complex value, except possibly zero, infinitely often or else is identically constant.

By replacing $\Theta = W^{(k)}$ in the Theorem 4, we get the following result

Corollary 3. Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ respectively. Then

$$T_0(r,W) \le \left(2+\frac{1}{k}\right) N_0\left(r,\frac{1}{W}\right) + \left(2+\frac{2}{k}\right)\overline{N}_0\left(r,\frac{1}{W^{(k)}-a}\right) + 2N_x(r,W) + S_0(r,W).$$

By replacing the value of $F = \dfrac{W-\omega_1}{\omega_2}$, where ω_1 and ω_2 be complex numbers $\omega_2 \ne 0$ and $T_0(r,F) = T_0(r,W) + O(1)$ in Theorem 4. Then we get the following result.

Corollary 4. (Hayman's Alternative on annuli.) Let $W(z)$ be an ν-valued algebroid function defined by (1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$ respectively. Then either W assumes every finite value infinitely often or $W^{(k)}$ assumes every finite value except possibly zero infinitely often in $\mathbb{A}$.

3. **Proof of Theorems**

3.1. **Proof of Theorem 1.**

From Lemma 2, we can get

$$S_0(r, W^{(k)}) = O(\log rT_0(r, W^{(k)})) = O(\log rT_0(r, W)) = S_0(r, W).$$

First of all, we consider the special case when $\Theta(z) = W^{(k)}(z)$. By Lemma 2.2, we have

$$m_0\left(r, \frac{W^{(k)}}{W}\right) = S_0(r, W), \tag{9}$$

and

$$\begin{aligned}
T_0(r, W^{(k)}) &= m_0(r, W^{(k)}) + N_0(r, W^{(k)}) \\
&\leq m_0(r, W) + m_0\left(r, \frac{W^{(k)}}{W}\right) + N_0(r, W^{(k)}) \\
&\leq m_0(r, W) + N_0(r, W) + k\overline{N}_0(r, W) \qquad (10)\\
&\quad +2(\nu-1)(2k-1)T_0(r, W) + S_0(r, W) \\
&\leq (k+1)T_0(r, W) + 2(\nu-1)(2k-1)T_0(r, W) + S_0(r, W) \\
&= [2\nu(2k-1) - 3(k-1)]T_0(r, W) + S_0(r, W).
\end{aligned}$$

In the following, we consider the general case. It is obvious that

$$\begin{aligned}
m_0\left(r, \frac{\Theta}{W}\right) &\leq \sum_{l=0}^{k} m_0\left(r, \frac{a_l W^{(l)}}{W}\right) + \log(k+1) \\
&\leq \sum_{l=0}^{k}\left[m_0(r, a_l) + m_0\left(r, \frac{W^{(l)}}{W}\right)\right] + \log(k+1) \\
&\leq S_0(r, W). \qquad (11)
\end{aligned}$$

Thus, we have

$$m_0(r, \Theta) \leq m_0\left(r, \frac{\Theta}{W}\right) + m_0(r, W) \leq m_0(r, W) + S_0(r, W). \tag{12}$$

On the other hand, we have

$$\begin{aligned}
N_0(r, \Theta) &\leq N_0\left(r, W^{(k)}\right) \leq N_0(r, W) + k\,\overline{N}_0(r, W) \qquad (13)\\
&\quad +2(\nu-1)(2k-1)T_0(r, W) + S_0(r, W).
\end{aligned}$$

it follows from (12) and (13) that

$$\begin{aligned}
T_0(r, \Theta) &= m_0(r, \Theta) + N_0(r, \Theta) \\
&\leq m_0(r, W) + N_0(r, W) + k\overline{N}_0(r, W) + S_0(r, W) \\
&\leq T_0(r, W) + k\,\overline{N}_0(r, W) + 2(\nu-1)(2k-1)T_0(r, W) + S_0(r, W) \\
&\leq [2\nu(2k-1) - 3(k-1)]T_0(r, W) + S_0(r, W).
\end{aligned}$$

Therefore

$$T_0(R, \Theta) \leq [2\nu(2k-1) - 3(k-1)]T_0(r, W) + S(r, W)$$

which completes the proof of Theorem 1.

3.2. **Proof of Theorem 2.** By the Second Fundamental theorem for algebroid functions on annuli, we have

$$m_0\left(r,\Theta\right)+m_0\left(r,\frac{1}{\Theta}\right)+m_0\left(r,\frac{1}{\Theta-a}\right)\leq(2\nu+1)\,T_0\left(r,\Theta\right)-N_0^{(1)}(r,W)-N_x(r,\Theta)+S_0(R,\Theta). \quad (14)$$

By the First Fundamental theorem for algebroid functions on annuli, we have

$$\begin{aligned}
&(2\nu+1)\,T_0\left(r,\Theta\right)-\ N_0^{(1)}(r,W)\\
=\ &m_0\left(r,\Theta\right)+\sum_{j=1}^{2\nu}m_0\left(r,a_j,\Theta\right)+N_0\left(r,\Theta\right)+\sum_{j=1}^{2\nu}N_0\left(r,a_j,\Theta\right)-\left[2\,N_0\left(r,\Theta\right)-\ N_0\left(r,\Theta'\right)+N_0\left(r,\frac{1}{\Theta'}\right)\right]\\
=\ &m_0\left(r,\Theta\right)+\ m_0\left(r,a,\Theta\right)+\ N_0\left(r,a_j,\Theta\right)-\ N_0\left(r,\frac{1}{\Theta'}\right)+N_0\left(r,\Theta'\right)-\ N_0\left(r,\Theta\right). \quad (15)
\end{aligned}$$

It is obvious that

$$\begin{aligned}
N_0\left(R,\Theta'\right)-\ N_0\left(R,\Theta\right)\ &\leq\ \overline{N}_0(r,\Theta)+N_x(r,\Theta)+O(1)\\
&\leq\ \overline{N}_0\left(r,W\right)+2N_x(r,W)+S_0(r,W) \quad (16)
\end{aligned}$$

and

$$\sum_{j=1}^{2\nu}N_0\left(r,\frac{1}{\Theta-a_j}\right)-\ N_0\left(r,\frac{1}{\Theta'}\right)\leq\sum_{j=1}^{2\nu}\overline{N}_0\left(R,\frac{1}{\Theta-a_j}\right)+N_x(r,W)-\ N_0^{(0)}\left(r,\frac{1}{\Theta'}\right). \quad (17)$$

Hence it follows from (14), (15), (16) and (17) that

$$m_0\left(r,\frac{1}{\Theta}\right)\leq\overline{N}_0\left(r,W\right)+\sum_{j=1}^{2\nu}\overline{N}_0\left(r,\frac{1}{\Theta-a}\right)+2N_x(r,W)-\ N_0^0\left(r,\frac{1}{\Theta'}\right)+\ S_0(r,\Theta). \quad (18)$$

From (2.8), we have

$$S_0(r,\Theta)=S_0(r,W).$$

By First fundamental Theorem for algebroid function on annuli, we have

$$\begin{aligned}
T_0\left(r,W\right)\ &=\ m_0\left(r,\frac{1}{W}\right)+\ N_0\left(r,\frac{1}{W}\right)+\ O(1)\\
&\leq\ m_0\left(r,\frac{1}{\Theta}\right)+\ m_0\left(r,\frac{\Theta}{W}\right)+\ N_0\left(r,\frac{1}{W}\right)+\ O(1)\\
&\leq\ m_0\left(r,\frac{1}{\Theta}\right)+\ N_0\left(r,\frac{1}{W}\right)+\ S_0(r,W). \quad (19)
\end{aligned}$$

From (18) and (19), we have

$$T_0\left(r,W\right)\leq\overline{N}_0\left(r,W\right)+\ N_0\left(r,\frac{1}{W}\right)+\sum_{j=1}^{2\nu}\overline{N}_0\left(r,\frac{1}{\Theta-a_j}\right)+2N_x(r,W)-\ N_0^{(0)}\left(r,\frac{1}{\Theta'}\right)+\ S_0(R,W)$$

which completes the Proof of Theorem 2.

3.3. **Proof of Theorem 3.** We first define the function

$$\widehat{W}(z)=\frac{\left(W^{(k+1)}\right)^{k+1}}{\left(a-W^{(k)}\right)^{k+2}}=\frac{(\Theta')^{k+1}}{(a-\Theta)^{k+2}}. \quad (20)$$

Suppose W has a simple pole at z_0, i,e $W(z)=b(z-z_0)^{-1}+O(1)$ for some $b\neq0$. Then differentiating k times,

$$W^{(k)}(z)=\frac{(-1)^k a\,k!}{(z-z_0)^{k+1}}\ \left(1+O((z-z_0)^{k+1})\right).$$

Differentiating again and then substituting it into $\widehat{W}$, we find that

$$\widehat{W}(z)=\frac{(-1)^k(k+1)^{k+1}}{a\,k!}\ \left(1+O((z-z_0)^{k+1})\right).$$

Thus, at a simple pole of W, $\widehat{W} \neq 0, \infty$, but $\widehat{W}'$ has a zero of order at least k. Now we apply first fundamental theorem for algebroid function on annulus to $\dfrac{\widehat{W}'}{\widehat{W}}$, assuming $\widehat{W}$ to be non constant, giving

$$\begin{aligned} m_0\left(r, \frac{\widehat{W}'}{\widehat{W}}\right) \quad - &\quad m_0\left(r, \frac{\widehat{W}}{\widehat{W}'}\right) + O(1) \\ = &\quad N_0\left(r, \frac{\widehat{W}}{\widehat{W}'}\right) - N_0\left(r, \frac{\widehat{W}'}{\widehat{W}}\right) \\ = &\quad N_0\left(r, \widehat{W}\right) + N_0\left(r, \frac{1}{\widehat{W}'}\right) - N_0\left(r, \widehat{W}'\right) - N_0\left(r, \frac{1}{\widehat{W}}\right) \\ = &\quad N_0\left(r, \frac{1}{\widehat{W}'}\right) - N_0\left(r, \frac{1}{\widehat{W}}\right) - \overline{N}_0\left(r, \widehat{W}\right) - N_x(r, W) \\ = &\quad N_0^{(0)}\left(r, \frac{1}{\widehat{W}'}\right) - \overline{N}_0\left(r, \frac{1}{\widehat{W}}\right) - \overline{N}_0\left(r, \widehat{W}\right) - N_x(r, W). \end{aligned} \tag{21}$$

Thus using (21) and the property that $m_0\left(r, \dfrac{\widehat{W}}{\widehat{W}'}\right)$ is non negative, we have

$$\begin{aligned} k\, N_0^1(r, W) \quad \leq &\quad N_0^{(0)}\left(r, \frac{1}{\widehat{W}'}\right) \leq \overline{N}_0\left(r, \frac{1}{\widehat{W}}\right) + \overline{N}_0\left(r, \widehat{W}\right) + N_x(r, W) + m_0\left(r, \frac{\widehat{W}'}{\widehat{W}}\right) + O(1) \\ \leq &\quad \overline{N}_0\left(r, \frac{1}{\widehat{W}}\right) + \overline{N}_0\left(r, \widehat{W}\right) + N_x(r, W) + S_0(r, \widehat{W}). \end{aligned} \tag{22}$$

By (22) and zeros and poles of g can only occur at multiple poles of f, a-points of Θ or zeros of Θ' which are not a-points of Θ and so

$$\overline{N}_0\left(R, \frac{1}{\widehat{W}}\right) + \overline{N}_0\left(R, \widehat{W}\right) \leq \overline{N}_0\left(r, \frac{1}{\Theta - a}\right) + \overline{N}_0^{(2}(r, W) + N_0^{(0)}\left(r, \frac{1}{\Theta'}\right).$$

Hence by (19), we have

$$k\, N_0^1(r, W) \leq \overline{N}_0^{(2}(r, W) + \overline{N}_0\left(r, \frac{1}{\Theta - a}\right) + N_0^{(0)}\left(r, \frac{1}{\Theta'}\right) + N_x(r, W) + S_0(r, W).$$

3.4. **Proof of Theorem 4.** We start by noting that in $N_0(R, f)$, multiple poles are counted at least twice and then apply (2.9)

$$N_0^1(r, W) + 2\,\overline{N}_0^{(2}(r, W) \leq T_0(r, W) \leq \overline{N}_0\,(r, f) + N_0\left(r, \frac{1}{W}\right) + \overline{N}_0\left(r, \frac{1}{\Theta - a}\right) - N_0^{(0)}\left(r, \frac{1}{\Theta'}\right) + 2N_x(r, W) + S_0(r, W). \tag{23}$$

Since $\overline{N}_0(r, W) = N_0^1(r, W) + \overline{N}_0^{(2}(r, W)$, hence by (3.15), we get

$$\overline{N}_0^{(2}(r, W) \leq N_0\left(r, \frac{1}{W}\right) + \overline{N}_0\left(R, \frac{1}{\Theta - a}\right) - N_0^{(0)}\left(R, \frac{1}{\Theta'}\right) + 2N_x(r, W) + S_0(r, W). \tag{24}$$

By (24) and (2.10), we get

$$\begin{aligned} k\, N_0^1(r, W) \quad \leq &\quad N_0\left(r, \frac{1}{W}\right) + \overline{N}_0\left(r, \frac{1}{\Theta - a}\right) - N_0^{(0)}\left(r, \frac{1}{\Theta'}\right) + \overline{N}_0\left(r, \frac{1}{\Theta - a}\right) + N_0^{(0)}\left(r, \frac{1}{\Theta'}\right) + S_0(r, W) \\ \leq &\quad N_0\left(r, \frac{1}{W}\right) + 2\,\overline{N}_0\left(r, \frac{1}{\Theta - a}\right) + S_0(r, W). \end{aligned} \tag{25}$$

By (24) and (25), we can write

$$\begin{aligned}\overline{N}_0(r,W) &= N_0^1(r,W) + \overline{N}_0^{(2}(r,W)\\ &\le \frac{1}{k}N_0\left(r,\frac{1}{W}\right) + \frac{2}{k}\,\overline{N}_0\left(r,\frac{1}{\Theta-a}\right) + N_0\left(r,\frac{1}{W}\right) + \overline{N}_0\left(r,\frac{1}{\Theta-a}\right)\\ &\quad - N_0^{(0)}\left(r,\frac{1}{\Theta'}\right) + N_x(r,W) + S_0(r,W)\end{aligned}$$

$$\overline{N}_0(r,W) \le \left(1+\frac{1}{k}\right) N_0\left(r,\frac{1}{W}\right) + \left(1+\frac{2}{k}\right)\overline{N}_0\left(r,\frac{1}{\Theta-a}\right) - N_0^{(0)}\left(r,\frac{1}{\Theta'}\right) + 2N_x(r,W) + S_0(r,W). \tag{26}$$

Since $N_0^{(0)}\left(r,\frac{1}{\Theta'}\right) \ge 0$, we substitute this and (26) into (2.9), we get

$$T_0(r,W) \le \left(2+\frac{1}{k}\right) N_0\left(r,\frac{1}{W}\right) + \left(2+\frac{2}{k}\right)\overline{N}_0\left(r,\frac{1}{\Theta-a}\right) + S_0(r,W).$$

4. **Open questions**

Can we use Milloux inequality and Hayman's alternative for algebroid functions for more general differential polynomials on annuli and use those to prove results related to sharing of two differential polynomials of algebroid functions on annuli.

Conflict of Interest

The authors declares that there is no conflict of interest regarding the publication of this paper.

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